Return to the equilibrium and pseudospectral estimates: a toy model

Francis Nier
IRMAR
Université de Rennes 1
Campus de Beaulieu
35042 Rennes, France
Francis.Nier@univ-rennes1.fr

Abstract

Several kinds of spectral quantities associated with semigroup generators are involved in the problem of the return to the equilibrium for parabolic or hypoelliptic type linear evolution equations: the numerical range, the spectrum and the pseudo-spectrum (or ϵ-spectrum). The distinction between the three spectral objects becomes crucial when the generator is a parameter-dependent differential operator. In a recent work with T. Gallay and I. Gallagher, we have studied a simple one dimensional model. It is a parameter dependent non self-adjoint perturbation of the harmonic oscillator hamiltonian, where the three spectral notions are related to various quantitative estimates. Such a simple model, originally arising from the study of the stability of Oseen vortices in fluid mechanics, shows a wide variety of phenomena. After introducing the motivations and the relationship between spectral quantitative estimates and quantitative estimates of the time decay, the analysis done in [6] is summarized.

1 Introduction

1.1 Motivation from fluid mechanics

The problem arose originally from works by T. Gallay and C.E. Wayne in [7][8] about the stability of Oseen vortices. Consider the incompressible 2D Navier-Stokes equation

$$\left\{\begin{array}{l} \partial_t u + u.\nabla u = \Delta u - \nabla p \\ \text{div } u = 0, \quad u = u(x,t) \in \mathbb{R}^2, \quad x \in \mathbb{R}^2, t > 0\,, \end{array}\right.$$

in the vorticity formulation with $\omega = \partial_1 u_2 - \partial_2 u_1$ with the Biot-Savart law

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\, dy =: (K_{BS} * \omega)(x)\,.$$

After introducing self-similar coordinates $\xi = \frac{x}{\sqrt{t}}$ and $\tau = \log t$, $\omega(x,t) = \frac{1}{t}w(\frac{x}{\sqrt{t}}, \log t)$ and $u(x,t) = \frac{1}{\sqrt{t}}v(\frac{x}{\sqrt{t}}, \log t)$, it is written

$$\partial_\tau w + v.\nabla w = \Delta_\xi w + \frac{1}{2}\xi.\nabla_\xi w + w \quad , \quad v = K_{BS} * w\,,$$

with the equilibrium solution

$$G(\xi) = \frac{1}{4\pi}e^{-\frac{|\xi|^2}{4}} \quad , \quad v^G(\xi) = \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}(1 - e^{-\frac{|\xi|^2}{4}})\,.$$

The linearized equation around αG (write $w = \alpha G + \tilde{w}$ and forget the second order corrections) is

$$\partial_\tau \tilde{w} = (\mathcal{L}_1 - \alpha\Lambda_1)\tilde{w}$$

with

$$\mathcal{L}_1\tilde{w} = \Delta_\xi \tilde{w} + \frac{1}{2}\xi.\nabla_\xi \tilde{w} + \tilde{w} \quad \text{and} \quad \Lambda_1 \tilde{w} = v^G.\nabla\tilde{w} + (K_{BS} * \tilde{w}).\nabla G\,,$$

studied in the natural space $L^2(\mathbb{R}^2, G^{-1}\; d\xi)^2$. A conjugation with $G^{1/2}$ gives

$$\begin{aligned}
\mathcal{L} &:= -G^{-1/2}\mathcal{L}_1 G^{1/2} = -\Delta_\xi + \frac{|\xi|^2}{16} - \frac{2}{4} \quad \text{harmonic oscillator}\\
\Lambda &:= -G^{-1/2}\Lambda_1 G^{1/2} = v^G.\nabla_\xi + 2(K_{BS} * G^{1/2}.)\nabla G^{1/2}\,,
\end{aligned}$$

in $L^2(\mathbb{R}^2, d\xi)^2$. The first spectral properties of those operators have been studied in [8][21]. The operator Λ is anti-adjoint and the rotational invariance allows to write

$$\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n \quad \text{and} \quad \Lambda = \oplus_{n\in\mathbb{Z}}\Lambda_n\,.$$

