Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

Reiichiro Kawai & Hiroki Masuda

MI 2010-19

( Received April 26, 2010 )
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling *

Reiichiro Kawai† and Hiroki Masuda‡

This version: April 26, 2010

Abstract

We prove the local asymptotic normality for the full parameters of the normal inverse Gaussian Levy process $X$, when we observe high-frequency data $X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{n\Delta_n}$ with sampling mesh $\Delta_n \to 0$ and the terminal sampling time $n\Delta_n \to \infty$. The rate of convergence turns out to be $(\sqrt{n\Delta_n}, \sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n})$ for the dominating parameter $(\alpha, \beta, \delta, \mu)$, where $\alpha$ stands for the heaviness of the tails, $\beta$ the degree of skewness, $\delta$ the scale, and $\mu$ the location. The essential feature in our study is that the suitably normalized increments of $X$ in small time is approximately Cauchy-distributed, which specifically comes out in the form of the asymptotic Fisher information matrix.

Keywords. High-frequency sampling, local asymptotic normality, normal inverse Gaussian Lévy process.

2010 Mathematics Subject Classification. 60G51, 62E20.

1 Introduction

Lévy processes have been recognized as building blocks for analyzing realistic data structure, which most often loses touch with the conventional Gaussianity especially when dealing with high-frequency data, such as intraday stock returns. For a stochastic-process model based on high-frequency data, one of the most fundamental, yet in no way obvious, issues is estimation of the dominating parameters involved in a Lévy process $X = (X_t)_{t \in \mathbb{R}^+}$ where we observe discrete-time sample $X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{n\Delta_n}$, where $\Delta_n \to 0$ denotes a diminishing sampling mesh. This often lead to a better understanding of estimation performance than in case of targeting the classical independent and identically distributed data with $\Delta_n \equiv \Delta > 0$, a fixed constant. Nevertheless, due to a wide variety of the class of Lévy processes, it is a rather difficult matter to formulate a parametric estimation for the whole class of Lévy processes. In this respect, specific studies become considerably important.

Among others, the normal inverse Gaussian (NIG) Lévy process exhibits attractive natures: the tractability and the availability of a simple simulation method at arbitrary sampling frequencies (i.e., for any $\Delta_n > 0$). The NIG distribution is a four-parameter family, derived as a special case of the five-parameter generalized hyperbolic (GH) distribution introduced by Barndorff-Nielsen [4] for investigating a distribution of size of wind-blown particles of sand. The GH distribution is known to be infinitely divisible (more strongly, selfdecomposable), hence we can associate the GH Lévy process such that its marginal distribution at time 1 is a GH distribution. However, the GH Lévy processes has a drawback for practical use; its marginal distribution at non-unit time may no longer belong to

---

*This work was partly supported by Grant-in-Aid for Young Scientists (B), Japan (H. Masuda).
†Email Address: reiichiro.kawai@gmail.com. Postal Address: Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK.
‡Email Address: hiroki@math.kyushu-u.ac.jp. Postal Address: Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan.
the GH family. Within the GH family, NIG and normal gamma (NG) distributions are known to have the reproducing property, which entails that, if the distribution at unit time for a Lévy process is NIG or NG, then its marginal distribution at any time belongs to the same distribution family. Under discrete sampling, the reproducing property combined with the Markov property helps to simplify the expression of the likelihood function, and its further asymptotic analysis as well.

Besides, toward optimal inference and testing hypothesis concerning $\theta$, a fundamental step is to investigate asymptotic behavior of the likelihood-ratio random fields based on an available data $X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{n\Delta_n}$. In this article, we investigate Local Asymptotic Normality (LAN) for NIG Lévy process observed at discrete time points under large-term and high-frequency sampling design, where $\Delta_n \to 0$ and $n\Delta_n \to \infty$. The concept of LAN was introduced by Lucien Le Cam (1924–2000) in [11] in order to study approximations (simplifications) of statistical tests for large sample, and nowadays has become a vital concept to establish asymptotic optimalities of estimation and test in large-sample framework. For a systematic account concerning the LAN theory, we refer to, among others, Le Cam and Yang [12], Strasser [18] and van der Vaart [19]. Also, Jacod [8] presents a nice concise review in this direction, with a particular focus on the case of diffusion processes. An earlier attempt at systematic study of the LAN for discretely observed Lévy processes was made by Woerner [20], where various LAN results were individually provided for each specific parameter, such as drift, diffusion, scale, and skewness. However, no systematic account for a full-parameter LAN even for NIG Lévy process in case of the high-frequency asymptotics was given. This is the objective of this article.

The rest of this paper is organized as follows. Section 2 is devoted to a brief review of basic facts on the normal inverse Gaussian Levy process and the LAN under high-frequency sampling. Section 3 states our main result, which provides the rate of convergence and the Fisher information matrix in closed form concerning the LAN for NIG Lévy processes discretely observed at high frequency. Also, we partly compare our result with the case of continuous observation, and clarify big differences between them. To maintain the flow of the paper, we collect proofs in Section 4. Our result requires rather lengthy proofs of somewhat routine nature. To avoid overloading the paper, we omit nonessential details in some instances.

2 Preliminaries

2.1 Basic notation

Throughout this article, the following basic notation is used:

- $I(A)$ denotes the indicator function of any event $A$;
- $\mathcal{L}(X)$ denotes the distribution of a random element $X$;
- $\varphi_a$ denotes the characteristic function of $a$, which indicates a distribution or a random variable;
- $\partial_z := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})^\top$ and $\partial_z^2 := \partial_z \partial_z^\top$ for a vector $x = (x_j)_{1 \leq j \leq k}$ with $\top$ denoting transpose, and also we sometimes use the notation $f'$ for the derivative of a function $f$, when no confusion may occur for the differentiating variable;
- $M^\otimes 2 := MM^\top$ for any matrix $M$;
- $C$ denotes a generic positive constant which may vary at each appearance;
- $a_n \lesssim b_n$ and $a_n \sim b_n$ indicate that $a_n \leq Cb_n$ for every $n$ large enough and that $a_n/b_n \to 1$ as $n \to \infty$, respectively.
2.2 Normal inverse Gaussian Lévy process

A univariate Lévy process $X = (X_t)_{t \in \mathbb{R}_+}$ with finite mean has the Lévy-Khintchine representation

$$\varphi_{X_t}(u) = \exp \left\{ t \left( iu \mu - \frac{1}{2} \sigma^2 u^2 + \int (e^{iuz} - 1 - iuz) \nu(dz) \right) \right\},$$

(1)

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\nu(dz)$ is a Lévy measure, i.e., a $\sigma$-finite measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int (1 \wedge |x|^2) \nu(dz) < \infty$. When the generating triplet $(\mu_0, \sigma^2, \nu(dz))$ depends on a finite-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^p$, we denote by $P_\theta$ the distribution of $X$ on the Skorohod space. We refer the reader to Sato [11] for a detailed account of Lévy processes.

The univariate normal inverse Gaussian (NIG) distribution, denoted by $NIG(\alpha, \beta, \delta, \mu)$, is the self-decomposable (hence infinitely divisible) distribution admitting a density

$$y \mapsto \frac{\alpha \delta}{\pi} \exp\{\delta \sqrt{\alpha^2 - \beta^2} + \beta(y - \mu)\} \frac{K_1(\alpha \sqrt{\beta^2 + (y - \mu)^2})}{\sqrt{\beta^2 + (y - \mu)^2}},$$

(2)

where $K_w(y)$, $w \in \mathbb{R}$, $y > 0$, denotes the modified Bessel function of the third kind with index $w$:

$$K_w(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{w-1} \exp\left\{-y \left( x + \frac{1}{x} \right) \right\} dx.$$  

(3)

We write

$$\theta = (\alpha, \beta, \delta, \mu) \in \Theta \subset \mathbb{R}^4,$$

where the parameter space $\Theta$ is a bounded convex domain such that

$$\Theta^- \subset \{(\alpha, \beta, \delta, \mu)|\alpha > 0, \alpha > |\beta| \geq 0, \delta > 0, \mu \in \mathbb{R}\}. \quad (4)$$

(Throughout, we rule out the case where $\alpha = |\beta| \geq 0$.) The distribution $NIG(\alpha, \beta, \delta, \mu)$ exhibits semi-heavy tails in the sense that the density behaves as a constant multiple of $|y|^{-3/2} \exp(-\alpha|y| + \beta y)$ for $|y| \to \infty$, so that moments of any order are finite. The mean and variance of $NIG(\alpha, \beta, \delta, \mu)$ are respectively given by

$$\mu + \frac{\beta \delta}{\sqrt{\alpha^2 - \beta^2}} \quad \text{and} \quad \frac{\alpha^2 \delta}{(\alpha^2 - \beta^2)^{1/2}},$$

and the characteristic function by

$$u \mapsto \exp \left\{ iu \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (iu + \beta)^2}) \right\}. \quad (5)$$

Now, the univariate NIG Lévy process is defined to be a Lévy process $X$ starting from the origin such that $\mathcal{L}(X_1) = NIG(\alpha, \beta, \delta, \mu)$. It is clear from [5] that for any $\Delta_n > 0$ and $a \neq 0$

$$\mathcal{L}(a(X_{\Delta_n} - \mu|\Delta_n)) = NIG\left(\frac{\alpha}{|a|}, \frac{\beta}{a}, \delta |a| \Delta_n, 0 \right). \quad (6)$$

