Mathematical analysis of the Navier-Stokes flow around a rotating obstacle

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Abstract

This lecture provides a survey of the progress in mathematical studies of incompressible viscous fluid motion in the exterior of a rotating obstacle. Our interest is focussed on the existence, stability and asymptotic profile of the solution.

1 Introduction

Let us consider the motion of an incompressible viscous fluid, which is governed by the well-known Navier-Stokes system, in an exterior domain $D \subset \mathbb{R}^3$ with smooth boundary ∂D. The case where the obstacle ($\equiv \mathbb{R}^3 \setminus D$), which consists of finite number of rigid bodies, moves in a prescribed way is of particular interest. In his series of famous papers, Robert Finn considered the problem with translating bodies and started its mathematical analysis. In this case we know rich results including qualitative behaviors of flows inside/outside wake region. We refer to [9], [10], [29], [31], [4], [33] and the references therein. When the rotation of the obstacle was also taken into account, however, few mathematical results were available in the 20th century and our knowledge of the motion of the fluid was far from complete. In order to understand the effect of rotation mathematically, in this lecture, we concentrate ourselves on the purely rotating problem without translation of the obstacle. In the reference frame, linear partial differential operator which appears in the reduced equation is

$$L = -\Delta - (\omega \times x) \cdot \nabla + \omega \times, \tag{1.1}$$

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see (1.3), where $\omega \in \mathbb{R}^3 \setminus \{0\}$ is a constant angular velocity of the obstacle and $\times$ stands for the usual exterior product. What is interesting is the presence of the drift operator $(\omega \times x) \cdot \nabla$ with unbounded coefficient because this is no longer subordinate to the Laplace operator even though $|\omega|$ is small. In fact, when we consider the operator L in the usual Lebesgue space $L_q(\mathbb{R}^3)$, $1 < q < \infty$, then the spectrum of $-L$ is given as follows and does not depend on $q \in (1, \infty)$ ([7], [8]):

$$\sigma(-L) = \{\lambda = \mu + ik|\omega|;\ \mu \leq 0, k \in \mathbb{Z}\}, \qquad i = \sqrt{-1}, \tag{1.2}$$

from which one can actually find hyperbolic effect of the operator $(\omega \times x) \cdot \nabla$. The picture of the spectrum concludes that the semigroup (2.7) generated by the operator $-L$ is never analytic on $L_q(\mathbb{R}^3)$ unlike the heat semigroup (2.8), and this is a remarkable point of the problem under consideration.

In the last several years, however, we have overcome the difficulty above to make progress in studies of:

(i) the existence of a unique steady Navier-Stokes flow, denoted by u_s, which decays like $1/|x|$ at infinity;

(ii) the stability of the steady flow u_s in the sense that a unique unsteady Navier-Stokes flow exists globally in time and goes to u_s as $t \to \infty$ whenever initial disturbance is small;

(iii) the asymptotic profile of the steady flow u_s for $|x| \to \infty$.

All of them are proved for small angular velocity ω. We note that the steady flow in the reference frame corresponds to the time-periodic flow with period $2\pi/|\omega|$ in the original frame, see (1.5). For the proof of the existence of a unique solution for all time $t \geq 0$, as usual, the first step is to find a basic flow around which the global solution might be constructed and, especially for the exterior problem, a steady flow with nice summability at infinity is a good candidate for the basic flow. So, the step (i) should be the starting point for us. The next step toward (ii) is the spectral analysis of the operator L given by (1.1). What is crucial is that, in spite of hyperbolic effect of the drift term $(\omega \times x) \cdot \nabla u$ such as (1.2), the semigroup generated by $-(Lu + \nabla p)$ over exterior domains, see (2.1), enjoys both a certain smoothing action near $t = 0$ ([20] in L_2, [17] in L_q) and decay properties as $t \to \infty$ of parabolic type (so-called L_p-L_q type, see [22]). As for (i) and (ii), after getting around special difficulty of the operator L, the final statements are eventually more or less similar to those for the usual Navier-Stokes flow ($\omega = 0$), while we deduce a different profile from the case $\omega = 0$ in (iii). We catch anisotropic decay structure for $|x| \to \infty$ arising from effect of rotation of the obstacle; in fact, it is made clear that the rotating axis plays an important role as preferred direction.