When the lower order term $2(K_{BS} * G^{1/2}.)\nabla G^{1/2}$ is neglected and with $\hat{w} = G^{-1/2}\tilde{w} = w_n(r)e^{in\theta}$ in polar coordinates, one is led to the operator

$$-\frac{1}{r^2}(r\partial_r)^2 + \frac{n^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\frac{\alpha n}{2\pi r^2}(1 - e^{-r^2/4})\,,$$

on $L^2(\mathbb{R}_+, rdr)$. The main difficulty occurs when $r \to \infty$ (the asymptotics $\alpha \to \infty$ is also of interest) and setting $r = 2\sqrt{|n|} + \rho$ leads to

$$-\partial_\rho^2 + \frac{\rho^2}{16}(1 + \mathcal{O}(\frac{\sqrt{|n|}}{\rho})) + i\frac{\alpha n}{2\pi\rho^2}(1 + \mathcal{O}(\frac{\sqrt{|n|}}{\rho})) - \rho^{-1}\partial_\rho - \frac{\alpha n e^{-(2\sqrt{|n|}+\rho)^2/4}}{(\rho + 2\sqrt{|n|})^2}\,. \tag{1.1}$$

The main term equals

$$-\partial_\rho^2 + \frac{\rho^2}{16} + \frac{if(\rho)}{\epsilon} \tag{1.2}$$

with $\epsilon = \frac{2\pi}{|n|\alpha}$, $f(\rho) \in \mathbb{R}$ bounded and$f(\rho) \sim \frac{\pm 1}{\rho^2}$ as $\rho \to \infty$.

1.2 Parameter dependent non self-adjoint spectral problems.

The spectral analysis of parameter dependent non self-adjoint operators has known recently a strong development (see for example [27, 3, 4, 26]) and more specifically for the exponential decay of contraction semigroups (see for example [14, 11, 17, 15, 16, 28, 29]).

Consider in a Hilbert space X with the scalar product $\langle\, ,\, \rangle$, a maximal accretive operator H_ϵ, with domain $D(H_\epsilon)$, depending on the parameter $\epsilon \to 0$. The maximal accretivity implies that the numerical range

$$\Theta(H_\epsilon) \;=\; \left\{\langle H_\epsilon u, u\rangle \in \mathbb{C}\,;\, u \in D(H_\epsilon)\,,\; \|u\|_{L^2} = 1\right\} \tag{1.3}$$

is contained in $\{z \in \mathbb{C},\, \mathrm{Re}\, z \geq 0\}$. Differentiating $\|e^{-tH_\epsilon}u\|^2$ with respect to $t \in \mathbb{R}_+$ yields

$$\|e^{-tH_\epsilon}\| \leq e^{-t\,\mathrm{dist}(i\mathbb{R}, \Theta(H_\epsilon))}\,. \tag{1.4}$$

Assume further that it is sectorial with the numerical range included in the sector $\{z \in \mathbb{C}\,;\, |\arg z| \leq \frac{\pi}{2} - 2\alpha\}$ for some $\alpha \in (0, \frac{\pi}{4}]$ which may depend on ϵ (the case $\alpha = 0$ can be included with some variations like in [14][11]) and that the resolvent $(1+H_\epsilon)^{-1}$ is compact so that the $\sigma(H_\epsilon)=\{\lambda_n(\epsilon),\, n\in\mathbb{N}\}$ is discrete. We set

$$\Xi(\epsilon)=\mathrm{dist}(i\mathbb{R},\Theta(H_\epsilon))=\inf\mathrm{Re}(\Theta(H_\epsilon))\,, \tag{1.5}$$

$$\Sigma(\epsilon) = \inf\mathrm{Re}(\sigma(H_\epsilon)) = \min_{n\in\mathbb{N}}\mathrm{Re}(\lambda_n(\epsilon))\,, \tag{1.6}$$

$$\Psi(\epsilon) = \left(\sup_{\lambda\in\mathbb{R}}\|(H_\epsilon-i\lambda)^{-1}\|\right)^{-1}\,. \tag{1.7}$$