The generating triplet of $X$ is given by $\mu_0 = \mu + \beta \delta / \sqrt{\alpha^2 - \beta^2}$, $\sigma^2 = 0$, and $\nu(dz; \theta) = g(z; \alpha, \beta, \delta)dz$ with

$$g(z; \alpha, \beta, \delta) = \frac{\alpha \delta}{\pi |z|^2} e^{\beta z} K_1(\alpha |z|), \quad z \neq 0. \quad (7)$$

One can consult Barndorff-Nielsen [34] [41] for more analytical facts concerning the NIG distribution and the NIG Lévy process.
2.3 LAN under high-frequency sampling

Fix a \( \theta \in \Theta \), and let \( X \) be a Lévy process observed at \( t_j = t^n_j \), \( j \leq n \), with \( t^n_0 < t^n_1 < \cdots < t^n_n \) for each \( n \). We denote by \( x_j = x_{nj} \) the successive increments:

\[
x_j = x_{nj} := X_{t_j} - X_{t_{j-1}}.
\]

Because of the independent-increments property of \( X \), the sequence \((x_j)_{j \leq n}\) for each \( n \) forms an independent array. For simplicity, we set \( t_j = j \Delta_n \) for some \( \Delta_n > 0 \), so that \( \mathcal{L}(x_j) = \mathcal{L}(X_{t_j}) \) under \( P_\theta \) for every \( j \leq n \). Then, we denote by \( P^{\theta\Delta}_n \) the distribution of \((X_{t_j})_{j \leq n}\) under \( P_\theta \).

Suppose that under \( P_\theta \), \( X \) admits an everywhere positive transition density with respect to the Lebesgue measure on \( \mathbb{R} \), which is of the class \( C^2(\Theta) \) as a function of \( \theta \). According to the stationarity and independence of increments of \( X \), the log-likelihood function takes the form

\[
\ell_n(\theta) = \sum_{j=1}^n \log p_{\Delta_n}(x_j; \theta),
\]

where \( p_{\Delta_n}(x; \theta) \) denotes the density of \( X_{\Delta_n} \) under \( P_\theta \).

Let \((r_n)\) be a nonrandom positive definite diagonal matrices tending to \( 0 \) in norm, and \( \mathcal{I}(\theta) \) a nonnegative definite symmetric \( \mathbb{R}^p \otimes \mathbb{R}^p \) matrix. Pick any \( h \in \mathbb{R}^p \). We may suppose that \( \theta_n := \theta + r_nh \in \Theta \). We say that LAN holds true at \( \theta \) with rate \( r_n \) and Fisher information matrix \( \mathcal{I}(\theta) \), if the stochastic expansion

\[
\log \frac{dP^{\theta\Delta}_n}{dP_\theta} = \ell_n(\theta_n) - \ell_n(\theta) = h^\top S_n(\theta) - \frac{1}{2} \mathcal{I}(\theta)[h] + o_{P_\theta}(1)
\]

holds true, where \( S_n(\theta) := \sum_{j=1}^n r_n \partial_\theta \ell_n(\theta) \to \mathcal{N}_p(0, \mathcal{I}(\theta)) \) weakly under \( P_\theta \), where \( \mathcal{N}_p(0, \mathcal{I}(\theta)) \) stands for the \( p \)-variate normal distribution with mean \( 0 \) and covariance \( \mathcal{I}(\theta) \). Let us note that, in order to apply the general asymptotic optimality theory based on the LAN, the matrix \( \mathcal{I}(\theta) \) has to be positive definite; if not, the LAN is not of much help to clarify asymptotic optimality criteria.

If we have the LAN, then it is known that general criteria for asymptotic optimality of estimation and testing hypotheses follows from the LAN. Here, let us briefly mention the following (see the references cited in Section 3 for more details): if one has asymptotically normally distributed estimator \( \hat{\theta}_n \) of \( \theta \), say \( c_n^{-1}(\hat{\theta}_n - \theta) \to \mathcal{N}_p(0, \mathcal{I}(\theta)) \) weakly under \( P_\theta \) where \( c_n^{-1} \to \infty \) and \( \mathcal{I}(\theta) \in \mathbb{R}^p \otimes \mathbb{R}^p \) is positive definite, then the maximal rate of convergence and the minimal asymptotic covariance matrix are given by \( r_n^{-1} \) and \( \mathcal{I}(\theta)^{-1} \), respectively. Namely, the optimal quantities are explicitly provided by the form of the LAN obtained. In our main result (Theorem 3.1 below), the rate and the Fisher information matrix are specified by (12) and (11), respectively, where the latter turns out to be positive definite for each \( \theta \in \Theta \).

3 Main result

Let \( X \) be a Lévy process such that \( \mathcal{L}(X_1) = NIG(\alpha, \beta, \delta, \mu) \) (recall (3)) and suppose that available data is \((X_{j\Delta_n})_{j \leq n}\) with

\[
\Delta_n \to 0 \quad \text{and} \quad n\Delta_n \to \infty.
\]

Define the matrix \( \mathcal{I}(\theta) = [\mathcal{I}_{kl}(\theta)]_{k,l=1}^4 \) for \( \theta = (\alpha, \beta, \delta, \mu) \in \Theta \) as follows:

\[
\mathcal{I}(\theta) = \begin{pmatrix}
\mathcal{I}_{11}(\theta) & \mathcal{I}_{12}(\theta) & 0 & 0 \\
\mathcal{I}_{21}(\theta) & \mathcal{I}_{22}(\theta) & 0 & 0 \\
\mathcal{I}_{33}(\theta) & 0 & \mathcal{I}_{33}(\theta) & 0 \\
\text{sym.} & \mathcal{I}_{44}(\theta)
\end{pmatrix},
\]

(11)
where

\[ I_{11}(\theta) := \frac{\delta}{\alpha^2} \int_0^\infty \left( e^{(\beta/\alpha)y} + e^{-(\beta/\alpha)y} \right) \frac{K_0(y)}{K_1(y)} dy, \]

\[ I_{12}(\theta) := -\frac{2\alpha \delta}{\pi (\alpha^2 - \beta^2)} \left\{ 1 + \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \arctan \left( \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \right) \right\}, \]

\[ I_{22}(\theta) := \frac{\alpha^2 \delta}{(\alpha^2 - \beta^2) \beta^2}, \]

\[ I_{33}(\theta) := \frac{1}{2\beta^2}, \]

\[ I_{44}(\theta) := \frac{1}{2\beta^2}. \]

(The integral in \( I_{11}(\theta) \) is indeed finite; see Lemma 11.) Let

\[ r_n = \text{diag}(r_{1n}, r_{2n}, r_{3n}, r_{4n}) := \text{diag} \left( \frac{1}{\sqrt{n} \Delta_n}, \frac{1}{\sqrt{n} \Delta_n}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right). \]  

(12)

Our main result is the following, which clarifies a crucial contrast between discrete and continuous observations (see Corollary 3 for the latter case).

**Theorem 3.1.** Let \( X \) be as above and suppose (4) and (10). Then LAN holds true at each \( \theta \in \Theta \) with rate \( r_n \) and the Fisher information matrix \( I(\theta) \). In particular, \( I(\theta) \) is positive definite for each \( \theta \in \Theta \).

Thus we have seen that the rate \( \sqrt{n} \) for \((\delta, \mu)\) is faster than \( \sqrt{n} \Delta_n \) for \((\alpha, \beta)\). Such a phenomenon is known to arise in some specific cases of Lévy processes under high-frequency sampling. A prime example is the scaled Wiener process with drift, say \( X_t = \mu t + \sqrt{\sigma} w_t \), where \( w \) denotes the standard Wiener process; in this case we have the LAN for each \((\mu, \sigma)\) at rate \((\sqrt{n} \Delta_n, \sqrt{n})\). The rate \( \sqrt{n} \Delta_n \) is the discrete-sampling analogue to \( \sqrt{T} \) in the case of continuous observation \((X_t)_{t \leq T} \) as \( T \to \infty \); see, e.g., Akritas and Johnson [3] for details. See also Masuda [13] for the cases of the gamma and the inverse Gaussian subordinators.

For non-Gaussian stable Lévy processes with drift and symmetric Lévy density, it turns out that the Fisher information matrix is singular at “every” \( \theta \) whenever both the stability index and scale parameters are included in theta (see Aït-Sahalia and Jacod [2] and Masuda [17] for details). In the present NIG case, normalized small time increment \((X_{\Delta_n} - \mu \Delta_n)/(\delta \Delta_n)\) is approximately Cauchy distributed (see Lemma 16 below). If \( X \) is the Cauchy Lévy process such that \( \mathcal{L}(X_t) \) admits the Lebesgue density \( x \mapsto (\delta/\pi) \left\{ \delta^2 + (x - \mu)^2 \right\}^{-1} \), then, by a direct application of Masuda [14] Theorem 2.1, we see that the LAN holds true at each \((\delta, \mu)\) with rate \( \sqrt{n} \) and Fisher information matrix \( \text{diag}(1/(2\delta^2), 1/(2\delta^2)) \); we here do not suffer from the singularity of Fisher information, since the stability index is fixed at 1 and is not the parameter to be estimated. Returning to the present NIG case, we note that the last expression is exactly the same as the lower right \(2 \times 2\) submatrix of \( I(\theta) \) in Theorem 3.1. Although we have additionally \( \alpha \) and \( \beta \), Theorem 3.1 implies that we can derive the LAN jointly for the full parameter \( \theta \) as soon as \( n \Delta_n \to \infty \). Moreover, in view of the block diagonal form of \( I(\theta) \), we may expect various possibilities of approximate conditional inference, simplified estimation procedure, and so on (see, e.g., Cox and Reid [7] and Jørgensen and Knudsen [9]).