Since the Navier-Stokes equation is rotationally invariant, without loss of generality, the rotating axis of the obstacle may be assumed to be y_3-axis so that the angular velocity is given by $\omega = ae_3$, where $a \in \mathbb{R} \setminus \{0\}$ is a constant and $e_3 = (0,0,1)^T$. In what follows all vectors are column ones and superscript-T denotes the transpose. The unknown velocity $v(y,t) = (v_1, v_2, v_3)^T$ and pressure $\pi(y,t)$ obey the Navier-Stokes equation

$$\partial_t v + v \cdot \nabla v = \Delta v - \nabla \pi, \qquad \mathrm{div}\ v = 0$$

for $y \in D(t)$ subject to the boundary condition

$$v|_{\partial D(t)} = \omega \times y \equiv a(-y_2, y_1, 0)^T, \qquad v \to 0 \quad \text{as } |y| \to \infty.$$

Note that we have imposed the usual no-slip boundary condition on the surface $\partial D(t)$ since $\omega \times y$ is the rotating velocity of the obstacle. Here, the domain $D(t)$ occupied by the fluid at time t and its boundary $\partial D(t)$ are given by

$$D(t) = \{y = O(at)x;\ x \in D\}, \qquad \partial D(t) = \{y = O(at)x;\ x \in \partial D\},$$

where

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We take the coordinate system attached to the obstacle to reduce the problem above to

$$\partial_t u + u \cdot \nabla u = \Delta u + (\omega \times x) \cdot \nabla u - \omega \times u - \nabla p, \qquad \mathrm{div}\ u = 0 \tag{1.3}$$

in D subject to

$$u|_{\partial D} = \omega \times x, \qquad u \to 0 \quad \text{as } |x| \to \infty, \tag{1.4}$$

where

$$u(x,t) = (u_1, u_2, u_3)^T = O(at)^T v(O(at)x, t), \quad p(x,t) = \pi(O(at)x, t), \tag{1.5}$$

see [2], [20], [13].

Mathematical analysis of the initial value problem for (1.3)–(1.4) was begun first by Wolfgang Borchers, who proved the existence of weak solutions; indeed, the only literature was his Habilitationsschrift [2] when I started the study of this problem (Professor Tetsuro Miyakawa kindly informed me of his result). As in the usual Navier-Stokes theory (traced back to Jean

Leray), we don't know the uniqueness of weak solutions. I constructed a unique solution locally in time within the L_2-theory ([20]) when the initial data possess regularity to some extent as in Fujita and Kato [11]. In order to remove the regularity of them, as was done by Giga and Miyakawa [18] and also by Kato [24], we need the L_q-theory. But the generation of the semigroup in L_q had been hard. Because, generally speaking, the Yosida-Hille theorem is not so useful when the semigroup is neither contractive nor analytic (even the usual Stokes semigroup is never contractive on account of the pressure unless $q = 2$). In [17] Geissert, Heck and Hieber succeeded in its proof; indeed, they constructed the semigroup concretely and then made sure of the domain of the generator finally, see (2.1). Later on, Shibata [34] gave another construction of the semigroup.

In [14] Galdi first proved that the steady problem

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p + u \cdot \nabla u = 0, \qquad \operatorname{div} u = 0 \tag{1.6}$$

in D subject to (1.4) has a unique solution (u, p) which satisfies

$$|u(x)| \leq \frac{C|\omega|}{|x|}, \qquad |\nabla u(x)| + |p(x)| \leq \frac{C|\omega|}{|x|^2} \tag{1.7}$$

for large $|x|$ provided $|\omega|$ is small enough. Later on, in [5] another outlook on his pointwise decay (1.7) was provided in terms of weak-L_q spaces:

$$u \in L_{3,\infty}(D), \qquad (\nabla u, p) \in L_{3/2,\infty}(D). \tag{1.8}$$

The proof was based on some L_q-estimates ([6], [21]) for the operator L in the whole space $\mathbb{R}^3$ with the aid of tools from harmonic analysis together with cut-off procedure developed by [35]. The class (1.8) or pointwise decay (1.7) of the steady flow is important to deduce its stability, which has been established by [15] and [22]. In the former it was shown that the disturbance goes to zero in the Dirichlet norm without any rate, while in the latter some definite convergence rates have been derived, see (2.3).

We finally address the leading term of (1.7), which decays exactly at the rate $1/|x|$ so that the remaining term decays faster. It is proved that the leading term is given by a member of the family of (-1)-homogeneous solutions, found first by Landau [27] and revisited by Šverák [36], for the usual Navier-Stokes equation

$$-\Delta u + \nabla p + u \cdot \nabla u = 0, \qquad \operatorname{div} u = 0 \qquad (x \in \mathbb{R}^3 \setminus \{0\}). \tag{1.9}$$

Note that, for (1.9), (-1)-homogeneity is equivalent to self-similarity. It is proved by [36] that the family of solutions constructed by Landau covers all self-similar solutions of (1.9). Each member of this family is parameterized by vector about which it is axisymmetric. For the leading term of the flow under consideration, this vector parameter is parallel to the angular velocity ω. Therefore, this leading term satisfies also (1.6) in $\mathbb{R}^3 \setminus \{0\}$ since the additional two terms vanish, see (3.9). This study is inspired by the recent work [26] due to Korolev and Šverák, in which the leading term of the usual exterior Navier-Stokes flow for the case $\omega = 0$ is provided; it is given by another member of the same family as above and possesses symmetry about the axis whose direction (vector parameter mentioned above) is the net force (3.8) of the given flow.

We remark that the leading profile is the Oseen fundamental solution (without effect of nonlinearity) when the obstacle is translating with constant velocity, see for instance [4], on account of better decay property outside wake region behind the obstacle. In the case where both translation and rotation of the obstacle are taken into account, a wake region was still found by [16]. In this case as well, very probably, the leading profile comes from the linear part unlike the purely rotating problem discussed in this lecture.