They satisfy

$$\Xi(\epsilon)\leq\Psi(\epsilon)\leq\Sigma(\epsilon)\,. \tag{1.8}$$

The role of $\Sigma(\epsilon)$ and $\Psi(\epsilon)$ in the exponential decay of $\|e^{-tH_\epsilon}\|$ occurs via the Laplace transform and the deformation contour in

$$e^{-tH_\epsilon}u=\frac{1}{2i\pi}\int_{+i\infty}^{-i\infty}\frac{e^{-tz}}{(z-H_\epsilon)^{-1}}u\;dz\,,\quad u\in D(H_\epsilon)\,.$$

More precisely the next general result can be added to (1.4).

Proposition 1.1 *Let A be a maximal accretive operator in a Hilbert space X, with numerical range contained in the sector $\{z\in\mathbb{C}\,;\,|\arg z|\leq\frac{\pi}{2}-2\alpha\}$ for some $\alpha\in(0,\frac{\pi}{4}]$. Assume that A is invertible and let*

$$\Sigma=\inf\mathrm{Re}(\sigma(A)) > 0\,,\quad \textit{and}\quad \Psi=\left(\sup_{\lambda\in\mathbb{R}}\|(A-i\lambda)^{-1}\|\right)^{-1}\,.$$

Then the following holds:

i) *If there exist $C\geq 1$ and $\mu>0$ such that $\|e^{-tA}\|\leq C\,e^{-\mu t}$ for all $t\geq 0$, then*

$$\Sigma\geq\mu\,,\quad \textit{and}\quad \Psi\geq\frac{\mu}{1+\log(C)}\,.$$

ii) *For any $\mu\in(0,\Sigma)$, we have $\|e^{-tA}\|\leq C(A,\mu)\,e^{-\mu t}$ for all $t\geq 0$, where*

$$C(A,\mu)=\frac{1}{\pi\tan\alpha}\Big(\mu N(A,\mu)+2\pi\Big)\,,\quad \textit{and}\quad N(A,\mu)=\sup_{\lambda\in\mathbb{R}}\|(A-\mu-i\lambda)^{-1}\|\,.$$

iii) *For $\mu\in(0,\Psi)$, the quantity $N(A,\mu)$ is not larger than $(\Psi-\mu)^{-1}$.*

iv) *For $\mu\in(0,\Sigma)$, the quantity $N(A,\mu)$ is bounded from below by $\frac{1}{\Psi}e^{\frac{\mu}{e\Psi}}$.*

This general result applied with $A=H_\epsilon$ is especially informative when $\alpha\propto\mathcal{O}(\epsilon^{\nu_0})$, $\nu_0\geq 0$, and

$$\Psi(\epsilon)\propto\epsilon^{-\nu_\psi}\ll\Sigma(\epsilon)\propto\epsilon^{-\nu_\sigma}\,,\quad \nu_\sigma>\nu_\psi>0\,.$$

($a(\epsilon)\propto b(\epsilon)$ means that $(\frac{a(\epsilon)}{b(\epsilon)})^{\pm 1}$ remains bounded as $\epsilon\to 0^+$)
The conclusion is then