**Remark 3.2.** Woerner [20, 22] previously derived the LAN for \( \beta \), which she termed “skewness parameter”, within a general framework of discretely observed Lévy processes. She supposed that all the other parameters (here \( \alpha \), \( \delta \), and \( \mu \)) are known. In contrast, our Theorem 3.1 provides the information of asymptotically optimal “full-parameter” estimation. Also, let us observe that \( \delta \) and \( \mu \) here express general location and scale parameters in the sense of Woerner [20], while the parameter \( \alpha \) has no
generic meaning in the entire class of Lévy processes. Recall \( [7] \), where \( \alpha \) is also involved inside the Bessel function \( K_1 \). The LAN for \( \alpha \) cannot be derived as any direct corollary of the general results presented in Woerner [20].

It is interesting to compare Theorem \( [3,1] \) with the case of continuous observation. In order to state the continuous-observation LAN result for the NIG Lévy processes, let us first recall a general characterization of the absolute continuity. Let \( X \) be a Lévy process admitting the Lévy-Khintchine representation \( (11) \) with \( (\mu_0, \sigma^2, \nu(dz)) = (\mu_0(\theta), \sigma^2(\theta), \nu(dz; \theta)) \) for \( \theta \in \Theta \subset \mathbb{R}^p \), and suppose that we observe \( (X_t)_{t \in [0,T]} \). Let \( P^{(T)}_\theta \) denote the restriction of \( P_\theta \) to \( \mathcal{F}_T \), the natural filtration generated by the continuous-time record \( (X_t)_{t \leq T} \). The local equivalence of \( P_\theta \) and \( P_{\theta'} \) for \( \theta \neq \theta' \) is characterized by the following proposition borrowed from Raible [16] Proposition 2.19.

**Proposition 3.3.** Fix any \( T > 0 \) and \( \theta, \theta' \in \Theta \). Then \( P^{(T)}_\theta \) and \( P^{(T)}_{\theta'} \) are equivalent iff the following conditions are fulfilled.

(a) \( \nu(dz; \theta') = \gamma(z; \theta') \nu(dz; \theta) \) for some Borel function \( \gamma(\cdot; \theta, \theta') : \mathbb{R} \to (0, \infty) \).

(b) \( \mu(\theta') = \mu(\theta) + \int_\mathbb{R} (\gamma(z; \theta, \theta') - 1) \nu(dz; \theta) + \sigma^2(\theta)b \) for some \( b \in \mathbb{R} \).

(c) \( \int_\mathbb{R} (1 - \sqrt{\gamma(z; \theta, \theta')})^2 \nu(dz; \theta) < \infty \).

(d) \( \sigma^2(\theta') = \sigma^2(\theta) \).

As a corollary to Raible [16] Proposition 2.20] based on Proposition \( [3,3] \) we have

**Corollary 3.4.** Let \( P_{\theta_k}, k = 1, 2, \) denote the distribution of the NIG Lévy process with parameters \( \theta_k = (\alpha_k, \beta_k, \delta_k, \mu_k) \in \Theta \), and fix any \( T > 0 \). Then \( P^{(T)}_{\theta_1} \) and \( P^{(T)}_{\theta_2} \) are equivalent iff \( \delta_1 = \delta_2 \) and \( \mu_1 = \mu_2 \).

Corollary \( [3,4] \) clears up an essential difference between the cases of continuous and high-frequency sampling for the NIG Lévy processes. Indeed, Corollary \( [3,4] \) enables us to study the LAN for the continuous-observation case, where the asymptotics are taken as \( T \to \infty \); we do not touch the details in order not to digress from the main topic, but only refer to Akritas and Johnson [3] for possible LAN for \( (\alpha, \beta) \) at rate \( \sqrt{T} \). On the contrary, as specified in Theorem \( [3,1] \), the likelihood function does exist when we deal with the high-frequency (discrete-time) sample, so that the maximum-likelihood estimation of \( (\delta, \mu) \) becomes meaningful. Finally, let us mention that, as the rate of convergence of \( (\delta, \mu) \) is \( \sqrt{n} \) free of \( \Delta_n \), it may not be necessary to impose that \( n\Delta_n \to \infty \) for estimating \( (\delta, \mu) \), with regarding \( (\alpha, \beta) \) as a nuisance parameter; of course, this is the case for estimation of \( \sigma \) in the aforementioned Wiener case.

### 4 Proof

We proceed as follows. First, in Section \( [4,1] \) we provide a useful general tool (Proposition \( [4,3] \)) for proving Theorem \( [3,1] \). Next, we prepare some preliminary lemmas in Section \( [4,2] \) for investigating the likelihood function in question, whose expression together with its derivatives up to the second order are specified in Section \( [4,3] \). Finally, Sections \( [4,4] \) to \( [4,7] \) are devoted to verifications of the conditions of Proposition \( [4,3] \).

#### 4.1 A tool for proving LAN under high-frequency sampling

In this section, as a continuation of Section \( [2,3] \) we prepare a useful tool for proving our main result. Our setup here covers general Lévy processes discretely observed at high frequency.
Write \( g_{nj}(\theta) = \partial_{\theta} \log p_{\Delta_n}(x_j; \theta) \). The random fields \( \log(dP_{\theta_n}^n / dP_{\theta}^n) \) on \( \Theta \) admits the asymptotically quadratic structure

\[
\log \frac{dP_{\theta_n}^n}{dP_{\theta}^n} = \sum_{j=1}^{n} r_n \{g_{nj}(\theta) - E_\theta[g_{nj}(\theta)]\} - \frac{1}{2} \sum_{j=1}^{n} E_\theta[\{r_n g_{nj}(\theta)\}^2] + o_{P_\theta}(1) \tag{13}
\]

if it holds that

\[
\limsup_{n \to \infty} \sum_{j=1}^{n} E_\theta[\{|r_n g_{nj}(\theta)|^2\}] < \infty, \tag{14}
\]

\[
\sum_{j=1}^{n} E_\theta \left[\left| r_n g_{nj}(\theta) \right|^2 I(\left| r_n g_{nj}(\theta) \right| \geq \epsilon) \right] \to 0 \quad \text{for every } \epsilon > 0, \tag{15}
\]

\[
\sum_{j=1}^{n} E_\theta \left[\left\{ \sqrt{\frac{p_{\Delta_n}(x_j; \theta_n)}{p_{\Delta_n}(x_j; \theta)}} - 1 + \frac{1}{2} h^\top r_n g_{nj}(\theta) \right\}^2 \right] \to 0. \tag{16}
\]

See, e.g., Strasser [13] Theorem 74.2 and Corollary 74.4 for details. Now we impose that

\[
\sum_{j=1}^{n} E_\theta[\{r_n g_{nj}(\theta)\}^2] \to \mathcal{I}(\theta), \tag{17}
\]

\[
\sum_{j=1}^{n} |r_n E_\theta[g_{nj}(\theta)]|^2 \to 0. \tag{18}
\]

Note that (17) implies (14).

To treat the first term in the right-hand side of (13), we prepare the following.

**Lemma 4.1.** Suppose the conditions (13), (17), and (18). Then the first term in the right-hand side of (13) weakly under \( P_\theta \) tends to \( N_p(0, \mathcal{I}(\theta)) \).

**Proof.** Introduce the centered variables \( \chi_{nj} = \chi_{nj}(\theta) := r_n \{g_{nj}(\theta) - E_\theta[g_{nj}(\theta)]\} \). By means of the central limit theorem for rowwise independent triangular arrays (e.g. Kallenberg [10] Theorem 5.12) combined with the Cramér-Wald device, the claim follows from the convergence of the cumulative variance and the Lindeberg condition, that is, \( \sum_{j=1}^{n} E_\theta[\chi_{nj}^2] \to \mathcal{I}(\theta) \) and \( \sum_{j=1}^{n} E_\theta[|\chi_{nj}|^2 I(|\chi_{nj}| \geq \epsilon')] \to 0 \) for every \( \epsilon' > 0 \), respectively. The former is obtained by noting that

\[
\sum_{j=1}^{n} E_\theta[\chi_{nj}^2] = \sum_{j=1}^{n} E_\theta[\{r_n g_{nj}(\theta)\}^2] - \sum_{j=1}^{n} \left| r_n E_\theta[g_{nj}(\theta)] \right|^2
\]

and then applying (17) and (18). Now fix any \( \epsilon' > 0 \). Since (18) entails that \( |\sqrt{n} r_n E_\theta[g_{nj}(\theta)]| \to 0 \), we can find \( \epsilon > 0 \) such that \( \epsilon' < \left| r_n E_\theta[g_{nj}(\theta)] \right| \) \( \epsilon > 0 \) for every \( n \) large enough. Accordingly,

\[
\sum_{j=1}^{n} E_\theta[|\chi_{nj}|^2 I(|\chi_{nj}| \geq \epsilon')]
\]

\[
\leq \sum_{j=1}^{n} E_\theta[|r_n g_{nj}(\theta)|^2 I(|r_n g_{nj}(\theta)| \geq \epsilon') + |r_n E_\theta[g_{nj}(\theta)]|^2] + \sum_{j=1}^{n} \left| r_n E_\theta[g_{nj}(\theta)] \right|^2
\]

\[
\leq \sum_{j=1}^{n} E_\theta[|r_n g_{nj}(\theta)|^2 I(|r_n g_{nj}(\theta)| \geq \epsilon')] + |\sqrt{n} r_n E_\theta[g_{nj}(\theta)]|^2 \to 0
\]

by virtue of (18). The proof is complete. \( \Box \)