The next section is devoted to the spectral analysis of the operator L given by (1.1). In the final section we find the asymptotic profile of the steady flow at infinity.

2 Spectral analysis of the operator L

The results of this section were obtained jointly with Yoshihiro Shibata [22]. We adopt the same symbols for vector and scalar function spaces. Let $C_0^\infty(D)$ consist of all C^∞-functions with compact support in D. For $1 \leq q \leq \infty$ and $0 \leq k \in \mathbb{Z}$, we denote by $W_q^k(D)$, with $W_q^0(D) = L_q(D)$, the usual L_q-Sobolev space of order k. Let $1 < q < \infty$ and $1 \leq r \leq \infty$. Then the Lorentz spaces are defined by

$$L_{q,r}(D) = \Big(L_1(D), L_\infty(D)\Big)_{1-1/q,r}$$

where $(\cdot,\cdot)$ is the real interpolation functor. It is well known that a measurable function f is in $L_{q,\infty}(D)$ if and only if

$$\sup_{\sigma>0} \sigma\,|\{x \in D; |f(x)| > \sigma\}|^{1/q} < \infty$$

and that $L_{q,\infty}(D)$ is the dual space of $L_{q/(q-1),1}(D)$. Note that $C_0^\infty(D)$ is not dense in $L_{q,\infty}(D)$. Let $C_{0,\sigma}^\infty(D)$ be the class of all C_0^∞-vector fields f which satisfy div $f = 0$ in D. For $1 < q < \infty$ we denote by $J_q(D)$ the completion of $C_{0,\sigma}^\infty(D)$ in $L_q(D)$. Then the Helmholtz decomposition of L_q-vector fields holds, see [12], [30]:

$$L_q(D) = J_q(D) \oplus \{\nabla\pi \in L_q(D); \pi \in L_{q,loc}(\overline{D})\}.$$

Let P denote the projection operator from $L_q(D)$ onto $J_q(D)$ associated with the decomposition. Then the Stokes operator $\mathcal{L}$ with rotation effect is defined by

$$\begin{aligned} &D(\mathcal{L}) = \{u \in J_q(D) \cap W_q^2(D);\, u|_{\partial D} = 0,\, (\omega \times x) \cdot \nabla u \in L_q(D)\}, \\ &\mathcal{L}u = PLu = -P[\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u]. \end{aligned} \tag{2.1}$$

It is proved by [17] and [34] that the operator $-\mathcal{L}$ generates a C_0-semigroup $\{T(t)\}_{t \geq 0}$ on the space $J_q(D), 1 < q < \infty$ (see also [19] for the case $q = 2$). But this is not an analytic semigroup. In fact, Farwig, Nečasová and Neustupa [7], [8] investigated the spectrum not only for the whole space problem (1.2) but also for the exterior problem; the essential spectrum $\sigma_{ess}(-\mathcal{L})$ is given by the RHS of (1.2) in $J_q(D)$. Nevertheless, the semigroup $T(t)$ possesses smoothing effect such as (2.4) for $t \to 0$.

Let u_s be the solution to the steady problem (1.6) subject to (1.4) and v the disturbance of u_s. In terms of the semigroup $T(t)$, the equation which v obeys is reduced to

$$v(t) = T(t)v_0 - \int_0^t T(t-\tau)P(u_s \cdot \nabla v + v \cdot \nabla u_s + v \cdot \nabla v)(\tau)\, d\tau, \tag{2.2}$$

where v_0 denotes the initial disturbance of u_s. The additional linear terms with coefficient u_s in the class (1.8) can be treated as perturbation when we essentially use the solenoidal Lorentz spaces $J_{q,r}(D)$ as in Yamazaki [37], where

$$J_{q,r}(D) = \Big(J_{q_0}(D), J_{q_1}(D)\Big)_{\theta,r}$$

with $1 < q_0 < q < q_1 < \infty, 1 \leq r \leq \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Note that $\{T(t)\}_{t \geq 0}$ is extended to the semigroup on the space $J_{q,r}(D)$.

Theorem 2.1 *Let $u_s \in L_{3,\infty}(D)$ and $v_0 \in J_{3,\infty}(D)$.*

1. *There is a constant $\delta > 0$ such that if*

$$\|u_s\|_{L_{3,\infty}(D)} + \|v_0\|_{L_{3,\infty}(D)} \leq \delta,$$

then the equation (2.2) possesses a unique global solution

$$v \in BC\left((0,\infty); J_{3,\infty}(D)\right) \quad \text{with} \quad w^*\text{-}\lim_{t \to 0} v(t) = v_0 \text{ in } J_{3,\infty}(D).$$

2. *Let $3 < q < \infty$. Then there is a constant $\widetilde{\delta}(q) \in (0, \delta]$ such that if*

$$\|u_s\|_{L_{3,\infty}(D)} + \|v_0\|_{L_{3,\infty}(D)} \leq \widetilde{\delta}(q),$$

then the solution $v(t)$ obtained above enjoys

$$\|v(t)\|_{L_r(D)} = O(t^{-1/2+3/2r}) \quad \text{as } t \to \infty \tag{2.3}$$

for every $r \in (3, q)$.