1. A uniform estimate $\|e^{-tH_\epsilon}\|\leq 1\times e^{-t\mu}$ holds for $\mu\leq\Xi(\epsilon)$. It makes sense for all $t\geq 0$.
2. An estimate $\|e^{-tH_\epsilon}\|\leq\epsilon^{-\nu_\mu}\times e^{-t\mu}$, for some $\nu_\mu\geq 0$, is possible for $\mu\leq\Psi(\epsilon)/2$. It makes sense for $t\gg\epsilon^{\nu_\psi}|\log\epsilon|$.
3. An estimate $\|e^{-tH_\epsilon}\|\leq C(H_\epsilon,\mu)\times e^{-t\mu}$, holds for $\mu\leq\Sigma(\epsilon)$. It makes sense for $t\gg\frac{\log C(H_\epsilon,\mu)}{\mu}$.
4. When $\mu\in(\Psi(\epsilon),\Sigma(\epsilon))$, the constant $C(H_\epsilon,\mu)$ is "exponentially large" $C(H_\epsilon,\mu)\geq e^{\frac{\mu-\Psi(\epsilon)}{\Psi(\epsilon)}}$ owing to i). Upper bounds are worse.

1.3 Pseudospectral nature of $\Psi(\epsilon)$.

For ϵ-dependent non self-adjoint differential or pseudo-differential operators, (usually written in the form $p(x, \epsilon D_x)$) it is important to distinguish in the complex plane the set of λ's for which the resolvent norm is polynomially bounded:

$$\exists N_\lambda \in \mathbb{R}, \quad \|p(x, \epsilon D_x) - \lambda)^{-1}\| = \mathcal{O}(\epsilon^{-N_\lambda}) .$$

The complement of this set is usually called the pseudospectrum or ϵ-spectrum (see [3][23][4] for example).
The dependence w.r.t $\epsilon > 0$ of our operator H_ϵ is a bit different and the notion can be refined by considering ϵ-dependent areas in the complex plane.

Definition 1.2 *Let $(\omega_\epsilon)_{\epsilon\in(0,1]}$ be a family of complex domains, i.e. $\omega_\epsilon \subset \mathbb{C}$ for all $\epsilon \in (0, 1]$. We say that ω_ϵ* meets the pseudospectrum *of H_ϵ as $\epsilon \to 0$ if*

$$\lim_{\epsilon\to 0} \epsilon^N \sup_{z\in\omega_\epsilon} \|(H_\epsilon - z)^{-1}\| = +\infty , \quad \text{for all } N \in \mathbb{N} .$$

On the contrary, we say that ω_ϵ avoids the pseudospectrum *of H_ϵ as $\epsilon \to 0$ if there exists $N \in \mathbb{N}$ such that*

$$\sup_{z\in\omega_\epsilon} \|(H_\epsilon - z)^{-1}\| = \mathcal{O}(\epsilon^{-N}) , \quad \text{as } \epsilon \to 0 .$$

The pseudopsectral nature of the quantity $\Psi(\epsilon)$ defined in (1.7) and which is so crucial in the exponential decay, appears in the next result.

Proposition 1.3
i) *For any $\kappa \in (0, 1)$, the domain $\{\mathrm{Re}(z) \le \kappa\Psi(\epsilon)\}$ avoids the pseudospectrum of H_ϵ as $\epsilon \to 0$.*
ii) *If $\mu_\epsilon \gg \Psi(\epsilon)(1 + \log \Psi(\epsilon) + \log(\epsilon^{-1}))$ in the sense that the ratio goes to $+\infty$ as $\epsilon \to 0$, then the domain $\{\mathrm{Re}(z) \le \mu_\epsilon\}$ meets the pseudospectrum of H_ϵ as $\epsilon \to 0$.*

2 Main results.

2.1 Assumptions.

Consider for a bounded real-valued function f

$$H_\epsilon = -\partial_x^2 + x^2 + \frac{i}{\epsilon} f(x) , \quad x \in \mathbb{R} , \tag{2.1}$$

acting on the Hilbert space $X = L^2(\mathbb{R})$, with domain $D(H_\epsilon) = \{u \in H^2(\mathbb{R})\, ;\, x^2u \in L^2(\mathbb{R})\}$. It satifies the assumptions of Subsection 1.2 and its numerical range is contained in the region $\mathcal{R}_\epsilon \subset \mathbb{C}$ defined by

$$\mathcal{R}_\epsilon = \left\{\lambda \in \mathbb{C}\, ;\, \mathrm{Re}(\lambda) \ge 1 ,\ \epsilon\, \mathrm{Im}(\lambda) \in \overline{f(\mathbb{R})}\right\} . \tag{2.2}$$

Here are the assumptions which fit with the analysis as $\rho \to \infty$ of our fluid mechanics example (1.1)(1.2).