Thus we have seen that the desired property (9) can be derived under (13), (16), (17), and (18). Nevertheless, it is convenient to replace (16) by an alternative, which is easier to verify. We prepare the following lemma.
Lemma 4.2. The condition \((17)\) holds true if
\[
\sum_{j=1}^{n} \sup_{\rho \in \Theta: |r_n^{-1}(\rho - \theta)| \leq a} \mathbb{E}_\rho \left[ \left| r_n \partial_\theta g_{n_j}(\rho)^T r_n \right|^2 + \left| r_n g_{n_j}(\rho) \right|^4 \right] \to 0.
\] (19)
for any \(a > 0\).

Proof. Write \(\sum_{j=1}^{n} e_{n_j}(\theta)\) for the left-hand side of \((19)\), and let \(H_n(x; \theta) := \{p_{\Delta_n}(x; \theta_n)^{1/2} - p_{\Delta_n}(x; \theta)^{1/2} - (\theta_n - \theta)^T \partial_\theta [p_{\Delta_n}(x; \theta)^{1/2}] \}^2\). Then
\[
e_{n_j}(\theta) = \mathbb{E}_\rho \left[ p_{\Delta_n}(x_j; \theta)^{-1} H_n(x_j; \theta) \right] = \int_{\mathbb{R}} H_n(x; \theta) dx.
\] (20)

On the other hand, noting that
\[
\left| r \partial_\theta^2 f(\theta)^{1/2} r \right|^2 \leq f(\theta) \{ |r | \partial_\theta^2 \log f(\theta) |r| \}^2 + |r \partial_\theta \log f(\theta)|^4
\]
for any nonnegative \(C^2(\Theta)\) function \(f\) and diagonal \(p \times p\) matrix \(r\), we get for each \(x\)
\[
H_n(x; \theta) \leq \int_0^1 |r_n \partial_\theta^2 [p_{\Delta_n}(x; \theta + s r_n h)^{1/2}] r_n| \, ds
\]
\[
\leq \int_0^1 \left\{ |r_n \partial_\theta^2 \log p_{\Delta_n}(x; \rho_n') r_n|^2 + |r_n \partial_\theta \log p_{\Delta_n}(x; \rho_n')|^4 \right\} p_{\Delta_n}(x; \rho_n') \, ds,
\] (21)
where we wrote \(\rho_n' = \theta + s r_n h\), which belongs to \(\Theta\) for every \(n\) large enough. Now, by substituting \((21)\) in \((20)\) and then applying Fubini’s theorem for interchanging the \(ds\) and \(dx\) integrals, we have
\[
\sum_{j=1}^{n} e_{n_j}(\theta) \leq \sup_{\rho \in \Theta: |r_n^{-1}(\rho - \theta)| \leq |h|} \int_{\mathbb{R}} \left\{ |r_n \partial_\theta^2 \log p_{\Delta_n}(x; \rho_n') r_n|^2 + |r_n \partial_\theta \log p_{\Delta_n}(x; \rho_n')|^4 \right\} p_{\Delta_n}(x; \rho_n') \, dx \to 0
\]
by means of \((19)\); recall that \(h \in \mathbb{R}^p\) here is fixed arbitrarily. This completes the proof. \(\square\)

To sum up we have derived the following proposition, which serves as our basic tool for proving LAN.

**Proposition 4.3.** Suppose that \((15)\), \((17)\), \((18)\), and \((19)\) hold true. Then we have \((9)\), that is, LAN holds true at each \(\theta\) with rate \(r_n\) and the Fisher information matrix \(\mathcal{I}(\theta)\).

**Remark 4.4.** Of course, the concept LAN is defined for much more general situations than ours, such as a discrete-time sample from an ergodic process. Let \(\{r_n(\theta)\}\) be a nonrandom positive definite diagonal matrices tending to \(0\) in norm for each \(\theta \in \Theta \subset \mathbb{R}^p\), and \(\mathcal{I}(\theta)\) a positive symmetric \(\mathbb{R}^p \otimes \mathbb{R}^p\) matrix as before. Fix any \(h \in \mathbb{R}^p\) and put \(\theta_n = \theta + r_n(\theta) h\). Let \(\ell_n(\theta)\) denote the log-likelihood function associated with any array of random vectors. Then, in a similar way, we say that LAN holds true at \(\theta\) with rate \(r_n\) and the Fisher information matrix \(\mathcal{I}(\theta)\), if the stochastic expansion of the form \((2)\) holds true with \(S_n(\theta) := \sum_{j=1}^{n} r_n \partial_\theta \ell_n(\theta) \to \mathcal{N}(0, \mathcal{I}(\theta))\) weakly under \(P_\theta\). Also in this case, as a useful tool for proving the LAN, we can provide an analogous set of conditions to Proposition 4.3.
4.2 Preliminary lemmas

For later use, we prepare some lemmas. We consistently use the notation \((8)\). For \(j \leq n\), we introduce

\[
\epsilon_{nj} = \epsilon_{nj}(\delta, \mu) := \frac{x_j - \mu \Delta_n}{\delta \Delta_n}.
\]

Clearly we have \(\mathcal{L}(\epsilon_{nj}) = \mathcal{L}(\epsilon_{n1})\) for each \(n \in \mathbb{N}\) and \(j \leq n\).

**Lemma 4.5.** It holds that for each \(n \in \mathbb{N}\), \(\mathcal{L}(\epsilon_{n1}) = NIG(\alpha \delta \Delta_n, \beta \delta \Delta_n, 1, 0)\).

**Proof.** Obvious from \((8)\) with taking \(a = (\delta \Delta_n)^{-1}\).

An important point in our study is that the normalized increments of \(X\) in small time can be approximated by the Cauchy distribution having the Blumenthal-Getoor index 1. In what follows, let

\[
\phi_1(y) := \frac{1}{\pi (1 + y^2)},
\]

the standard symmetric Cauchy density corresponding to the characteristic function \(u \mapsto \exp(-|u|)\).

**Lemma 4.6.** Denote by \(f_{\Delta_n} : \mathbb{R} \rightarrow (0, \infty)\) the smooth density of \(\mathcal{L}(\epsilon_{n1})\). For any nonnegative integer \(k\), we have

\[
\lim_{\Delta_n \to 0} \sup_{y \in \mathbb{R}} |\partial^k_y f_{\Delta_n}(y) - \partial^k_y \phi_1(y)| = 0.
\]

**Proof.** In view of \((13)\), we have \(\varphi_{\epsilon_{n1}}(u) = \exp\{\delta \Delta_n \sqrt{m} - \sqrt{(\alpha \Delta_n)^2 - (iu + \beta \Delta_n)^2}\}\). Here and in what follows, we write

\[
m = \alpha^2 - \beta^2 > 0.
\]

Clearly, we have \(\varphi_{\epsilon_{n1}}(u) \to \exp(-|u|)\) for each \(u \in \mathbb{R}\). Also, simple manipulation of complex numbers gives the estimate

\[
|\varphi_{\epsilon_{n1}}(u)| \lesssim \exp \left\{ -\frac{1}{\sqrt{2}} \left( (\delta \Delta_n)^2 m + u^2 + \sqrt{(\delta \Delta_n)^2 m + u^2} + (2 \beta \delta \Delta_n u)^2 \right)^{1/2} \right\} \lesssim e^{-C|u|}. \tag{22}
\]

On the other hand, by means of the Fourier inversion formula we have

\[
\sup_{y \in \mathbb{R}} |\partial^k_y f_{\Delta_n}(y) - \partial^k_y \phi_1(y)| \lesssim \int |u|^k |\varphi_{\epsilon_{n1}}(u) - e^{-|u|}| \, du. \tag{23}
\]

Under \((22)\), we can apply the dominated convergence theorem to the upper bound of \((23)\). This completes the proof.

In particular, note that the limit of \(\mathcal{L}(\epsilon_{n1})\) in total variation is symmetric even if \(\beta \neq 0\). As a matter of fact, since the Lévy density \(g\) of \(NIG(\alpha, \beta, \delta, \mu)\) admits the expansion \(z^2 g(z) = (1/\pi) + (\delta \beta / \pi) |z| + o(|z|)\) as \(|z| \to 0\) (see \((7)\) together with \((25)\) below, or, more generally, Raible \(16\) Proposition 2.18)), Lemma 4.6 can also be deduced from the behavior of the Lévy density of \(\mathcal{L}(\epsilon_{n1})\) around the origin; note that the standard Cauchy Lévy density equals \(z \mapsto (1/\pi) |z|^{-2}\).