What is essential for the proof of Theorem 2.1 is to establish the L_p-L_q estimates of the semigroup $T(t)$. In the following theorem, we have to take care of the dependence of $T(t)$ on the angular velocity $\omega = ae_3$, and thus we write $T_a(t)$.

Theorem 2.2 *Suppose that*

$$\begin{cases} 1 < p \le q \le \infty \quad (p \ne \infty) & \text{for } j = 0, \\ 1 < p \le q \le 3 & \text{for } j = 1, \end{cases}$$

and let $a_0 > 0$ be arbitrary. Set

$$\kappa = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

Then there is a constant $C = C(p, q, a_0) > 0$ such that

$$\|\nabla^j T_a(t) f\|_{L_q(D)} \le C t^{-j/2-\kappa} \|f\|_{L_p(D)} \tag{2.4}$$

for all $t > 0$, $f \in J_p(D)$ and ω with $|\omega| = |a| \le a_0$.

Estimate (2.4) with $p = q$ concludes the uniform boundedness of the semigroup in t on $J_q(D)$, which was not shown in [17], while the semigroup is contractive on $J_2(D)$, see [19]. The restriction $q \le 3$ for the gradient estimate, which was first proved by Iwashita [23] for the case of the usual Stokes semigroup ($\omega = 0$), arises from the fact that the effect from the solution to the whole space problem remains near the boundary. In fact, Maremonti and Solonnikov [28] pointed out that one cannot avoid that restriction. In view of their proof, this is also related to the decay structure of steady solutions.

The origin of strategy toward (2.4) based on spectral analysis together with cut-off technique was found in [32] by Shibata for hyperbolic equation with a dissipation term in exterior domains and, later on, the framework of argument was well developed by Iwashita [23] and also by Kobayashi and Shibata [25] for the Stokes and Oseen semigroups. We now fix $R > 0$ such that $\mathbb{R}^3 \setminus D \subset B_R$, and set $D_R = D \cap B_R$. The most difficult step is to derive the so-called local energy decay estimate

$$\|T(t)Pf\|_{W^1_q(D_R)} \le C t^{-3/2} \|f\|_{L_q(D)} \qquad (t \ge 1) \tag{2.5}$$

for all $f \in L_{q,[d]}(D)$, where $1 < q < \infty$ and

$$L_{q,[d]}(D) = \{f \in L_q(D);\ f(x) = 0 \text{ a.e. } |x| \geq d\}, \qquad d > 0.$$

Indeed there is no relationship between the analyticity of semigroup and local energy decay properties, but we have actually some difficulties caused by lack of analyticity of $T(t)$. For instance, we have no information about the behavior of the resolvent $(\lambda I + \mathcal{L})^{-1}$ for $|\lambda| \to \infty$ along the imaginary axis, and thus it is no longer obvious to understand the representation of the semigroup

$$T(t)Pf = \frac{-1}{2\pi i t}\int_{-\infty}^{\infty} e^{i\tau t}\partial_\tau(i\tau I + \mathcal{L})^{-1}Pf\,d\tau, \tag{2.6}$$

which is formally obtained from the inverse Laplace transform of the resolvent when we shift the path of integration to the imaginary axis ($\lambda = i\tau$) after integration by parts with respect to λ.

The large time behavior of the semigroup $T(t)$ is closely related to the regularity for small λ of the resolvent $(\lambda I + \mathcal{L})^{-1}$. The first step is the analysis of the resolvent problem

$$\lambda u + Lu + \nabla p = f, \qquad \operatorname{div} u = 0 \qquad (x \in \mathbb{R}^3).$$

The solution, which we denote by $A_{\mathbb{R}^3}(\lambda)f$, is described as the Laplace transform of the semigroup

$$(S(t)f)(x) = O(at)^T \left(e^{t\Delta}f\right)(O(at)x) \tag{2.7}$$

in the whole space, where

$$\left(e^{t\Delta}f\right)(x) = (G(\cdot,t) * f)(x), \qquad G(x,t) = (4\pi t)^{-3/2}e^{-|x|^2/(4t)}. \tag{2.8}$$

The Fourier transform of (2.7) is given by

$$\widehat{(S(t)f)}(\xi) = O(at)^T e^{-|\xi|^2 t}\widehat{f}\,(O(at)\xi)\,,$$

and thus we have

$$\begin{aligned}
u(x,\lambda) &= (A_{\mathbb{R}^3}(\lambda)f)(x) \\
&= \frac{1}{(2\pi)^3}\int_0^\infty\int_{\mathbb{R}^3} e^{-(\lambda+|\xi|^2)t}e^{ix\cdot\xi}\,O(at)^T\Pi(O(at)\xi)\widehat{f}(O(at)\xi)\,d\xi dt \\
&= \frac{1}{(2\pi)^3}\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-(\lambda+|\xi|^2)t}e^{i(O(at)x-y)\cdot\xi}\,O(at)^T\Pi(\xi)f(y)\,dyd\xi dt
\end{aligned}$$

for $\operatorname{Re}\lambda \geq 0$ and $f \in L_q(\mathbb{R}^3)$, where $\Pi(\xi) = I - \xi\otimes\xi/|\xi|^2$. If in particular $f \in L_{q,[d]}(\mathbb{R}^3)$, then $A_{\mathbb{R}^3}(\lambda)f$ possesses a certain regularity in the space $W_q^2(B_R)$. In fact, we find

$$\partial_\lambda A_{\mathbb{R}^3}(\lambda)f \sim |\lambda - ika|^{-1/2}, \qquad k = 0, \pm 1$$

near $\lambda = 0, \pm ia$ and $A_{\mathbb{R}^3}(\cdot)f$ is of class C^1 on $\overline{\mathbb{C}_+} \setminus \{0, \pm ia\}$ with values in $W^2_q(B_R)$. Furthermore, we observe

$$\begin{aligned} \partial_\lambda^2 A_{\mathbb{R}^3}(\lambda)f &\sim |\lambda - ika|^{-3/2}, \qquad k = 0, \pm 1 \\ \partial_\lambda^2 A_{\mathbb{R}^3}(\lambda)f &\sim |\lambda - ika|^{-1/2}, \qquad k = \pm 2, \pm 3 \end{aligned}$$

near $\Lambda = \{0, \pm ia, \pm 2ia, \pm 3ia\}$ and $A_{\mathbb{R}^3}(\cdot)f$ is of class C^2 on $\overline{\mathbb{C}_+} \setminus \Lambda$ with values in $W^2_q(B_R)$. We know that each of ika is the end point of the spectrum $\sigma(-L)$, see (1.2). The observations above, however, tell us that the order of singularity at $\lambda = ika$ depends on $k \in \mathbb{Z}$. Further, we can justify the regularity $C^{3/2}$ of the resolvent $A_{\mathbb{R}^3}(\lambda)$ in the sense that

$$\int_{-K}^{K} \|(\partial_\lambda A_{\mathbb{R}^3})(i(\tau+h))f - (\partial_\lambda A_{\mathbb{R}^3})(i\tau)f\|_{W^2_q(B_R)} d\tau \le C|h|^{1/2}\|f\|_{L_q(\mathbb{R}^3)} \quad (2.9)$$

for $|h| \le 1$ and $f \in L_{q,[d]}(\mathbb{R}^3)$, where $K > 0$ is a fixed large number.

The analysis of the resolvent for large λ is also quite important in our problem because of lack of analyticity of the semigroup. The main point is that, by integration by parts with respect to t, the resolvent $A_{\mathbb{R}^3}(\lambda)$ can be divided into two parts. The first term arising from $t = 0$ is something like parabolic part, that is, its analytic continuation into a sectorial subset of the left half complex plane is possible, while the second term decays rapidly as $|\lambda| \to \infty$ in $\overline{\mathbb{C}_+}$, even along the imaginary axis. To be precise, given arbitrary $N \in \mathbb{N}$, we have the representation

$$\begin{aligned} &(A_{\mathbb{R}^3}(\lambda)f)(x) \\ &= \sum_{k=0}^{N-1} M^k(\lambda I - \Delta_{\mathbb{R}^3})^{-(k+1)} P_{\mathbb{R}^3} f(x) \\ &\quad + \frac{1}{(2\pi)^3} \int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-(\lambda+|\xi|^2)t} e^{ix\cdot\xi}}{(\lambda+|\xi|^2)^N} O(at)^T \left[\widetilde{M}^N(\Pi \widehat{f})\right](O(at)\xi)\, d\xi dt \end{aligned}$$

in $W^2_q(B_R)$ for all $f \in L_{q,[d]}(\mathbb{R}^3)$, where

$$M = (\omega \times x)\cdot \nabla - \omega \times, \qquad \widetilde{M} = (\omega \times \xi)\cdot \nabla_\xi - \omega \times .$$

Note the relation $(\widehat{M\psi})(\xi) = \widetilde{M}\widehat{\psi}(\xi)$.

We next derive a representation of the resolvent $(\lambda I + \mathcal{L})^{-1}$ in exterior domains. We employ a cut-off technique to construct a parametrix (v, π) by use of resolvents in the whole space $\mathbb{R}^3$ and in the bounded domain D_R together with the Bogovskiĭ operator ([1]) to recover the solenoidal condition. By $A(\lambda)$ we denote the linear mapping: $f \mapsto v(\cdot, \lambda) = A(\lambda)f$, where f is a given external force. The pair (v, π) should obey

$$(\lambda + L)v + \nabla\pi = f + R(\lambda)f, \qquad \text{div } v = 0 \qquad (x \in D)$$

subject to $v|_{\partial D} = 0$, where $R(\lambda)f$ denotes the remainder term arising from the cut-off procedure. The operator $R(\lambda)$ is divided into two parts

$$R(\lambda) = R_1 + R_2(\lambda)$$

where R_1 is independent of λ and consists of pressure in the whole space $\mathbb{R}^3$ and λ-independent part (arising from the Helmholtz decomposition) of pressure in the bounded domain D_R. As usual, a compactness argument implies the existence of the bounded inverse $(I + R(\lambda))^{-1}$; however, the behavior of $(I + R(\lambda))^{-1}$ for large λ is not clear. We thus reconstruct this inverse of the form

$$\begin{aligned}(I + R(\lambda))^{-1} &= \left[I + (I + R_1)^{-1}R_2(\lambda)\right]^{-1}(I + R_1)^{-1} \\ &= \sum_{k=0}^{\infty}\left\{-(I + R_1)^{-1}R_2(\lambda)\right\}^k (I + R_1)^{-1}\end{aligned}$$

for large λ by using $R_2(\lambda) \to 0$ as $|\lambda| \to \infty$. A key step is thus to show the invertibility of $I + R_1$, which is reduced to the uniqueness of the Helmholtz decomposition. As a consequence, the operator norm of the inverse $(I + R(\lambda))^{-1}$ is uniformly bounded in λ; therefore, both the behavior for large λ and the regularity for small λ of the resolvent

$$(\lambda I + \mathcal{L})^{-1}Pf = A(\lambda)(I + R(\lambda))^{-1}f$$

are governed by those of the resolvent in the whole space $\mathbb{R}^3$. In fact, as long as we take f from the space $L_{q,[d]}(D)$, we have the following behavior of the resolvent:

$$\|\partial_\lambda^j(\lambda I + \mathcal{L})^{-1}Pf\|_{W_q^1(D_R)} \le C\,|\lambda|^{-j-1/2}\|f\|_{L_q(D)}, \quad j = 0, 1, 2 \tag{2.10}$$

in $\{\lambda \in \overline{\mathbb{C}_+}; |\lambda| \ge K\}$, where $K > 0$ is a fixed large number, and

$$\|\partial_\lambda(\lambda I + \mathcal{L})^{-1}Pf\|_{W_q^1(D_R)} \le C\,|\lambda - ika|^{-1/2}\|f\|_{L_q(D)}, \quad k = 0, \pm 1 \tag{2.11}$$

in $\{\lambda \in \overline{\mathbb{C}_+}; |\lambda - ika| \le |a|/4\}$. Moreover, one can justify the formula (2.6) in the space $W_q^1(D_R)$ for $f \in L_{q,[d]}(D)$ as well as the $C^{3/2}$-regularity of $(\lambda I + \mathcal{L})^{-1}P$ in the same sense as in (2.9). We now consider

$$\int_{-\infty}^{\infty} e^{i\tau t}\partial_\tau(i\tau I + \mathcal{L})^{-1}Pf\,d\tau$$

in $W_q^1(D_R)$ and split it into integrals for large τ and for finite τ. Then the former decays like t^{-1} that follows from (2.10) for $j = 1, 2$ by integration by parts once more, while the latter decays like $t^{-1/2}$ that follows from the relationship between the regularity of a function and the decay property of its inverse Fourier image; thus, we conclude (2.5).

3 Asymptotic profile of the steady flow

The results of this section have been recently obtained jointly with Reinhard Farwig (the paper is now in preparation). The first step should be the study of the associated Stokes problem

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p = f, \qquad \mathrm{div}\, u = 0 \tag{3.1}$$

in D subject to (1.4). Even for this linear problem, it is no longer clear what the leading term of the Stokes flow is. In addition, anisotropic decay structure arising from effect of rotation must be observed at the level of the linear problem. We thus intend to derive such structure from asymptotic representation of the Stokes flow for $|x| \to \infty$. The results obtained in [5] and [21] suggest that the optimal rate of decay of the solution to (3.1) is $1/|x|$ in general even though the external force is very nice such as, for instance, $f = \mathrm{div}\, F$ with $F \in C_0^\infty(D)^{3\times 3}$. Theorem 3.1 below provides its rigorous explanation when we look at the leading term. For the sake of simplicity to catch the profile, the external force is of the form $f = \mathrm{div}\, F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$, the restriction of $F \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$ to $\overline{D}$ (although divergence form is not needed). We will look for not only the leading term ($\sim 1/|x|$) but also the second one ($\sim 1/|x|^2$).