Hypothesis 2.1 *We assume that $f \in C^3(\mathbb{R}, \mathbb{R})$ has the following properties:*
i) All critical points of f are non-degenerate; i.e., $f'(x) = 0$ implies $f''(x) \ne 0$.
ii) There exist positive constants C and k such that, for all $x \in \mathbb{R}$ with $|x| \ge 1$,

$$\left|\partial_x^\ell \left(f(x) - \frac{1}{|x|^k}\right)\right| \le \frac{C}{|x|^{k+\ell+1}} , \quad \text{for } \ell = 0, 1, 2, 3 . \tag{2.3}$$

2.2 Results.

Theorem 2.2 *If f satisfies Hypothesis 2.1, there exists $C_\psi \geq 1$ such that, for all $\epsilon \in (0,1]$,*

$$\frac{1}{C_\psi\, \epsilon^{\nu_\psi}} \leq \Psi(\epsilon) \leq \frac{C_\psi}{\epsilon^{\nu_\psi}}\ , \quad \textit{where} \quad \nu_\psi = \frac{2}{k+4}\ . \tag{2.4}$$

This provides also a lower bound for $\Sigma(\epsilon)$. But an accurate analysis of $\Sigma(\epsilon)$ requires the control of exponentially large quantities and is achieved after a complex deformation argument. Hence the assumption of f have to be strengthened. The next example, which is again related to (1.1)(1.2), shows that $\Sigma(\epsilon) \gg \Psi(\epsilon)$ occurs.

Theorem 2.3 *Fix $k > 0$ and assume that*

$$f(x) = \frac{1}{(1+x^2)^{k/2}}\ , \quad x \in \mathbb{R}\ . \tag{2.5}$$

Then there exists a constant $C_\sigma > 0$ such that the lowest real part of the spectrum satisfies, for all $\epsilon \in (0,1]$,

$$\Sigma(\epsilon) \geq \frac{C_\sigma}{\epsilon^{\nu_\sigma}}\ , \quad \textit{where} \quad \nu_\sigma = \min\left\{\frac{1}{2}, \frac{2}{k+2}\right\}\ . \tag{2.6}$$

Theorem 2.2 can be refined in a form which shows that the lower and upper bounds for $\Psi(\epsilon)$ result from the competition of various phenomena. Under Hypothesis 2.1, the function f has only a finite number of critical points. The finite set of critical values of f is denoted by

$$\mathrm{cv}(f) = \left\{ f(x)\,;\, x \in \mathbb{R}\,,\ f'(x) = 0 \right\}\ .$$

For any $\lambda \in \mathbb{R}$ and any $\epsilon \in (0,1)$, we define

$$\kappa(\epsilon, \lambda) = \|(H_\epsilon - i\lambda)^{-1}\|\ . \tag{2.7}$$

The following proposition gives accurate bounds on $\kappa(\epsilon,\lambda)$ in various parameter regimes:

Proposition 2.4 *For $\epsilon \in (0,1)$ and $\lambda \in \mathbb{R}$, the quantity $\kappa(\epsilon,\lambda)$ defined in (2.7) satisfies the following estimates:*

i) *If $\mathrm{dist}(\epsilon\lambda, f(\mathbb{R})) \geq \delta > 0$, then $\kappa(\epsilon,\lambda) \leq \epsilon/\delta$.*

ii) *If $\mathrm{dist}(\epsilon\lambda, \mathrm{cv}(f) \cup \{0\}) \geq \delta > 0$, then $\kappa(\epsilon,\lambda) \leq C_\delta\, \epsilon^{2/3}$.*