We introduce the following functions defined on \([0, \infty)\):

\[
\eta(y) := \phi'_1(y)/\phi_1(y), \\
\zeta(y) := K'_1(y)/K_1(y), \\
H(y) := y^{-1} (1 + y \zeta(y)) = -K_0(y)/K_1(y), \tag{24}
\]

where we used the identity \(K'_w(y) = -K_{w-1}(y) - (w/y)K_w(y)\) for \((24)\).
Lemma 4.7.  (a) The functions \( y \mapsto \eta(y), \) \( y\eta(y), \) and \( y^2\eta''(y) \) are bounded in \( \mathbb{R}. \)

(b) \( y \mapsto H(y) \) is bounded and continuous in \([0, \infty). \) Moreover, \( H(y) \sim -y \log(1/y) \) as \( y \to 0 \) and \( H(y) = -1 + 1/(2y) + 3/(8y^2) + O(y^{-3}) \) as \( y \to \infty. \)

(c) \( H'(y) \sim -\log(1/y) \) as \( y \to 0 \) and \( y^2H'(y) = -1/2 + O(y^{-1}) \) as \( y \to \infty. \) In particular, \( y \mapsto yH'(y) \) is bounded and continuous in \([0, \infty). \)

**Proof.** The claim (a) readily follows from the well known fact
\[
\sup_{y \in \mathbb{R}} |y|^k |\partial_y^k \phi_1(y)| < \infty
\]
for each \( k \in \mathbb{Z}_+. \) This is valid too for \( \phi_1 \) replaced by the a general symmetric non-Gaussian \( \beta \)-stable density.

As for (b), the continuity of \( H \) is clear. We first note the asymptotic behaviors:
\[
K_w(y) \sim \begin{cases} 
\log(1/y) + \log 2 - \mathcal{C} & \text{if } w = 0, \\
\Gamma(|w|) \left| y^{-|w|} \right| & \text{if } w \neq 0, 
\end{cases} \quad \text{as } y \to 0, 
\]
\[
K_w(y) = \frac{\pi}{2y} e^{-y} \left( 1 + \frac{k - 1}{8y} + \frac{(k - 1)(k - 9)}{(8y)^2} + O(y^{-3}) \right) \quad \text{as } y \to \infty,
\]
where \( \mathcal{C} \approx 0.5772 \) is the Euler-Mascheroni constant and \( \kappa := 4w^2. \) The desired behavior of \( H(y) \) as \( y \to 0 \) is trivial from (25). Next, by applying (26) for \( \nu = 0, 1 \) and then expanding the fraction \(-K_0(y)/K_1(y)\) as a power series of \( y^{-1}. \) straightforward computations lead to the desired behavior of \( H(y) \) as \( y \to \infty. \) Now the boundedness of \( H \) is trivial.

Using the known identity \( K_w(y) = K_{-w}(y) \) valid for each \( w, y > 0, \) we get
\[
H'(y) = 1 + H(y)/y - \{H(y)\}^2,
\]
and so \( y^2H'(y) = y^2 + yH(y) - y^2\{H(y)\}^2. \) These expressions combined with (b) lead to the claims. \( \square \)

Now we define
\[ q_{n1} = q_{n1}(\alpha, \delta, \mu) := \alpha \delta \Delta_n(1 + \epsilon_{n1}^2)^{1/2} \]
and
\[
A_k(\theta) := (-1)^k \frac{\alpha \delta}{\pi} \int_0^\infty (e^{(\beta/\alpha)y} + e^{-(\beta/\alpha)y}) y^{k-1} K_1(y) \left( \frac{K_0(y)}{K_1(y)} \right)^k dy, \\
A'(\theta) := -\frac{1}{\pi} \int_0^\infty (e^{(\beta/\alpha)y} + e^{-(\beta/\alpha)y}) y K_0(y) dy.
\]
We need the following lemmas to specify the Fisher information matrix \( \mathcal{I}(\theta), \) and to estimate the remainder term in the stochastic expansion of the likelihood ratio random fields (see (27) below).

**Lemma 4.8.** For any \( k \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} E_\theta [\{q_{n1} H(q_{n1})\}^k] = A_k(\theta),
\]
with \( A_k(\theta) \) being finite. In particular, \( \limsup_{n \to \infty} \Delta_n^{-1} E_\theta [||q_{n1} H(q_{n1})||] < \infty. \)

**Proof.** Reminding Lemma 4.3 and (24), we have
\[
\frac{1}{\Delta_n} E_\theta [\{q_{n1} H(q_{n1})\}^k] = \int_{\mathbb{R}} \left\{ \alpha \delta \Delta_n \sqrt{1 + x^2} H \left( \alpha \delta \Delta_n \sqrt{1 + x^2} \right) \right\}^k \frac{\alpha \delta \Delta_n}{\pi} e^{\alpha \delta \Delta_n \sqrt{1 + x^2}} \left( \frac{K_0(\alpha \delta \Delta_n \sqrt{1 + x^2})}{K_1(\alpha \delta \Delta_n \sqrt{1 + x^2})} \right)^k \frac{dx}{\sqrt{1 + x^2}} \\
= \frac{1}{\Delta_n^{1/2}} \int_{\mathbb{R}} \left\{ \alpha \delta \Delta_n \sqrt{1 + x^2} H \left( \alpha \delta \Delta_n \sqrt{1 + x^2} \right) \right\}^k \frac{\alpha \delta \Delta_n}{\pi} e^{\alpha \delta \Delta_n \sqrt{1 + x^2}} \left( \frac{K_0(\alpha \delta \Delta_n \sqrt{1 + x^2})}{K_1(\alpha \delta \Delta_n \sqrt{1 + x^2})} \right)^k \frac{dx}{\sqrt{1 + x^2}} \\
= (-1)^k \frac{\alpha \delta}{\pi} e^{\alpha \delta \Delta_n \sqrt{1 + x^2}} B_{\Delta_n}^{(k)} \sim (-1)^k \frac{\alpha \delta}{\pi} B_{\Delta_n}^{(k)}.
\]
Write $B_{\Delta}^{(k)} = \int_{0}^{\infty} + \int_{-\infty}^{0} =: B_{\Delta}^{(k)+} + B_{\Delta}^{(k)-}$. 

First let us look at $B_{\Delta}^{(k)+}$. The change of variables $y = \alpha \delta \Delta_n (\sqrt{1 + x^2} - 1)$ leads to $B_{\Delta}^{(k)+} = \int_{0}^{\infty} b_{\Delta}^{(k)+}(y)dy$, where

$$b_{\Delta}^{(k)+}(y) = e^{(\beta/\alpha)\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} (y + \alpha \delta \Delta_n)^{k}} \frac{K_0(y + \alpha \delta \Delta_n)}{K_1(y + \alpha \delta \Delta_n)} K_1(y + \alpha \delta \Delta_n).$$

Obviously, for each $y \in (0, \infty)$

$$b_{\Delta}^{(k)+}(y) \rightarrow e^{(\beta/\alpha)\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} (y + \alpha \delta \Delta_n)^{k}} K_1(y + \alpha \delta \Delta_n).$$

(28)

In order to apply the dominated convergence theorem, we have to look at the behaviors of $b_{\Delta}^{(k)+}(y)$ as $y \rightarrow 0$ and $y \rightarrow \infty$ uniformly in small $\Delta_n$, say $\Delta_n \in (0, 1]$. First, by means of Lemma 4.10 (b) we can derive as $y \rightarrow \infty$

$$\sup_{\Delta_n \leq 1} |b_{\Delta}^{(k)+}(y)| \lesssim e^{(\beta/\alpha)\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} (y + \alpha \delta \Delta_n)^{k}} K_1(y + \alpha \delta \Delta_n)$$

$$\lesssim e^{-(1-\beta/\alpha)\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} y^{-3/2}},$$

(29)

the upper bound being Lebesgue integrable at infinity; here the assumption $|\beta| < \alpha$ comes into effect.

On the other hand, on account of (25) and Lemma 4.10 (b), it holds that $y^{k-1/2} \{K_0(y)/K_1(y)\}^k K_1(y) \sim C y^{2k-3/2} \log(1/y) \rightarrow 0$ as $y \rightarrow 0$. This leads to $\sup_{y \in [0, 1]} y^{k-1/2} \{K_0(y)/K_1(y)\}^k K_1(y) < \infty$, so that, as $y \rightarrow 0$

$$\sup_{\Delta_n \leq 1} |b_{\Delta}^{(k)+}(y)| \lesssim y^{-1/2} \sup_{\Delta_n \leq 1} \left(\frac{y + \alpha \delta \Delta_n}{\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} \alpha \delta \Delta_n}^{k-1/2} \left\{\frac{K_0(y + \alpha \delta \Delta_n)}{K_1(y + \alpha \delta \Delta_n)}\right\} K_1(y + \alpha \delta \Delta_n)\right)$$

$$\lesssim y^{-1/2},$$

(30)

the upper bound being Lebesgue integrable near the origin. Having (28), (29) and (30) in hand, the dominated convergence theorem yields that $B_{\Delta}^{(k)+} \rightarrow \int_{0}^{\infty} b_{0}^{(k)+}(y)dy < \infty$.