Theorem 3.1 *Let $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$. Given $f = \mathrm{div}\, F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$, let (u, p) be the solution to* (3.1) *subject to* (1.4)*. Then it is represented as*

$$u(x) = U_{1st}(x) + U_{2nd}(x) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right),$$

$$p(x) = P_{1st}(x) + O\left(\frac{1}{|x|^3}\right)$$

for $|x| \to \infty$ with

$$\begin{aligned} U_{1st}(x) &= \frac{1}{8\pi} \int_{\partial D} (\nu \cdot (T + F))_3 \, d\sigma_y \left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3}\right) \\ &= E_{St}(x) \begin{pmatrix} 0 \\ 0 \\ \int_{\partial D} (\nu \cdot (T + F))_3 \, d\sigma_y \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} U_{2nd}(x) &= \frac{1}{8\pi|x|^3} \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{3\,(x \otimes x)}{8\pi|x|^5} \begin{pmatrix} \frac{\alpha'}{2} x_1 \\ \frac{\alpha'}{2} x_2 \\ \alpha_3 x_3 \end{pmatrix} \\ &= \frac{\beta(e_3 \times x)}{8\pi|x|^3} + \left\{\alpha - \frac{3\left(\frac{\alpha'}{2}|x'|^2 + \alpha_3 x_3^2\right)}{|x|^2}\right\} \frac{x}{8\pi|x|^3}, \end{aligned}$$

$$P_{1st}(x) = \int_{\partial D} \{(\nu \cdot (\Delta u))\, y - p\nu + \nu \cdot F\}\, d\sigma_y \cdot Q_{St}(x).$$

Here, ν is the exterior unit normal to the boundary ∂D, the pair of

$$E_{St}(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}\mathbb{I} + \frac{x \otimes x}{|x|^3}\right), \qquad Q_{St}(x) = \nabla\left(\frac{-1}{4\pi|x|}\right) = \frac{x}{4\pi|x|^3} \tag{3.2}$$

is the usual Stokes fundamental solution, $\mathbb{I}$ is the 3×3 unity matrix, $x \otimes x = (x_i x_j)_{1 \le i,j \le 3}$,

$$T = T(u,p) = \nabla u + (\nabla u)^T - p\mathbb{I}$$

is the Cauchy stress tensor, and

$$\begin{aligned}
&\alpha = -\int_{\partial D} y \cdot (\nu \cdot (T + F))\, d\sigma_y + \int_D trF\, dy = \alpha' + \alpha_3,\\
&\alpha' = -\int_{\partial D} y' \cdot (\nu \cdot (T + F))'\, d\sigma_y + \int_D (F_{11} + F_{22})\, dy,\\
&\alpha_3 = -\int_{\partial D} y_3 (\nu \cdot (T + F))_3\, d\sigma_y + \int_D F_{33}\, dy,\\
&\beta = e_3 \cdot \int_{\partial D} y \times (\nu \cdot (T + F))\, d\sigma_y + \int_D (F_{12} - F_{21})\, dy,\\
&\nu \cdot (T + F) = \Big((\nu \cdot (T + F))', (\nu \cdot (T + F))_3\Big)^T\\
&x = (x', x_3)^T, \quad y = (y', y_3)^T.
\end{aligned}$$

The proof relis upon a detailed analysis of the fundamental solution $\{\Gamma(x,y), Q(x,y)\}$ of the equation (3.1) in the whole space $\mathbb{R}^3$. In terms of the heat kernel $G(x,t)$, see (2.7)–(2.8), we find

$$\Gamma(x,y) = \Gamma^0(x,y) + \Gamma^1(x,y)$$

with

$$\Gamma^0(x,y) = \int_0^\infty O(at)^T G(O(at)x - y, t)\, dt, \tag{3.3}$$

$$\begin{aligned}
\Gamma^1(x,y) &= -\int_0^\infty \int_0^s \nabla_x \nabla_y [G(O(at)x - y, s)]\, dtds\\
&= \int_0^\infty (4\pi s)^{-3/2} \int_0^s e^{-|O(at)x - y|^2/(4s)}\\
&\qquad \left\{\frac{\left(x - O(at)^T y\right) \otimes (O(at)x - y)}{4s^2} - \frac{1}{2s}\, O(at)^T\right\} dtds,
\end{aligned} \tag{3.4}$$

and also

$$Q(x,y) = \nabla_y \frac{1}{4\pi|x-y|} = Q_{St}(x-y),$$

see (3.2).

By Theorem 3.1 we know what kind of effect on the profile the rotation of the body causes. For the Stokes flow, the leading profile is the third column vector of the usual Stokes fundamental solution (3.2) and possesses

(i) the symmetry about the rotating axis (x_3-axis);

(ii) (-1)-homogeneity.

We are thus actually informed of the important role of the rotating axis and this knowledge is useful in finding the leading term of the Navier-Stokes flow. Furthermore, the quantity which controls the rate of decay is $e_3 \cdot N$, where

$$N = \int_{\partial D} \nu \cdot T(u,p)\, d\sigma \tag{3.5}$$

is the net force (case $F = 0$ in Theorem 3.1). Thus, it is reasonable to expect that the leading term U of the Navier-Stokes flow for (1.6) subject to (1.4) still keeps the properties (i), (ii) above and solves

$$-\Delta U - (\omega \times x) \cdot \nabla U + \omega \times U + \nabla P + U \cdot \nabla U = (e_3 \cdot N)\, e_3 \delta, \quad \operatorname{div} U = 0 \tag{3.6}$$

in $\mathcal{D}'(\mathbb{R}^3)$, where δ denotes the Dirac measure at 0. The present section concludes that this is correct. Here, we should note the relation

$$e_3 \cdot N = e_3 \cdot \widetilde{N}, \tag{3.7}$$

which is the consequence of $u|_{\partial D} = \omega \times x$ together with $e_3 \cdot (\omega \times x) = 0$, where

$$\widetilde{N} = \int_{\partial D} \nu \cdot \{T(u,p) - u \otimes u\}\, d\sigma. \tag{3.8}$$

Note also that

$$(e_3 \times x) \cdot \nabla U - e_3 \times U = 0 \tag{3.9}$$

holds for all vector fields which are symmetric about x_3-axis. Further, (3.9) holds in $\mathcal{D}'(\mathbb{R}^3)$ when $U \sim 1/|x|$ around $x = 0$. Thus the candidate above for the leading term solves (3.11) with $k = e_3 \cdot N$ and, due to [36], it must be a member of the family of the Landau solutions explained below.

Let $b \in \mathbb{R}^3$ be a prescribed vector, that we call the Landau parameter. Then, among nontrivial smooth solutions of (1.9), Landau [27] found an exact solution, called the Landau solution, which satisfies:

- the symmetry about the axis $\mathbb{R}b$;
- the homogeneity

$$u(x) = \frac{1}{|x|}\, u\left(\frac{x}{|x|}\right), \qquad p(x) = \frac{1}{|x|^2}\, p\left(\frac{x}{|x|}\right);$$

- $-\Delta u + \nabla p + u \cdot \nabla u = b\delta$ in $\mathcal{D}'(\mathbb{R}^3)$.

When b is parallel to e_3, the Landau solution is of the form

$$\begin{cases} u(x) = \dfrac{2}{|x|}\Big[\dfrac{c\sigma_3 - 1}{(c-\sigma_3)^2}\,\sigma + \dfrac{1}{c-\sigma_3}\,e_3\Big], \\ p(x) = \dfrac{4(c\sigma_3 - 1)}{|x|^2(c-\sigma_3)^2} \end{cases} \tag{3.10}$$

with parameter $c \in (-\infty, -1) \cup (1, \infty)$, where $\sigma = x/|x|$. Further, (u, p) satisfies

$$-\Delta u + \nabla p + u \cdot \nabla u = ke_3\delta, \qquad \text{div } u = 0 \tag{3.11}$$

in $\mathcal{D}'(\mathbb{R}^3)$, where k is given by

$$k = k(c) = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\log\frac{c+1}{c-1}\right). \tag{3.12}$$

For this calculation we refer to [3]. Since the function $k(\cdot)$ is monotonically decreasing on each of $(-\infty, -1)$ and $(1, \infty)$, and fulfills

$$k(c) \to 0 \quad (|c| \to \infty); \quad k(c) \to -\infty \quad (c \to -1); \quad k(c) \to \infty \quad (c \to 1),$$

there is a unique $c \in (-\infty, -1) \cup (1, \infty)$ such that $k(c) = k$ for every $k \in \mathbb{R} \setminus \{0\}$. When $k = 0$, we may understand $(u, p) = (0, 0)$ as the solution (3.10) with $|c| \to \infty$.

Now, given smooth solution (u, p) of the Navier-Stokes problem (1.6) with (1.4), we take N and $\widetilde{N}$ as in (3.5) and (3.8). Let (U, P) be the Landau solution with the Landau parameter

$$b = (e_3 \cdot N)e_3 = (e_3 \cdot \widetilde{N})e_3,$$

see (3.7). Namely, (U, P) is given by (3.10) with c which is determined by $k(c) = e_3 \cdot N$ (it is the trivial solution in case $e_3 \cdot N = 0$), where $k(\cdot)$ is as in (3.12). By (3.9), (U, P) solves (3.6) in $\mathcal{D}'(\mathbb{R}^3)$ as well. We are in a position to give our result.

Theorem 3.2 *Let $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$. For each $q_0 \in (3/2, 3)$ there exists a constant $\eta = \eta(q_0) > 0$ such that if u is a smooth solution to* (1.6) *subject to* (1.4) *and satisfies*

$$\sup_{x \in D} |x||u(x)| + |e_3 \cdot N| \le \eta,$$

then, for every $q \in (q_0, 3)$, we have

$$u - U|_D \in L_q(D), \qquad \|u - U\|_{L_q(D)} \le C(|a|^{-3/q+1} + 1) \tag{3.13}$$

with some $C = C(q) > 0$, where U is the Landau solution as above.

This theorem tells us that the remainder $u - U$ possesses better summabilty (which suggests the pointwise decay $1/|x|^2$) at infinity; in this sense, the Landau solution U is the leading term of small solution u. Because the leading term of the usual Navier-Stokes flow ($\omega = 0$) found by Korolev-Šverák [26] is different (it is the Landau solution with $b = \widetilde{N}$), it is reasonable that our remainder possesses singular behavior for $a \to 0$ as in (3.13).

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