iii) *If $\lambda = \lambda(\epsilon)$ is such that $\lim_{\epsilon\to 0} \epsilon\lambda(\epsilon) = c \in \mathrm{cv}(f) \setminus \{0\}$, then $\limsup_{\epsilon \to 0} \epsilon^{-1/2}\kappa(\epsilon, \lambda(\epsilon)) \leq C$.*

iv) *For $\lambda = 0$, the quantity $\kappa(\epsilon, 0)$ satisfies*

$$\kappa(\epsilon, 0) \leq \begin{cases} C\,\epsilon^{\frac{2}{k+2}} & \textit{if} \quad 0 \notin f(\mathbb{R})\ , \\ C\,\epsilon^{\min\{\frac{2}{k+2}, \frac{2}{3}\}} & \textit{if} \quad 0 \in f(\mathbb{R}) \setminus \mathrm{cv}(f)\ , \\ C\epsilon^{\min\{\frac{2}{k+2}, \frac{1}{2}\}} & \textit{if} \quad 0 \in \mathrm{cv}(f)\ . \end{cases}$$

v) *There exists $C > 1$ such that $\kappa(\epsilon,\lambda) \leq C\epsilon^{\frac{2}{k+4}}$ for all $(\epsilon,\lambda) \in (0,1) \times \mathbb{R}$. Moreover, if $\kappa(\epsilon,\lambda) \geq C^{-1}\epsilon^{\frac{2}{k+4}}$, then λ is comparable to $\epsilon^{-\frac{4}{k+4}}$.*

*Finally all estimates in **i)**, **ii)**, **iii)**, **iv)**, and **v)** are optimal, in the sense that one can find $\lambda = \lambda(\epsilon)$ so that the pair $(\epsilon, \lambda(\epsilon))$ satisfies the required conditions as $\epsilon \to 0$ and so that $\kappa(\epsilon, \lambda(\epsilon))$ is comparable to the upper bound in this limit.*

Proposition 2.4 also allows to localize the pseudospectrum of H_ϵ accurately. This is summarized in the next picture.

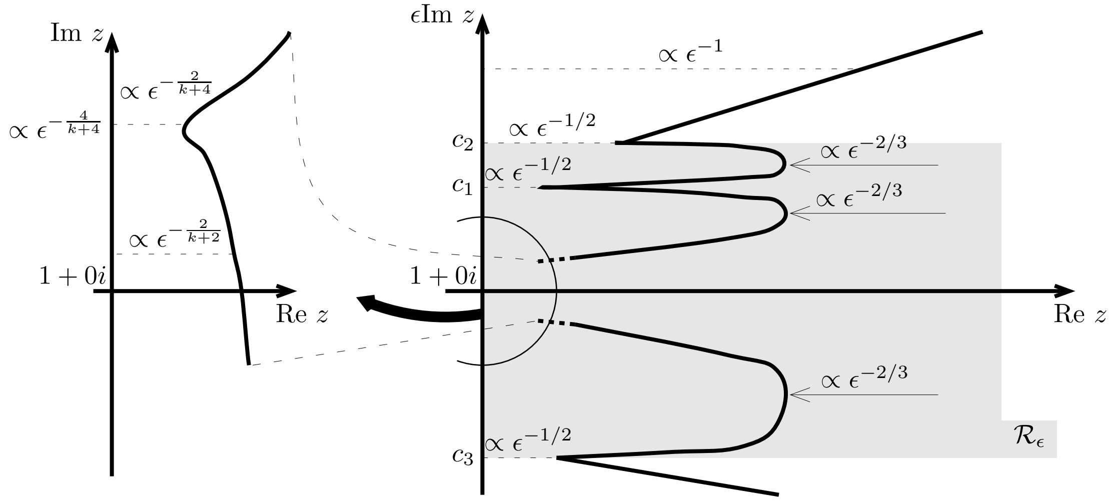

Fig. 2: The domain ω_ϵ on the left-hand side of the solid curve avoids the pseudospectrum of H_ϵ as $\epsilon \to 0$. The picture on the right shows the geometry at the scale $\epsilon z = \mathcal{O}(1)$, while the left picture focuses on the region where ϵz is small. Here $R_\epsilon = \{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \geq 0\,,\ \min f \leq \epsilon\, \mathrm{Im}\, z \leq \max f\}$ and $\mathrm{cv}(f) = \{c_1, c_2, c_3\}$.

2.3 Summarized proofs.

Theorem 2.2 is a straightforward consequence of Proposition 2.4 which also provides the pseudospectral geometry in Figure 1.

Sketch of the proof of Proposition 2.4: Owing to the symbol type behaviour assumed for $\partial_x^\alpha f(x)$ in Hypothesis 2.1, the two asymptotics $\epsilon \to 0$ and $x \to \infty$ are better handled by introducing a dyadic partition of unity

$$1 = \sum_{j=0}^{\infty} \chi_j(x)^2 = \chi_0(x)^2 + \sum_{j=1}^{\infty} \tilde{\chi}\Big(\frac{x}{2^j}\Big)^2 ,$$

where $\chi_0, \tilde{\chi} \in C_0^\infty(\mathbb{R})$ satisfy

$$\chi_0(x) = \begin{cases} 1 & \text{if} \quad |x| \leq \frac{3}{4} , \\ 0 & \text{if} \quad |x| \geq 1 , \end{cases} \qquad \tilde{\chi}(x) = \begin{cases} 1 & \text{if} \quad \frac{1}{2} \leq |x| \leq \frac{3}{4} , \\ 0 & \text{if} \quad |x| \leq \frac{3}{8} \text{ or } |x| \geq 1 . \end{cases}$$

Then the problem is reduced to finding regularity lower bounds for local problems which are parametrized by $(\epsilon, 2^j, \lambda)$:

Lemma 2.5 *For $j \in \mathbb{N}$, $\epsilon > 0$, and $\lambda \in \mathbb{R}$, consider the operator*

$$P_{j,\epsilon,\lambda} = -2^{-2j}\partial_x^2 + 2^{2j}x^2 + \frac{i}{\epsilon} f(2^j x) - i\lambda , \tag{2.8}$$

and let

$$C_j(\epsilon, \lambda) = \inf\Big\{ \|P_{j,\epsilon,\lambda} u\| \,;\, u \in C_0^\infty(\mathbb{R}) ,\ \ \mathrm{supp}\, u \subset K_j ,\ \|u\| = 1 \Big\} , \tag{2.9}$$

where $K_0 = [-1, 1]$ and $K_j = [-1, -1/4] \cup [1/4, 1]$ for any $j > 0$. Then the quantity $\kappa(\epsilon, \lambda) = \|(H_\epsilon - i\lambda)^{-1}\|$ satisfies

$$\Big(\inf_{j \in \mathbb{N}} C_j(\epsilon, \lambda)\Big)^{-1} \leq \kappa(\epsilon, \lambda) \leq C\Big(\inf_{j \in \mathbb{N}} C_j(\epsilon, \lambda)\Big)^{-1} , \tag{2.10}$$

for some constant $C \geq 1$ independent of ϵ, λ.

Essentially three cases have to be considered

1. j is bounded and $\epsilon\lambda \notin \mathrm{cv}(f)$. The term $2^{2j}x^2$ can be forgotten and one is reduced with the (micro)-local model
$$\tilde{P}_{j;\lambda;\epsilon} = -\partial_y^2 + \frac{i}{\epsilon}y$$
which is unitarily equivalent to $-\epsilon^{-2\alpha}\partial_y^2 + i\epsilon^{-1+\alpha}y$. Taking $\alpha = 1/3$ yields the lower bound $C_j(\epsilon, \lambda) \propto \epsilon^{-2/3}$.