Let $b_{0}^{(k)+}(y) := e^{-(\beta/\alpha)\sqrt{\sqrt{y^2 + 2 \alpha \delta \Delta_n}} (y + \alpha \delta \Delta_n)^{k}} K_1(y + \alpha \delta \Delta_n)$. In the same manner as before, we can deduce that $B_{\Delta}^{(k)+} \rightarrow \int_{0}^{\infty} b_{0}^{(k)+}(y)dy$. Thus $B_{\Delta}^{(k)+} \rightarrow \int_{0}^{\infty} \{b_{0}^{(k)+}(y) + b_{0}^{(k)-}(y)\}dy$, completing the proof of the first half of the claims. The last half is obvious from (27) and what we have seen above. The proof is complete.

\[ \text{Lemma 4.9. It holds that} \]
\[ \lim_{n \rightarrow \infty} E_{\theta}[\epsilon_{n1}q_{n1}1H(q_{n1})] = A' (\theta), \]

with $A' (\theta)$ being finite.

\[ \text{Proof.} \] The lemma can be deduced in an analogous way to the proof of Lemma 4.8 so we omit the details.

\[ \text{The Lemma 4.10 below provides the fully closed form of} A' (\theta), \text{which directly leads to the closed} \]
\[ \text{form of} Z_{\Delta 2}(\theta), \text{as we will see in Section 4.5. We set aside the integral form of} A' (\theta) \text{for later convenience} \]
\[ \text{in the proof of positive definiteness of} Z(\theta). \]

\[ \text{Lemma 4.10. It holds that} \]
\[ A' (\theta) = - \frac{2 \alpha^2}{\pi m} \left(1 + \frac{\beta}{\sqrt{m}} \arctan \left(\frac{\beta}{\sqrt{m}}\right)\right) . \]
We note that \((4.9)\) can effectively be applied. (33)\(\frac{31}{32}\) (4.7)\(\frac{34}{35}\)\(\theta\) follows:

The expression (based on which we write down \(\ell\) density of \(\ell\) 4.3 Likelihood, score, and observed information in question

Using this fact with \(b = \beta/\alpha\) and \(-\beta/\alpha\), it is straightforward to deduce the claim.

\[\boxed{\text{Proof.}}\]

Lemma
deriving various limiting values as well as deducing estimates of stochastically small terms, to which
\(\phi\) introduction of the standard Cauchy density be obviously simplified. However, we have meaningly transformed as just described. In fact, the
\(\delta\) \(\partial\) \(n\) \(\sum\) \(\int\)

\[\frac{\alpha \delta \Delta_n}{\pi} \exp(\delta \Delta_n \sqrt{m} + \beta(x_j - \mu \Delta_n)) \frac{K_1(\alpha \sqrt{(\delta \Delta_n)^2 + (x_j - \mu \Delta_n)^2})}{\sqrt{(\delta \Delta_n)^2 + (x_j - \mu \Delta_n)^2}},\]

based on which we write down \(\ell_n(\theta)\) in terms of \((\epsilon_{nj})_{j \leq n}\) as

\[\ell_n(\theta) = \sum_{j=1}^{n} \left\{ \log \alpha + \delta \Delta_n (\sqrt{m} + \beta \epsilon_{nj}) + \log \phi_1(\epsilon_{nj}) + \frac{1}{2} \log(1 + \epsilon_{nj}^2) + \log K_1 \left( \alpha \delta \Delta_n \sqrt{1 + \epsilon_{nj}^2} \right) \right\}.\]  

(31)

The expression (31) may look unnecessarily lengthy, as the term \(\log \phi_1(\epsilon_{nj}) + 2^{-1} \log(1 + \epsilon_{nj}^2)\) can be obviously simplified. However, we have meaningly transformed as just described. In fact, the
\(\delta\) \(\partial\) \(n\) \(\sum\) \(\int\)Lemma(4.3) to (4.9) can effectively be applied.

For studying LAN, we need to look at the score \(\theta \mapsto \partial_{\theta} \ell_n(\theta)\) and the observed information
\(\theta \mapsto -\partial_{\theta}^2 \ell_n(\theta)\). Note that \(\partial_{\theta} \epsilon_{nj} = -\delta^{-1}, \partial_{\theta}^2 \epsilon_{nj} = 0, \partial_{\beta} \epsilon_{nj} = -\delta^{-1} \epsilon_{nj},\) and \(\partial_{\beta}^2 \epsilon_{nj} = 2 \delta^{-2} \epsilon_{nj},\) and \(\partial_{\delta} \partial_{\theta} \epsilon_{nj} = \delta^{-2}.\) In terms of (31), the first-order partial derivative of \(\theta \mapsto \ell_n(\theta)\) are explicitly given as follows:

\[\partial_{\alpha} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ \frac{\alpha \delta \Delta_n}{\sqrt{m}} + \frac{1}{\alpha} \epsilon_{nj} H(q_{nj}) \right\},\]  

(32)

\[\partial_{\beta} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ \delta \Delta_n \left( \epsilon_{nj} - \beta \sqrt{m} \right) \right\},\]  

(33)

\[\partial_{\delta} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\frac{1}{\delta} (\epsilon_{nj} \eta(\epsilon_{nj}) + 1) + \Delta_n \left( \sqrt{m} + \frac{\alpha}{\sqrt{1 + \epsilon_{nj}^2}} H(q_{nj}) \right) \right\},\]  

(34)

\[\partial_{\mu} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\frac{1}{\delta} \eta(\epsilon_{nj}) - \Delta_n \left( \beta + \frac{\alpha \epsilon_{nj}}{\sqrt{1 + \epsilon_{nj}^2}} H(q_{nj}) \right) \right\}.\]  

(35)
We also need to look at the Hessian matrix $\partial^2_\theta \ell_n(\theta)$: the diagonal elements are

$$
\partial^2_\theta \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\frac{\beta^2 \Delta_n}{m^{3/2}} + \frac{q_{nj}^2}{\alpha^2} H'(q_{nj}) \right\},
$$

(36)

and the off-diagonal ones are

$$
\partial_{\alpha} \partial_{\beta} \ell_n(\theta) = \sum_{j=1}^{n} \frac{\alpha \beta \delta \Delta_n}{m^{5/2}},
$$

(40)

$$
\partial_{\alpha} \partial_{\delta} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ \frac{\alpha \Delta_n}{\sqrt{m}} + \frac{\Delta_n}{\sqrt{1 + \epsilon_{nj}^2}} (H(q_{nj}) + q_{nj} H'(q_{nj})) \right\},
$$

(41)

$$
\partial_{\alpha} \partial_{\mu} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\Delta_n \frac{\epsilon_{nj}}{\sqrt{1 + \epsilon_{nj}^2}} (H(q_{nj}) + q_{nj} H'(q_{nj})) \right\},
$$

(42)

$$
\partial_{\beta} \partial_{\delta} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\beta \Delta_n \frac{\mu}{\sqrt{m}} \right\} = -\frac{\beta n \Delta_n}{\sqrt{m}},
$$

(43)

$$
\partial_{\beta} \partial_{\mu} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ -\Delta_n \right\} = -n \Delta_n,
$$

(44)

$$
\partial_{\beta} \partial_{\mu} \ell_n(\theta) = \sum_{j=1}^{n} \left\{ \frac{1}{\delta^2} (\eta(\epsilon_{nj}) + \epsilon_{nj} H'(\epsilon_{nj})) - \frac{\alpha \Delta_n}{\delta} \frac{\epsilon_{nj}}{1 + \epsilon_{nj}^2} (q_{nj} H'(q_{nj}) - H(q_{nj})) \right\}.
$$

(45)

In what follows, we complete the proof of Theorem 3.1 by verifying the conditions (15), (17), (18), and (19) given in Section 2.3 with taking $r_\kappa = (r_{\kappa n})_{n=1}^{k}$ as in (12).

### 4.4 Lindeberg condition

First we look at (15). As is well known, (15) is implied by the Lyapunov condition: there exists a constant $\epsilon' > 0$ such that $\sum_{j=1}^{n} E_\theta ||r_{\kappa n} g_{\kappa}(\theta)||^{2+\epsilon'} \to 0$. Here, we set $\epsilon' = 2$ and prove

$$
\sum_{k=1}^{4} \sum_{j=1}^{n} E_\theta ||r_{\kappa n} g_{k,n}(\theta)||^{4} \to 0,
$$

(46)

where $g_{k,n}(\theta)$ denotes the $k$th component of $g_{\kappa}(\theta)$. Using Lemma 4.3 we get

$$
\sum_{j=1}^{n} E_\theta ||r_{1n} g_{1,n}(\theta)||^{4} \lesssim \frac{1}{n \Delta_n} \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\Delta_n} \left\{ \Delta_n^4 + E_\theta [[q_{nj} H(q_{nj})]]^{4} \right\} \lesssim \frac{1}{n \Delta_n} \to 0.
$$

Next, noting that $r_{2n} g_{2,n}(\theta) = (n \Delta_n)^{-1/2} (x_j - \mu \Delta_n - \beta \delta \Delta_n / \sqrt{m})$ and that $E_\theta ||x_j||^{q} \lesssim \Delta_n$ for every $q \geq 2$, we have

$$
\sum_{j=1}^{n} E_\theta ||r_{2n} g_{2,n}(\theta)||^{4} \lesssim \frac{1}{n \Delta_n} \cdot \frac{1}{n} \sum_{j=1}^{n} \left\{ \Delta_n^{-1} E_\theta ||x_j||^{4} + \Delta_n^{4} \right\} \lesssim \frac{1}{n \Delta_n} \to 0.
$$
In view of Lemma 4.7(a) and 4.7(b), it is clear that \( \sum_{j=1}^{n} E_{\theta}[|r_{3n}g_{3,n,j}(\theta)|^{4}] + \sum_{j=1}^{n} E_{\theta}[|r_{4n}g_{4,n,j}(\theta)|^{4}] \lesssim 1/n \rightarrow 0 \). Thus (46), hence (15), has been obtained.