2. j is bounded and $\epsilon\lambda \in \mathrm{cv}(f)$. Then the (micro)-local model is
$$\tilde{P}_{j;\lambda;\epsilon} = -\partial_y^2 + \frac{i}{\epsilon}y^2$$
which is unitarily equivalent to $-\epsilon^{-2\alpha}\partial_y^2 + i\epsilon^{-1+2\alpha}y^2$. Taking $\alpha = 1/4$ yields $C_j(\epsilon, \lambda) \propto \epsilon^{-1/2}$.

3. $j \to \infty$, $\epsilon\lambda \to 0$. Several regimes have to be discussed. When $|x|$ or 2^j is very large, the real part $-\partial_x^2 + x^2$ alone brings the lower bound for $C_j(\epsilon, k)$. Owing to the homogeneity $f(x) \sim \frac{1}{|x|^k}$ as $x \to \infty$, the critical regime occurs when $h^2 := \epsilon 2^{(k-2)j} = \mathcal{O}\left(\epsilon^{\frac{6}{k+4}}\right)$ and the (micro)-local model is
$$\frac{1}{\epsilon 2^{kj}}(-h^2\partial_y^2 + iy) \quad \text{with} \quad h^2 \;=\; \epsilon 2^{(k-2)j}\ .$$
This corresponds to the regime in $x \propto 2^j \propto \epsilon^{-\frac{1}{k+4}}$ which specifies the position in terms of ϵ where the main phenomenon occurs. For those worst indices j's, this provides the behaviour $C_j(\epsilon, k) \propto \epsilon^{-\frac{2}{k+4}}$

4. Those lower bounds can be proved to be optimal by constructing approximate quasi-modes with the (micro)-local models.

□

Sketch of the proof of Theorem 2.3: The approach is similar to the analysis of resonances for Schrödinger operators (see[1][2][18][13]). Consider the change of variable $(U_\theta\phi)(x) = e^{\theta/2}f(e^\theta x)$ which defines a unitary operator U_θ when $\theta \in \mathbb{R}$. The operator
$$H_\epsilon(\theta) = U_\theta H_\epsilon U_{-\theta} = -e^{-2\theta}\partial_x^2 + e^{2\theta}x^2 + \frac{i}{\epsilon(1 + e^{2\theta}x^2)^{k/2}}$$
defines an analytic family of type (A) of operators. Hence its spectrum does not depend on θ, $|\operatorname{Im}\theta| < \pi/4$, and it has to be included in the intersection the ϵ-spectra of all the $H_\epsilon(\theta)$. By taking $\theta = it_k$ with $t_k = \frac{\pi}{4(k+2)}$, the operator $H_\epsilon(it_k)$ behaves like a sectorial operator in a region $\left\{z \in \mathbb{C},\, |z| \leq c\epsilon^{-1}\right\}$ with $c > 0$ small enough. Combined with the pseudospectral estimate of Proposition 2.4 summarized in Figure 1, this yields the result. □

2.4 Comments.

1. In the example (1.1)(1.2), the assumption says $k = 2$ and this leads to
$$\Psi(\epsilon) \propto \epsilon^{-1/3} \quad \text{and} \quad \Sigma(\epsilon) \geq C^{-1}\epsilon^{-1/2}\,,$$
which corresponds to numerical observations done before this analysis.

2. The competition of several microlocal models according to the size of $\epsilon\lambda$ and illustrated in Figure 1, can be observed numerically.

3. Additional proofs and results are given in [6]. One of them concerns an hypocoercivity approach adapted from [28][29]. It gives a slightly weaker result but with possibly more flexibility.

4. Several things are still to be studied:
   - Give an accurate description of the spectrum around $\{C^{-1}\Sigma(\epsilon) \leq \mathrm{Re}\, z \leq C\Sigma(\epsilon)\}$. Only a few elements are given in [6] showing that the lower bound of $\Sigma(\epsilon)$ should be optimal.
   - Complete the analysis of (1.1)(1.2) after including the neglected terms and possibly adding the lower order pseudodifferential term $2(K_{BS} * G^{1/2}.\ )\nabla G^{1/2}$.
   - Finally, exploit those linear results for improving the nonlinear stability analysis of Oseen vortices.

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