For later use, we note the stronger convergence

\[
\sum_{j=1}^{n} \sup_{\theta \in \Theta} E_{\theta}[|r_{n}g_{n,j}(\theta)|^{4}] \rightarrow 0, \tag{47}
\]

which directly follows from (46) and the boundedness of \( \Theta \).

### 4.5 Fisher information matrix

Next we look at (17) and the positive definiteness of the Fisher information matrix \( \mathcal{I}(\theta) \).

First we prove (17), which amounts to proving that

\[
\mathcal{I}_{kl}(\theta) = \lim_{n \to \infty} \sum_{j=1}^{n} E_{\theta}[r_{kn}g_{k,n,j}(\theta)r_{ln}g_{l,n,j}(\theta)], \quad 1 \leq k \leq l \leq 4.
\]

Prior to computing the limits, let us recall the expressions (32) to (35), and the notation \( A_k(\theta) \) in Lemma 4.3.

We begin with the diagonal elements. First, we observe that

\[
\sum_{j=1}^{n} E_{\theta}[(r_{1n}g_{1,n,j}(\theta))^{2}] = O(\Delta_{n}) + \frac{2\delta}{\sqrt{m}} \frac{1}{n} \sum_{j=1}^{n} E_{\theta}[q_{n,j}H(q_{n,j})] + \frac{1}{\alpha^{2}} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\Delta_{n}} E_{\theta}[(q_{n,j}H(q_{n,j}))^{2}]
\]

\[
= O(\Delta_{n}) + \frac{2\delta}{\sqrt{m}} \frac{1}{\Delta_{n}} \sum_{j=1}^{n} E_{\theta}[q_{n,1}H(q_{n,1})] + \frac{1}{\alpha^{2}} \frac{1}{\Delta_{n}} E_{\theta}[(q_{n,1}H(q_{n,1}))^{2}]
\]

\[
\rightarrow \frac{1}{\alpha^{2}} A_{2}(\theta) = \mathcal{I}_{11}(\theta).
\]

Noting that \( \mathcal{L}(\epsilon_{n,j}) = NIG(\alpha \delta \Delta_{n}, \beta \Delta_{n}, 1, 0) \) and \( E_{\theta}[\epsilon_{n,j} - \beta/\sqrt{m}]^{2} = (\delta \Delta_{n})^{-1} \alpha^{2}/m^{3/2} \), we get

\[
\sum_{j=1}^{n} E_{\theta}[(r_{2n}g_{2,n,j}(\theta))^{2}] = \delta \alpha^{2}/m^{3/2} = \mathcal{I}_{22}(\theta).
\]

We have known from Lemma 4.7(b) that the random variables \( H(q_{n,j})/\sqrt{1 + \epsilon_{n,j}^{2}} \) in (32) are essentially bounded. Therefore, the bounded convergence theorem yields that

\[
\sum_{j=1}^{n} E_{\theta}[(r_{3n}g_{3,n,j}(\theta))^{2}] = \frac{1}{\alpha^{2} \delta^{2}} \sum_{j=1}^{n} E_{\theta}[|\epsilon_{n,j}\eta(\epsilon_{n,j}) + 1|^{2}] + O(\Delta_{n}) = \frac{1}{\delta^{2}} E_{\theta}[|\epsilon_{n,1}\eta(\epsilon_{n,1}) + 1|^{2}] + O(\Delta_{n}).
\]

Building on Lemma 4.6 and Lemma 4.7(a), we can apply the bounded convergence theorem to the last expectation, so that

\[
\sum_{j=1}^{n} E_{\theta}[(r_{3n}g_{3,n,j}(\theta))^{2}] \rightarrow \frac{1}{\delta^{2}} \int_{\mathbb{R}} \left( \frac{y \phi_{1}(y)}{\phi_{1}(y)} + 1 \right)^{2} \phi_{1}(y) dy
\]

\[
= \frac{1}{2\pi \delta^{2}} \left[ \frac{y - y^{2}}{(1 + y^{2})^{2}} + \arctan y \right]_{y=\infty}^{y=-\infty} = \frac{1}{\delta^{2}} = \mathcal{I}_{33}(\theta).
\]

In a similar manner, based on the expression (35) we can deduce

\[
\sum_{j=1}^{n} E_{\theta}[(r_{4n}g_{4,n,j}(\theta))^{2}] \rightarrow \frac{1}{\delta^{2}} \int_{\mathbb{R}} \left( \frac{\phi_{1}'(y)}{\phi_{1}(y)} \right)^{2} \phi_{1}(y) dy
\]

\[
= \frac{1}{2\pi \delta^{2}} \left[ \frac{y^{3} - y}{(1 + y^{2})^{2}} + \arctan y \right]_{y=\infty}^{y=-\infty} = \frac{1}{\delta^{2}} = \mathcal{I}_{44}(\theta).
\]
Now we turn to the off-diagonal elements. First, by means of Lemmas 4.8, 4.9 and 4.10, we get
\[ \sum_{j=1}^{n} E_{\theta}[r_{1n, g_{1,nj}(\theta)r_{2n, g_{2,nj}(\theta)}] = \delta \frac{1}{\alpha} \sum_{j=1}^{n} E_{\theta} \left[ \left( \epsilon_{nj} - \frac{\beta}{\sqrt{m}} \right) \left( \frac{\alpha \Delta_{n}}{\sqrt{m}} + \frac{1}{\alpha} q_{nj} H(q_{nj}) \right) \right] \]
\[ = \delta \frac{1}{\alpha} \sum_{j=1}^{n} E_{\theta} \left[ \epsilon_{nj} - \frac{\beta}{\sqrt{m}} \right] q_{nj} H(q_{nj}) \]
\[ = \delta \frac{1}{\alpha} \sum_{j=1}^{n} E_{\theta} \left[ \epsilon_{nj} q_{nj} H(q_{nj}) \right] + O(\Delta_{n}) \rightarrow \frac{\delta}{\alpha} A'(\theta) = I_{12}(\theta). \]

Next, it follows from Lemma 4.7 that
\[ E_{\theta}[r_{2n, g_{2,nj}(\theta)r_{3n, g_{3,nj}(\theta)}] \leq \frac{1}{n} \left| E_{\theta} \left[ \sqrt{\Delta_{n}} \left( \epsilon_{n1} - \frac{\beta}{\sqrt{m}} \right) (1 + \epsilon_{n1} \eta(\epsilon_{n1})) \right] \right| + \delta \Delta_{n} \left| E_{\theta} \left[ \sqrt{\Delta_{n}} \left( \epsilon_{n1} - \frac{\beta}{\sqrt{m}} \right) \left( \sqrt{m} + \frac{\alpha}{\sqrt{1 + \epsilon_{n1}}^2} H(q_{nj}) \right) \right] \right| \]
\[ \leq \frac{1}{n} E_{\theta} \left[ \sqrt{\Delta_{n}} \left( \epsilon_{n1} - \frac{\beta}{\sqrt{m}} \right) \right] =: \frac{1}{n} E_{\theta}[\xi_{n}]. \]

Since \( \xi_{n} = o \rho_{n}(1) \) and \( \sup_{n} E_{\theta}[\xi_{n}^2] = \alpha^2/\delta m^{3/2} < \infty \) (cf. Lemmas 4.5 and 4.6), we deduce that
\[ E_{\theta}[\xi_{n}] \rightarrow 0. \]
Thus
\[ \left| \sum_{j=1}^{n} E_{\theta}[r_{2n, g_{2,nj}(\theta)r_{3n, g_{3,nj}(\theta)}] \right| \lesssim E_{\theta}[\xi_{n}] \rightarrow 0 = I_{23}(\theta). \]

Now let us note that \( \int_{\mathbb{R}} \phi_{1}(y)dy = 0 \), and that \( \int_{\mathbb{R}} y \{ \phi_{1}(y)/\phi_{1}(y) \}^{2} \phi_{1}(y)dy = 0 \) since the integrand is odd and behaves like \( y^{-3} \) up to multiplicative constant at infinity. Hence
\[ \sum_{j=1}^{n} E_{\theta}[r_{3n, g_{3,nj}(\theta)r_{4n, g_{4,nj}(\theta)}] \rightarrow \frac{1}{\beta^2} \int_{\mathbb{R}} \phi_{1}(y) \left( 1 + y \frac{\phi_{1}(y)}{\phi_{1}(y)} \right) \phi_{1}(y)dy = 0 = I_{34}(\theta). \]

The proofs for \( I_{43}(\theta) = 0 \) for the remaining \( (k, l) \)s are easier, and we omit them.

Summarizing the above now yields (17).

We now turn to prove the positive definiteness of \( I(\theta) \) for each \( \theta \in \Theta \). In view of the form (11), \( I(\theta) \) is positive definite as soon as so is the second principal submatrix, say \( I^{\alpha, \beta}(\theta) \). Obviously,
\[ \det [I^{\alpha, \beta}(\theta)] \] is symmetric as a function of \( \beta \). Hence, it suffices to prove that, given any \( \alpha > 0 \), the function \( \beta \rightarrow \det [I^{\alpha, \beta}(\theta)] = I_{11}(\theta) I_{22}(\theta) - (I_{12}(\theta))^{2} \) is positive for \( \beta \in [0, \alpha] \). Fix \( \alpha, \delta > 0 \) in the sequel. It is convenient to introduce the notation:
\[ C(\beta) = \int_{0}^{\infty} (e^{(\beta/\alpha)y} + e^{-(\beta/\alpha)y}) y K_{1}(y)dy, \]
\[ \Xi(y; \beta) = C(\beta)^{-1}(e^{(\beta/\alpha)y} + e^{-(\beta/\alpha)y}) y K_{1}(y), \quad y > 0. \]

Then \( y \mapsto \Xi(y; \beta) \) for each \( \beta \in [0, \alpha] \) acts as a probability density function on \( (0, \infty) \).

As in the proof of Lemma 4.10, we can derive
\[ C(\beta) = \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} y \left( e^{-((x+1/\alpha)^{2} - \beta/\alpha)x} + e^{-(x+1/\alpha)^{2} + \beta/\alpha)x} \right) dy dx \]
\[ = \frac{1}{2} \left[ \int_{0}^{\infty} \left\{ \frac{1}{2} \left( x + \frac{1}{x} \right) - \frac{\beta}{\alpha} \right\} ^{-2} dx + \int_{0}^{\infty} \left\{ \frac{1}{2} \left( x + \frac{1}{x} \right) + \frac{\beta}{\alpha} \right\} ^{-2} dx \right] \]
\[ = \frac{\alpha^{3} \pi}{m^{3/2}}, \]
where we used the identity valid for any $|b| < 1$:

$$
\int_0^\infty \left\{ \frac{1}{2} \left( x + \frac{1}{x} \right) + b \right\}^{-2} dx = \frac{2}{1 - b^2} \left[ \frac{1}{\sqrt{1 - b^2}} \arctan \left( \frac{x + b}{\sqrt{1 - b^2}} \right) - \frac{x - 2b^2x - b}{x^2 + 2bx + 1} \right]_{x=0}^{x=\infty} = \frac{2}{1 - b^2} \left[ \frac{\pi}{2} - \arctan \left( \frac{b}{\sqrt{1 - b^2}} \right) \right] - b.
$$

In particular, we have $C(0) = \pi$. Then, some elementary manipulations and Cauchy-Schwarz's inequality lead to

$$
det [I^{\alpha,\beta}(\theta)] = \alpha^{-2} A_2(\theta) \cdot \alpha^2 \delta m^{-3/2} - \alpha^{-2} \delta^2 \{A'(\theta)\}^2
= \frac{\alpha^4 \delta^2}{m^3} \left\{ \int_0^\infty \Xi(y; \beta) \left( \frac{K_0(y)}{K_1(y)} \right)^2 dy \cdot \int_0^\infty \Xi(y; \beta) dy - \left( \int_0^\infty \Xi(y; \beta) \frac{K_0(y)}{K_1(y)} dy \right)^2 \right\}
> 0,
$$

where the last strict inequality does hold true since $y \mapsto K_0(y)/K_1(y)$ is not a constant on $(0, \infty)$. This completes the proof of the positive definiteness of $I(\theta)$ for each $\theta \in \Theta$.

### 4.6 Negligibility of the centering

Turning to verification of (18), it suffices to see that $r_{kn} E_\theta[g_{k,n}(\theta)] = o(1/\sqrt{n})$ for each $k$.

Thanks to Lemma [13] we have $r_{1n} E_\theta[g_{1,n}(\theta)] = n^{-1/2} \{ O(\Delta_n) + E_\theta[q_{1n} H(q_{1n})] \} = O(\Delta_n/\sqrt{n}) = o(1/\sqrt{n})$, and it is obvious that $r_{2n} E_\theta[g_{2,n}(\theta)] = 0$. It follows from Lemmas [4.4] and [4.7] that $E_\theta[q_{1n} \Psi_{\eta}(\epsilon_{n1})] \to \int_R qy\phi_1'(y)dy = -1$ and $E_\theta[qy\epsilon(\epsilon_{n1})] \to \int_R qy\phi_1'(y)dy = 0$. Therefore $r_{3n} E_\theta[g_{3,n}(\theta)] = n^{-2/3} \{ o(1) + O(\Delta_n) \} = o(1/\sqrt{n})$ and similarly, $r_{3n} E_\theta[g_{3,n}(\theta)] = o(1/\sqrt{n})$. Thus we get (18).

### 4.7 Mean-square differentiability

Finally, we verify (19). For this it only remains to show that $\sum_{j=1}^n \sup_{\theta \in \Theta} E_\theta[r_{jk} \partial_{\theta_j} g_{nj}(\theta) r_{kn}^2] \to 0$ since we already have (17). To do this, we recall (12), and also the summands of the expressions (35) to (45) for $\partial_{\theta_j} g_{nj}(\theta) = \partial_{\theta_j}^2 \log \psi_{nj}(\theta)$. It suffices to estimate

$$
B_{kl,n}(\theta) := E_\theta \left[ |r_{kn} r_{nl} \partial_{\theta_k} \partial_{\theta_l} \log \psi_{nj}(\theta)|^2 \right]
$$

for $k, l \in \{1, 2, 3, 4\}$ individually, where we wrote $\theta := (\alpha, \beta, \delta, \mu) := (\theta_1, \theta_2, \theta_3, \theta_4)$ for convenience.

Invoking the boundedness of $y \mapsto y H'(y)$ (cf. Lemma [4.4] c), we get

$$
\sup_{\theta \in \Theta} B_{11,n}(\theta) \lesssim \frac{1}{n^2} \{ 1 + E_\theta[|\eta_{nj}|^2] \} \lesssim \frac{1}{n^2} \{ 1 + E_\theta[\eta_{nj}^2] \} \lesssim \frac{1}{n^2 \Delta_n} = o\left( \frac{1}{n} \right)
$$

as soon as $n \Delta_n \to \infty$, so that $\sum_{j=1}^n \sup_{\theta \in \Theta} B_{11,n}(\theta) \to 0$ according to the boundedness of $\Theta$. For the others, reminding Lemma [4.7] it is not difficult to deduce that

$$
\sup_{\theta \in \Theta} B_{kl,n}(\theta) \lesssim \begin{cases} O(1/n) & \text{for } k = l \neq 1 \text{ and for } \{k, l\} = \{1, 2\} \text{ or } \{3, 4\}, \\ O(\Delta_n/n) & \text{for the rest.} \end{cases} = o\left( \frac{1}{n} \right).
$$

Therefore $\sum_{j=1}^n \sup_{\theta \in \Theta} B_{kl,n}(\theta) \to 0$ for each $(k, l)$, completing the proof of (19).

### 5 Concluding remarks

In this article, we obtained the LAN for the statistical experiments consisting of the NIG Lévy process discretely observed at high frequency. The rate in the LAN are of two kind: $\sqrt{n}$ for $(\delta, \mu)$, while $\sqrt{n} \Delta_n$ for $(\alpha, \beta)$. Furthermore, the Fisher information matrix $I(\theta)$ turned out to be block-diagonal and
always positive-definite. Only the element $I_{11}(\theta)$ involves the integral, however, given any admissible parameter values, we can evaluate it numerically in a small amount of time.

One of important future tasks is construction of an estimator $\hat{\theta}_n$ of $\theta$, which is asymptotically optimal in the sense that, in view of Theorem 3.1, the normalized estimator $r_n^{-1}(\hat{\theta}_n - \theta)$ is asymptotically distributed as $N_4(0, I(\theta)^{-1})$ under the true measure. The maximum likelihood estimator is the first candidate. Nevertheless, direct simultaneous optimization for the four parameters might entail numerical difficulties; see Prause [15, Section 1]. It would then be more convenient to provide an rate-optimal estimator of $\theta$ (initial estimator) at first, and then execute the one-step improvement in order to attain the minimal asymptotic variance $I(\theta)^{-1}$. Those issues will be addressed in subsequent papers.

References


List of MI Preprint Series, Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1  Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2  Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost
Hermitian manifolds

MI2008-3  Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4  Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-
curve finite element scheme

MI2008-5  Yoshiyasu OZEKI
Torsion points of abelian varieties with values in finite extensions over a p-
adic field

MI2008-6  Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7  Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8  Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9  Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials
MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the $L^2$ a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the $L_1$ regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings
MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack’s formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric $\mathbb{G}$-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on $L^p$ spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain
MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for $H^2_0$-projection
MI2009-26 Manabu YOSHIDA  
Ramification of local fields and Fontaine’s property (Pm)

MI2009-27 Yu KAWAKAMI  
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA  
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA  
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA  
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI  
Hecke’s zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA  
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA  
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA  
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI  
Finite element computation for scattering problems of micro-hologram using DtN map
MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups

MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments
MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling