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On intersection properties of extremal ternary codes

Michio Ozeki

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Abstract. H. Koch [7],[8] derived formulas which describe design structure supported by the codewords of fixed weight in the extremal binary self-dual doubly even codes. His method uses the lattices (not extremal lattices) constructed from such codes and the modular forms associated with these lattices. The formulas imply the so called Assmus-Mattson theorem for binary codes as partial results plus an extra formula which is not obtainable from the original Assmus-Mattson theorem. In the present paper we develop a similar method to derive the formulas for the ternary self-dual extremal codes. The method also uses the lattice theory and the modular form theory. However in using the lattice theory the present paper differes largely from the ones by Koch.

Keywords. ternary extremal code, design equation

1. Introduction

For all the definitions and the notations, the reader may refer to the preliminaries below. Let $\mathbf{C}$ be an extremal self-dual binary (resp. ternary) code of length n, and $\mathcal{K}_m$ (resp. $\overline{\mathcal{K}_m}$)be the subset of $\mathbf{C}$ consisting of all the codewords of weight m (resp. a quotient set of the subset $\mathcal{K}_m$ of weight m codewords). Then by a theorem of Assmus-Mattson [1], $\mathcal{K}_m$ (resp. $\overline{\mathcal{K}_m}$) holds a $5-a$ design, where $a = \frac{n}{4} - 6\left[\frac{n}{24}\right]$ (resp. $a = \frac{n}{2} - 6\left[\frac{n}{12}\right]$) . Here [ ] is the Gaussian symbol and the weight m should receive a restriction according to the length n of the code $\mathbf{C}$. In [16] and [18] Mendelsohn derived interesting conditions for general t-designs. Combining these results, we can obtain the conditions for the intersection properties of extremal codes.

On the other hand, H.Koch ([7],[8]) derived the conditions for the intersection properties of binary extremal codes. Roughly speaking, Koch's method is explained as follows. First he derives inner product relations for the even unimodular lattice ,which is constructed from binary extremal codes, by way of theta series with spherical functions. Then he washed out the vectors in the lattice from these inner product relations, and the remainings are the intersection properties of such codes. However, most of Koch's conditions may be proved to be equivalent to those of Mendelsohn in the situation of codes. One exception is the condition coming from the final inner product relation (See , for example [8] page 461, formula (5)). It seems that Koch has not been aware of the works of Mendelsohn. One strong significance in Koch's derivation of intersection properties is that it does not depend on the above Assmus-Mattson theorem, and it may give a deeper insight for the interrelation between design structures of various $\mathcal{K}_m$'s.

In the present paper we develop a similar method to Koch's in ternary extremal codes case. Thus the present method together with Koch's work will cope with the Assmus-Mattson theorem in establishing design structures in coding theory. In developing our method, some ideas are parallel with binary code case. But other serious difficulties also arise in ternary code. These difficulties must be overcome properly.

2. Preliminary 1. Definitions from coding theory

2.1. Some common knowledge in coding theory

Let $\mathbf{F}_3 = GF(3)$ be the field of three elements. Let $V = \mathbf{F}_3^n$ be the vector space of dimension n over $\mathbf{F}_3$. A ternary linear $[n,k]$ code $\mathbf{C}$ is a subspace of V of dimension k. In V , the inner product , which is denoted by $(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y}$ in V, is defined as usual. The dual code $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^\perp = \{\mathbf{u} \in V \mid (\mathbf{u},\mathbf{v}) = \mathbf{0}\ \forall\ \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^\perp$. The code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^\perp$. Ternary self-dual codes exist only if $n \equiv 0 \pmod 4$ and $k = \frac{n}{2}$ .An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. Let $\mathbf{x}$ be a codeword of a linear $[n,k]$ code $\mathbf{C}$, then the Hamming weight $wt(\mathbf{x})$ of the codeword

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance d on $\mathbf{C}$ is also defined by $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} - \mathbf{y})$. A codeword $\mathbf{x}$ in $\mathbf{C}$ has minimal weight if it satisfies

$$\min_{\mathbf{0} \neq \mathbf{u} \in \mathbf{C}} wt(\mathbf{u}) = wt(\mathbf{x})$$

Such codeword $\mathbf{x}$ will be a minimal (weight) codeword. A ternary self-dual $[n, \frac{n}{2}]$ code $\mathbf{C}$ is called extremal if the minimal codeword $\mathbf{x}$ of $\mathbf{C}$ satisfies

$$wt(\mathbf{x}) = 3\left[\frac{n}{12}\right] + 3.$$

Here we give examples of ternary extremal codes of various lengths. Some of these will play an important role in the development of our present work. For the reader's sake, in one of two appendices we give generator matrices of the ternary Golay code, the unique [16,8,6] code, and one of six extremal [20,10,6] codes respectively.

Ex.1 The Golay ternary [12,6,6] code. The generator matrix G_6 is given in [15].
Ex.2 The code $2f_8$ in [3] is the unique [16,8,6] code. Its generator matrix is given there.
Ex.3 There are six extremal [20,10,6] codes. They are given in Pless and al. [25].

Ex.4 There are two extremal [24,12,9] codes. They are extended quadratic residue code $\mathbf{QR}_{24}$ and the Pless symmetry code P_{24}.(cf. [24])

Ex.5 There are some extremal [28,14,9] codes. (cf. [22],[6])

Ex.6 There are many extremal [32,16,9] codes. (cf.[6])

Ex.7 There is an extremal [36,18,12] code P_{36} . (cf.[24])

Ex.8 There are two extremal code of length 48, namely, the extended quadratic residue code $\mathbf{QR}_{48}$ and the Pless code P_{48}. (cf. [24])

2.2. Notions on intersections

Let $\mathbf{C}$ and be a ternary self-dual extremal code of length n, which is embedded in $\mathbf{F}_3^n$. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbf{F}_3^n$, then the number of the indices i such that $u_i \neq 0$ and $v_i \neq 0$ for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. We call $\mathbf{u} * \mathbf{v}$ the binary intersection number of the ternary vectors $\mathbf{u}$ and $\mathbf{v}$.

Now we introduce the notion of the ternary intersection index $\mathbf{u} \natural \mathbf{v} = ((\mathbf{u} \natural \mathbf{v})_1, (\mathbf{u} \natural \mathbf{v})_2)$ for the ternary vectors $\mathbf{u}$ and $\mathbf{v}$. Here $(\mathbf{u} \natural \mathbf{v})_1$ is the number of i's such that $u_i = v_i \neq 0$ and $(\mathbf{u} \natural \mathbf{v})_2$ is the number of i's such that $u_i \neq v_i, u_i \neq 0$ and $v_i \neq 0$. By definition it is clear that the equality

$$(\mathbf{u} \natural \mathbf{v})_1 + (\mathbf{u} \natural \mathbf{v})_2 = \mathbf{u} * \mathbf{v}. \tag{2.1}$$

holds. Note that $\mathbf{u} * \mathbf{u} = wt(\mathbf{u})$. We use the subsets $\mathcal{K}_m$ of the code $\mathbf{C}$ defined by

$$\mathcal{K}_m = \{\mathbf{u} \in \mathbf{C} \mid wt(\mathbf{u}) = m\}.$$

It is known that in self-dual ternary codes, $\mathcal{K}_m$ exists only when $m \equiv 0 \pmod 3$. The multiplicative group of $\mathbf{F}_3$ acts naturally on $\mathcal{K}_m$, and we denote by $\overline{\mathcal{K}_m}$ the quotient of $\mathcal{K}_m$ by this action. Usually, a codeword of weight 3 is called a trio, a codeword of weight 6 is called a sextet. Let $\mathbf{a}$ be any vector in $\mathbf{F}_3^n$, then intersection property is written according to the length of n.

3. Preliminary 2. Design structure for codes

3.1. A description of the design

A t-design is an ordered pair $(X, \mathbb{B})$ with the conditions below. $X = \{x_1, x_2, \cdot, x_v\}$ is a finite set of v points. $\mathbb{B} = \{B_1, B_2, \cdots, B_b\}$ is a set of subsets of X without repetition. A member B_i of $\mathbb{B}$ is called a block. A subset of X with the cardinality t is called a t-subset. With these terminologies, the design conditions for $(X, \mathbb{B})$ are stated as follows:

(1) Every t-subset of X is contained in exactly $\lambda_t = \lambda$ blocks of $\mathbb{B}$.
(2)Each block has a constant cardinality k.

If the pair $(X, \mathbb{B})$ satisfies the above two conditions, then $(X, \mathbb{B})$ is called a t-(v, k, λ) design. Usually some restrictions on v, k, t are imposed in order to avoid the triviality,namely, $v > k > t$. One important propertiey of a t-(v, k, λ) design is :

If $s \leq t$, then the t-(v, k, λ) design forms a s-(v, k, λ_s) design , where λ_s is given by

$$\lambda_s = \lambda_t \cdot \frac{\begin{pmatrix} v-s \\ t-s \end{pmatrix}}{\begin{pmatrix} k-s \\ t-s \end{pmatrix}} \tag{3.1}$$

An element $x_i \in X$ is incident to $B_j \in \mathbb{B}$ if and only if $x_i \in B_j$.

3.2. Mendelsohn equations

There are equations, due to Mendelsohn (”Intersection number of t-designs, Notices Amer. Math. Soc. 16 (1969) 984”). We give these equations here (but without proof), because they might not be well-known.
Let $(X, \mathbb{B})$ be a t-(v, k, λ) design, and B_0 be a fixed block of $\mathbb{B}$, then we define the numbers $\xi_0, \xi_1, \xi_2, \cdots, \xi_k$ by

$$\xi_i = \mid \{B \in \mathbb{B} \mid \mid B \cap B_0 \mid = i\} \mid,$$

where $|\{A\}|$ is the cardinality of the set $\{A\}$. Mendelsohn equations read

$$\xi_i + \binom{i+1}{i}\xi_{i+1} + \cdots + \binom{k}{i}\xi_k = \lambda_i \binom{k}{i} \quad 0 \le i \le t$$

Later R.J.Wilson [32],[33] generalized Mendelsohn's equations to the case when B_0 is any subset of X.

We now give a version of the Mendelsohn-Wilson equations as a lemma. Here we give a general system of equations for 5-degign which come from ternary codes. These are always applied to any 5-design. The parameters λ_i involved are the same as in (3.1).

Lemma 1. *Suppose the set $\overline{\mathcal{K}_m}$ holds a 5-design. Let $\mathbf{a}$ be any ternary vector of weight s ($= (\mathbf{a} * \mathbf{a})$). Then we have*

$$\sum_{\mathbf{u}\in\overline{\mathcal{K}_m}} (\mathbf{u} * \mathbf{a}) = \lambda_1(\mathbf{a} * \mathbf{a}) \tag{3.2}$$

$$\sum_{\mathbf{u}\in\overline{\mathcal{K}_m}} (\mathbf{u} * \mathbf{a})^2 = \lambda_2(\mathbf{a} * \mathbf{a})^2 + (\lambda_1 - \lambda_2)(\mathbf{a} * \mathbf{a}) \tag{3.3}$$

$$\sum_{\mathbf{u}\in\overline{\mathcal{K}_m}} (\mathbf{u} * \mathbf{a})^3 = \tag{3.4}$$

$$\lambda_3(\mathbf{a} * \mathbf{a})^3 + 3(\lambda_2 - \lambda_3)(\mathbf{a} * \mathbf{a})^2 + (\lambda_1 - 3\lambda_2 + 2\lambda_3)(\mathbf{a} * \mathbf{a})$$

$$\sum_{\mathbf{u}\in\overline{\mathcal{K}_m}} (\mathbf{u} * \mathbf{a})^4 \tag{3.5}$$

$$\begin{aligned} = \; & \lambda_4(\mathbf{a} * \mathbf{a})^4 + 6(\lambda_3 - \lambda_4)(\mathbf{a} * \mathbf{a})^3 \\ & +(7\lambda_2 - 18\lambda_3 + 11\lambda_4)(\mathbf{a} * \mathbf{a})^2 \\ & +(\lambda_1 - 7\lambda_2 + 12\lambda_3 - 6\lambda_4)(\mathbf{a} * \mathbf{a}) \end{aligned}$$

$$\sum_{\mathbf{u}\in\overline{\mathcal{K}_m}} (\mathbf{u} * \mathbf{a})^5 \tag{3.6}$$

$$\begin{aligned} = \; & \lambda_5(\mathbf{a} * \mathbf{a})^5 + 10(\lambda_4 - \lambda_5)(\mathbf{a} * \mathbf{a})^4 \\ & +5(5\lambda_3 - 12\lambda_4 + 7\lambda_5)(\mathbf{a} * \mathbf{a})^3 \\ & +5(3\lambda_2 - 15\lambda_3 + 22\lambda_4 - 10\lambda_5)(\mathbf{a} * \mathbf{a})^2 + \\ & (\lambda_1 - 15\lambda_2 + 50\lambda_3 - 60\lambda_4 + 24\lambda_5)(\mathbf{a} * \mathbf{a}) \end{aligned}$$

Since the proof is done by routine manipulations starting from the Mendelsohn-Wilson equations, we omit the proof.

Remark 1. (i) The equations in the above lemma are also applicable for t'-designs with $t' \le 5$. For example, if $(X, \mathbb{B})$ is a 3-design then the equations (3.2)∼(3.4) are valid. (ii) If we use $\mathcal{K}_m$ instead of $\overline{\mathcal{K}_m}$ in the equations (3.2) ∼ (3.6), we have only to multiply by 2 to the right-hand sides of each equation. (iii) The above equations do not suffice to prove that a system of subsets to form a 5-design. To assert it, we may need further restriction. But in luckier cases, the equations suffice to prove that.

4. Preliminary 3. Some definitions from lattice theory

4.1. Some terminologies

Let $\mathbb{Z}$ be the ring of rational integers and $\mathbb{Q}$ the field of rational numbers. A finitely generated $\mathbb{Z}$-module L in $\mathbb{Q}^n$ with a positive definite metric is called a positive definite quadratic lattice. Since we treat only the positive definite quadratic lattices, we shall omit the adjectives "positive definite quadratic". A lattice L is integral if L satisfies $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$ for any $\mathbf{x}, \mathbf{y} \in L$ where $(\; , \;)$ is the bilinear form associated to the metric. Two integral lattices L_1 and L_2 are said to be isomorphic if and only if there exists a bijective mapping from L_1 to L_2 preserving the metric. The dual lattice $L^{\#}$ of L is defined by

$$L^{\#} = \{\mathbf{y} \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}, \forall \mathbf{x} \in L\}.$$

A lattice L is called even if it holds that $(\mathbf{x}, \mathbf{x}) \equiv 0 \pmod 2$ for all $\mathbf{x} \in L$, and L is unimodular if $L = L^{\#}$. The maximal number of linearly independent vectors over $\mathbb{Q}$ in L is called the rank of L. It is known that the rank of an even unimodular lattice is divisible by 8. In an even lattice L, for the non-zero vector $\mathbf{x}$ in L $(\mathbf{x}, \mathbf{x})$ is an even integer, and we say that $\mathbf{x}$ is a 2m-vector if $(\mathbf{x}, \mathbf{x}) = 2m$ holds for some natural number m. Let $\Lambda_{2m}(L)$ be the set defined by

$$\Lambda_{2m}(L) = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2m\}. \tag{4.1}$$

4.2. Construction of even unimodular lattice of type nA_2 ($n \equiv 0 \pmod 4$)

We briefly give a construction of even unimodular lattice $\mathcal{L}(n, A_2)$ of type nA_2 ($n \equiv 0 \pmod 4$) using ternary extremal code of length n. Therefore the rank of $\mathcal{L}(n, A_2)$ is $2n$.

Let A_2 be the root lattice of type A_2. Since we use the structure of A_2 heavily, we describe a precise shape of A_2. Let $\mathbf{e}_j$ ($1 \le j \le 3$) be orthonormal vectors in the 3-dimensional Euclidean space $\mathbb{R}^3$. A_2 is spanned by the vectors $\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{e}_2 - \mathbf{e}_3$. The rank of A_2 is 2. A_2 contains six 2-vectors $\pm(\mathbf{e}_i - \mathbf{e}_j)$ ($1 \le i < j \le 3$). The dual lattice $A_2^{\#}$ is spanned by the vectors

$$\varphi_1 = \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3) \text{ and } \varphi_2 = \frac{1}{3}(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3).$$

The quotient $A_2^{\#}/A_2$ is isomorphic to $\mathbf{F}_3$. We easily verify that $(\varphi_1, \varphi_1) = (\varphi_2, \varphi_2) = 2/3$. It can be shown that any non-zero vector ξ in $A_2^{\#}$ satisfies the inequality $(\xi, \xi) \ge 2/3$ and we say a vector ξ in $A_2^{\#}$ minimal if $(\xi, \xi) = 2/3$. There are 6 minimal vectors in $A_2^{\#}$:

$$\varphi_1 = \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3), \varphi_2 = \frac{1}{3}(\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3), \tag{4.2}$$

$$\varphi_3 = \frac{1}{3}(-2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \varphi_4 = -\varphi_1, \varphi_5 = -\varphi_2, \varphi_6 = -\varphi_3.$$

The first three are equivalent to φ_1 modulo A_2, and the last three are equivalent to φ_4 modulo A_2. Let

$$M_n = \bigoplus_n A_2 \tag{4.3}$$

be an orthogonal sum of n copies of A_2. Then $M_n^\#/M_n$ is isomorphic to $(A_2^\#/A_2)^n \cong \mathbf{F}_3^n$. A vector $\mathbf{x} \in M_n^\#$ is called a minimal representative modulo M_n if it satisfies

$$(\mathbf{x}, \mathbf{x}) \le (\mathbf{y}, \mathbf{y})$$

for all $\mathbf{y} \in M_n^\#$ such that $\mathbf{y} \equiv \mathbf{x} \pmod{M_n}$.

We now give an explicit isomorphism ρ from $M_n^\#/M_n$ to $\mathbf{F}_3^n$. We label $A_2^{(i)}$ to the i-th component in (4.3) of M_n. The minimal vectors in $(A_2^{(i)})^\#/A_2^{(i)}$ are written as $\varphi_t^{(i)}$ $(1 \le t \le 6)$. Then the mapping ρ is defined by

$$\begin{array}{rcl} \rho : M_n^\#/M_n & \longrightarrow & \mathbf{F}_3^n \\ \Cup & & \Cup \\ \xi = (\xi_1, \cdots, \xi_n) & \longmapsto & \mathbf{u} = (u_1, \cdots, u_n), \end{array}$$

where $u_i = 1 \in \mathbf{F}_3$ if ξ_i is equivalent to one of $\varphi_t^{(i)}$ $(1 \le t \le 3)$ modulo $A_2^{(i)}$, , $u_i = 2 \in \mathbf{F}_3$ if ξ_i is equivalent to one of $\varphi_t^{(i)}$ $(4 \le t \le 6)$ modulo $A_2^{(i)}$, and $u_i = 0 \in \mathbf{F}_3$ if ξ_i belongs to $A_2^{(i)}$. We write as $\mathbf{u}$= t-supp(ξ), and we will call $\mathbf{u}$ the ternary support of $\xi \in M_n^\#/M_n$ (or even $\xi \in M_n^\#$). One may remark that 3^n minimal vectors in $M_n^\#$ correspond to one vector $\mathbf{u} \in \mathbf{F}_3^n$ by this ρ. Let

$$\varpi : M_n^\# \longrightarrow M_n^\#/M_n$$

be the canonical projection, and $\Phi = \rho \circ \varpi$ be the composition of ρ and ϖ. Let $\mathbf{C}$ be a self-dual ternary code of length n. We define a lattice $\mathcal{L}(n, A_2)$ as $\Phi^{-1}(\mathbf{C})$. We prove

Proposition 1. *Let the notations be as above, then we have :*
(i) The lattice $\mathcal{L}(n, A_2)$ is an even unimodular lattice,
(ii) if $n \ge 12$ and $\mathbf{C}$ is extremal, then

$$\Lambda_2(\mathcal{L}(n, A_2)) = \Lambda_2(M_n).$$

Proof of (i). Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a vector in $\mathcal{L}(n, A_2)$. We can take another vector $\mathbf{x}' = (x_1', \cdots, x_n')$ which is equivalent to $\mathbf{x}$ modulo M_n , and each component x_i' of $\mathbf{x}'$ is minimal or zero. $\mathbf{x}'$ can be written as

$$\mathbf{x}' = \sum_{i=1}^{n} a_i \varphi^i \ ,$$

where a_i $(1 \le i \le n)$ are either 0 or 1 and φ^i is one of six vectors $\varphi_t^{(i)}$ $(1 \le t \le 6)$. If we could observe that the number of i's such that $a_i \ne 0$ equals the weight of ternary support of $\mathbf{x}'$, then we see that

$$(\mathbf{x}', \mathbf{x}') = \frac{2}{3} wt(t\text{-supp}(\mathbf{x}')) \in 2\mathbb{Z},$$

because the weight of any codeword in $\mathbf{C}$ is divisible by 3. Thus $\mathcal{L}(n, A_2)$ is even. Next we see that

$$[M_n^\# : M_n] = [A_2^\# : A_2]^n = 3^n$$

and

$$\mathfrak{L}(n, A_2)/M_n \cong \mathbf{C}.$$

Therefore we have

$$[\mathfrak{L}(n, A_2) : M_n] = 3^{\frac{n}{2}}$$

and

$$[M_n^\# : \mathcal{L}(n, A_2)^\#] = 3^{\frac{n}{2}}.$$

From these we conclude $\mathcal{L}(n, A_2)^\# = \mathcal{L}(n, A_2)$. Thus $\mathcal{L}(n, A_2)$ is unimodular.

Proof of (ii). Let $\mathbf{x}$ be an element of $\mathcal{L}(n, A_2) - M_n$. Without loss of generality, we may take $\mathbf{x}$ to be a minimal representative of $\mathcal{L}(n, A_2)$ modulo M_n. Such an $\mathbf{x}$ is equal to $\mathbf{x}'$ in the proof of (i). Since $\mathbf{C}$ is extremal, $wt(t\text{-supp}(\mathbf{x}')) \ge 6$, so that we have $(\mathbf{x}', \mathbf{x}') \ge 4$. Thus $\Lambda_2(\mathcal{L}(n, A_2)) = \Lambda_2(M_n)$. Q.E.D.

Remark 2. If the length of $\mathbf{C}$ is 4, then $\Lambda_2(\mathcal{L}(n, A_2))$ is greater than $\Lambda_2(M_4)$. Indeed, $\Lambda_2(\mathcal{L}(4, A_2))$ forms a root system of type E_8.

4.3. Important Examples

By Proposition 1, the set of 2-vectors of $\mathcal{L}(n, A_2)$ $(n \ge 12)$ equals $\Lambda_2(M_n)$. However, when $n = 12$, 16 , 20, the set of 4-vectors $\Lambda_4(\mathcal{L}(n, A_2))$ consists of two kinds of subsets $\Lambda_{4,1}$ and $\Lambda_{4,2}$. $\Lambda_{4,1}$ consists of all 4-vectors coming from M_n. $\Lambda_{4,2}$ consists of 4-vectors obtained from the sextets in the ternary $[n, \frac{n}{2}, 6]$ code $\mathbf{C}$. A general form of $\mathbf{x}$ in $\Lambda_{4,2}$ takes

$$\mathbf{x} = \sum_{g=1}^{6} \varphi_t^{(i_g)},$$

where $i_1 \le i_2 < \cdots < i_6 \le n$ and t takes the values 1,2,3 or 4,5,6 according to the value of the coordinate of t-supp $(\mathbf{x}) \in \mathbf{C}$. For a fixed codeword $\mathbf{u}$ of weight 6 in $\mathbf{C}$, there correspond 3^6 $\mathbf{x}$'s in $\Lambda_{4,2}$ such that t-supp$(\mathbf{x}) = \mathbf{u}$. In other words, we see that

$$\bigcup_{\mathbf{u} \in \mathcal{K}_6} \Phi^{-1}(\mathbf{u}) \cap \Lambda_4 = \Lambda_{4,2}.$$

Here we summarize some features of the lattices constructed above.

(I) $12A_2$ type even unimodular lattice L

Λ_2 : the set of 2-vectors in L.

$\Lambda_{4,1}$: the set of 4-vectors in L coming from $12A_2$.

$\Lambda_{4,2}$: the set of 4-vectors in L coming from sextets.

$| \Lambda_2 |= 12 \cdot 6 = 72.$

$| \Lambda_{4,1} |= 66 \cdot 36 = 2376.$

$| \Lambda_{4,2} |= 264 \cdot 3^6 = 192456.$

$| \Lambda_4 |= 194832.$

(II) $16A_2$ type even unimodular lattice L.

Λ_2 : the set of 2-vectors in L.

$\Lambda_{4,1}$: the set of 4-vectors in L coming from $16A_2$.

$\Lambda_{4,2}$: the set of 4-vectors in L coming from sextets.

$| \Lambda_2 |= 16 \cdot 6 = 96.$

$| \Lambda_{4,1} |= 120 \cdot 36 = 4320.$

$| \Lambda_{4,2} |= 224 \cdot 3^6 = 163296.$

$| \Lambda_4 |= 167616.$

(III) $20A_2$ type even unimodular lattice L.

Λ_2 : the set of 2-vectors in L.

$\Lambda_{4,1}$: the set of 4-vectors in L coming from $20A_2$.

$\Lambda_{4,2}$: the set of 4-vectors in L coming from sextets.

$| \Lambda_2 |= 20 \cdot 6 = 120.$

$| \Lambda_{4,1} |= 190 \cdot 36 = 6840.$

$| \Lambda_{4,2} |= 120 \cdot 3^6 = 87480.$

$| \Lambda_4 |= 94320.$

(IV) $24A_2$ type even unimodular lattice L.

Λ_2 : the set of 2-vectors in L.

$\Lambda_{4,1}$: the set of 4-vectors in L coming from $20A_2$.

$\Lambda_{6,1}$: the set of 6-vectors in L coming from $20A_2$.

$\Lambda_{6,2}$: the set of 6-vectors in L coming from nintets.

$| \Lambda_2 |= 24 \cdot 6 = 120.$

$| \Lambda_4 |= 9936.$

$| \Lambda_{6,1} |= 190 \cdot 36 = 437328.$

$| \Lambda_{6,2} |= 4048 \cdot 3^9 = 79676784.$

$| \Lambda_6 |= 80114112.$

5. Preliminary 4. Some knowledge from modular form theory

5.1. Modular forms of degree 1

Let $\mathfrak{H}_1 = \{z = x + yi \in \mathbb{C} \mid y > 0 \}$ be the complex upper-half plane. A complex valued function $f(z)$ on $\mathfrak{H}_1$ is called a modular form of (even) weight $2h$ belonging to the full modular group $SL_2(\mathbb{Z})$ if it satisfies

$$\text{(i) } f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2h} f(z), \; with \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

(ii) $f(z)$ behaves well at the cusp $i\infty$.

The condition (ii) means that when f(z) is expanded as a Fourier series (this expansion is guarranteed by the condition (i)):

$$f(z) = \sum_{n=-\infty}^{n=+\infty} a_n(f) \exp(2\pi inz),$$

then the terms $\sum_{n<0} a_n \exp(2\pi inz)$ vanish.

Let M(1,2h) be the vector space of all modular forms of weight 2h belonging to the full modular group. A modular form $f(z) \in M(1, 2h)$ is said to be a cusp form if the Fourier coefficient $a_0(f)$ of $f(z)$ is zero. Let S(1,2h) be the vector subspace of M(1,2h), consisting of all cusp forms in M(1,2h). For the basis and the dimension of each space $M(1, 2h)$ or $S(1, 2h)$, one may refer to the book of Schoeneberg [27] (especially Theorem 18 in page 47). Here we only give some informations on the three fundamental modular forms, namely, $E_4(z)$, Eisenstein series of weight 4, $E_6(z)$, Eisenstein series of weight 6, and $\Delta_{12}(z)$, the cusp form of weight 12. The Fourier expansions of them are quite well-known, and we briefly give them:

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) \exp(2\pi inz),$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) \exp(2\pi inz),$$

$$\Delta_{12}(z) = \sum_{n=1}^{\infty} \tau(n) \exp(2\pi inz),$$

where $\sigma_r(n) = \sum_{d|n} d^r$ and $\tau(n)$ is the Ramanujan's tau function. Some of the values are given the Table 1 below.

Table1.

n	$240 \times \sigma_3(n)$	$504 \times \sigma_5(n)$	$\tau(n)$
1	240	504	1
2	2160	16632	-24
3	6720	122976	252
4	17520	532728	-1472

The first few Fourier coefficients of from $\Delta_{12}(z)E_4(z)$ up to $\Delta_{12}(z)E_6(z)(E_4(z))^2$ are easily calculated from the above Table 1.

We collect two important facts about theta-series associated with the lattice. Before doing so, we need some preliminaries.

Let L be an even unimodular lattice of rank 8k, then the theta series for L is defined by

(5.1) $$\vartheta(z, L) = \sum_{\mathbf{x}\in L} \exp(\pi i(\mathbf{x}, \mathbf{x})z).$$

This series is rewritten as

(5.2) $$\vartheta(z, L) = \sum_{m=1}^{\infty} a(2m, L)\exp(2\pi imz),$$

where $a(2m, L) = |\Lambda_{2m}(L)|$.

To define theta series with spherical function, some informations on the Gegenbauer polynomials are necessary. The Gegenbauer polynomial $H_\nu(u)$ of degree ν is a solution of the differential equation of the second order:

$$(1-u^2)\frac{d^2H_\nu}{du^2} - u(8k-1)\frac{dH_\nu}{du} + \nu(8k+\nu-2)H_\nu = 0.$$

The explicit form of the polynomial $H_\nu(u), (\nu \equiv 0 \pmod 2)$ is given by

$$H_\nu(u) = \sum_{r=0}^{\nu/2} \frac{(-1)^r \begin{pmatrix} \nu \\ 2r \end{pmatrix} (2r-1)!!}{\prod_{i=1}^{r}(8k+2\nu-4-2(r-i))} u^{\nu-2r},$$

Here we understand that $\displaystyle\prod_{i=1}^{r}(8k+2\nu-4-2(r-i)) = 1$ and $(2r-1)!! = 1$ when $r = 0$, where $(2r-1)!!$ is the product of odd integers from 1 up to $2r-1$. We write first few polynomials $H_\nu(u)$, although we will later use up to $H_{14}(u)$ (in the appendix B we give $H_{10}(u), H_{12}(u)$, and $H_{14}(u)$ respectively):

(5.3) $$H_2(u) = u^2 - \frac{1}{8k}$$

(5.4) $$H_4(u) = u^4 - \frac{6}{8k+4}u^2 + \frac{3}{(8k+4)(8k+2)}$$

(5.5) $$H_6(u) = u^6 - \frac{15}{8k+8}u^4 + \frac{45}{(8k+8)(8k+6)}u^2 - \frac{15}{(8k+8)(8k+6)(8k+4)}$$

(5.6) $$H_8(u) = u^8 - \frac{28}{8k+12}u^6 + \frac{210}{(8k+12)(8k+10)}u^4 - \frac{420}{(8k+12)(8k+10)(8k+8)}u^2 + \frac{105}{(8k+12)(8k+10)(8k+8)(8k+6)}$$

Using $H_\nu(u)$, the spherical function $P_\nu(\mathbf{x}, \alpha)$ of degree ν is defined by

$$P_\nu(\mathbf{x};\alpha) = H_\nu\left(\frac{(\mathbf{x},\alpha)}{\sqrt{(\mathbf{x},\mathbf{x})(\alpha,\alpha)}}\right)((\mathbf{x},\mathbf{x})(\alpha,\alpha))^{\frac{\nu}{2}}$$

Theta series with the spherical function is defined by

(5.7) $$\vartheta(z, P_\nu, L) = \sum_{\mathbf{x}\in L} P_\nu(\mathbf{x};\alpha)\exp(\pi i(\mathbf{x},\mathbf{x})z).$$

where α is any vector in $\mathbb{R}^{8k}$ with $8k = rank(L)$. This series is rewritten as

(5.8) $$\vartheta(z, P_\nu, L) = \sum_{m=1}^{\infty}\sum_{\mathbf{x}\in\Lambda_{2m}(L)} P_\nu(\mathbf{x};\alpha)\exp(2\pi imz)$$

With the above preliminaries, we state the facts:

Fact 1. When L is an even unimodular lattice of rank 8k, then

(5.9) $$\vartheta(z, L) \in M(1, 4k).$$

Fact 2. When L is an even unimodular lattice of rank 8k, then

(5.10) $$\vartheta(z, P_\nu, L) \in S(1, 4k+\nu) \text{ for } \nu > 0.$$

Combining these facts with the values of the Fourier coefficients of the series of the basis in the space and the equation (5.8), we get some relations among the quantities $\sum_{\mathbf{x}\in\Lambda_{2m}} P_\nu(\mathbf{x};\alpha)$ $(m = 1, 2)$. After that we get another type of relations among the quantities $\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^h$ and $\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{x},\alpha)^h$.

(I) Let $\mathcal{L}(12, A_2)$ be the even unimodular lattice of rank 24 of type $12A_2$ constructed in the previous section, then we have :

(5.11) $$\sum_{\mathbf{x}\in\Lambda_2} P_2(\mathbf{x};\alpha) = 0,$$

(5.12) $$\sum_{\mathbf{y}\in\Lambda_4} P_2(\mathbf{y};\alpha) = 0,$$

(5.13) $$\sum_{\mathbf{y}\in\Lambda_4} P_4(y;\alpha) = 216\sum_{\mathbf{x}\in\Lambda_2} P_4(\mathbf{x};\alpha),$$

(5.14) $$\sum_{\mathbf{y}\in\Lambda_4} P_6(\mathbf{y};\alpha) = -528\sum_{\mathbf{x}\in\Lambda_2} P_6(\mathbf{x};\alpha),$$

(5.15) $$\sum_{\mathbf{y}\in\Lambda_4} P_8(\mathbf{y};\alpha) = 456\sum_{\mathbf{x}\in\Lambda_2} P_8(\mathbf{x};\alpha),$$

$$(5.16)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_{10}(\mathbf{y};\alpha) = -288\sum_{\mathbf{x}\in\Lambda_2} P_{10}(\mathbf{x};\alpha) \text{ and}$$

$$(5.17)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_{14}(\mathbf{y};\alpha) = -48\sum_{\mathbf{x}\in\Lambda_2} P_{14}(\mathbf{x};\alpha).$$

From the equation (5.11) with the explicit form of $P_\nu(\mathbf{x};\alpha)$, we get

$$(5.18)\qquad \sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^2 = 6(\alpha,\alpha).$$

Similarly from (5.12) we get

$$(5.19)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^2 = 32472(\alpha,\alpha).$$

The next formula is obtained by using not only the equation (5.13) also the formulas (5.18) and (5.19):

$$(5.20)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^4 = 14688(\alpha,\alpha)^2 + 216\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^4$$

Likewise from (5.14) and the formulas obtained so far we get

$$(5.21)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^6$$

$$= 9720(\alpha,\alpha)^3 + 900\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^4(\alpha,\alpha) -528\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^6,$$

From (5.15) and others,

$$(5.22)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^8$$

$$= 7560(\alpha,\alpha)^4 + 2520(\alpha,\alpha)^2\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^4 -2352(\alpha,\alpha)\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^6 + 456\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^8,$$

From (5.16),

$$(5.23)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^{10}$$

$$= 5670(\alpha,\alpha)^5 + 6300\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^4(\alpha,\alpha)^3 -7560\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^6(\alpha,\alpha)^2 +2700\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^8(\alpha,\alpha) - 288\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^{10}.$$

From (5.17),

$$(5.24)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^{14}$$

$$= \frac{91}{12}(\alpha,\alpha)\sum_{y\in\Lambda_4} (y,\alpha)^{12} + \frac{39422565}{874}(\alpha,\alpha)^6 -\frac{315315}{4}(\alpha,\alpha)^5\sum_{x\in\Lambda_2} (x,\alpha)^4 + 105105(\alpha,\alpha)^4\sum_{x\in\Lambda_2} (x,\alpha)^6 -45045(\alpha,\alpha)^3\sum_{x\in\Lambda_2} (x,\alpha)^8 + 6006(\alpha,\alpha)^2\sum_{x\in\Lambda_2} (x,\alpha)^{10} +182(\alpha,\alpha)\sum_{x\in\Lambda_2} (x,\alpha)^{12} - 48\sum_{x\in\Lambda_2} (x,\alpha)^{14}.$$

(II) Let $\mathcal{L}(16, A_2)$ be an even unimodular lattice of rank 32 of type $16A_2$, then we have :

$$(5.25)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_2(\mathbf{y};\alpha) = -528\sum_{\mathbf{x}\in\Lambda_2} P_2(\mathbf{x};\alpha),$$

$$(5.26)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_4(\mathbf{y};\alpha) = 456\sum_{\mathbf{x}\in\Lambda_2} P_4(\mathbf{x};\alpha),$$

$$(5.27)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_6(\mathbf{y};\alpha) = -288\sum_{\mathbf{x}\in\Lambda_2} P_6(\mathbf{x};\alpha),$$

$$(5.28)\qquad \sum_{\mathbf{y}\in\Lambda_4} P_{10}(\mathbf{y};\alpha) = -48\sum_{\mathbf{x}\in\Lambda_2} P_{10}(\mathbf{x};\alpha).$$

In a similar manner as (I), from (5.25) with other knowledge we get

$$(5.29)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^2 = -528\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^2 + 24120(\alpha,\alpha).$$

From (5.26) we get

$$(5.30)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^4$$

$$= 9936(\alpha,\alpha)^2 - 504(\alpha,\alpha)\cdot\sum_{\mathbf{x}\in\Lambda_2} (x,\alpha)^2 +456\sum_{x\in\Lambda_2} (x,\alpha)^4.$$

From (5.27) ,

$$(5.31)\qquad \sum_{\mathbf{y}\in\Lambda_4} (\mathbf{y},\alpha)^6 = 6480(\alpha,\alpha)^3 - 540(\alpha,\alpha)^2\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^2 + 900(\alpha,\alpha)\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^4 - 288\sum_{\mathbf{x}\in\Lambda_2} (\mathbf{x},\alpha)^6.$$

From (5.28),

(5.32)
$$\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{y},\alpha)^{10}$$
$$= \frac{15}{4}(\alpha,\alpha)\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{y},\alpha)^8 - 15120(\alpha,\alpha)^5$$
$$+1575(\alpha,\alpha)^4\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^2 - 3150(\alpha,\alpha)^3\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^4$$
$$+1260(\alpha,\alpha)^2\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^6 + 90(\alpha,\alpha)\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^8$$
$$-48\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^{10}.$$

(III) Let $\mathcal{L}(20, A_2)$ be an even unimodular lattice of rank 40 of type $20A_2$, then we have :

$$\sum_{\mathbf{y}\in\Lambda_4} P_2(\mathbf{y};\alpha) = -288\sum_{\mathbf{x}\in\Lambda_2} P_2(\mathbf{x};\alpha), \tag{5.33}$$

$$\sum_{\mathbf{y}\in\Lambda_4} P_6(\mathbf{y};\alpha) = -48\sum_{\mathbf{x}\in\Lambda_2} P_6(\mathbf{x};\alpha). \tag{5.34}$$

From (5.33) we get

$$\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{y},\alpha)^2 = 11160(\alpha,\alpha) - 288\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^2. \tag{5.35}$$

From (5.34),

(5.36)
$$\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{y},\alpha)^6$$
$$= \frac{5}{4}(\alpha,\alpha)\sum_{\mathbf{y}\in\Lambda_4}(\mathbf{y},\alpha)^4 - 2700(\alpha,\alpha)^3 +$$
$$90(\alpha,\alpha)^2\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^2$$
$$+30(\alpha,\alpha)\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^4 - 48\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^6.$$

6. Main Results

6.1. Auxiliary Results

We put

$$\alpha = \varphi_1^{(i_1)} + \varphi_1^{(i_2)} + \cdots + \varphi_1^{(i_s)}\ , \tag{6.1}$$

where $\varphi_1^{(i_1)}$ is the same as in the section 4, and $\{i_1,\ i_2, \cdots, i_s\}$ is any non-empty subset of $\{1, 2, \cdots, n\}$.

Let $\Lambda_2 = \Lambda_2(\mathcal{L})$ be the set of 2-vectors in a lattice $\mathcal{L}$. We determine the value of $\sum_{\mathbf{x}\in\Lambda_2}(\mathbf{x},\alpha)^{2\nu}$ for α of the form (6.1) and for each one of three even unimodular lattices $\mathcal{L}(n, A_2)$ $(n = 12, 16, 20)$.

Lemma 2. *Let L be an even unimodular lattice $\mathcal{L}(n, A_2)$ and Λ_2 as above. Then it holds that*

$$\sum_{\mathbf{x}\in\Lambda_2}(\alpha,\mathbf{x})^{2\nu} = 4s \quad \nu \ge 1, \tag{6.2}$$

where α is a vector of the form (6.1).

Proof. 2-vectors $\mathbf{x}$ in the i_1-th component $A_2^{(i_1)}$ of $M_n \subset \mathcal{L}$ are 6 in number. Four of them satisfy $(\mathbf{x},\alpha) = \pm1$, and the other two of them satisfy $(\mathbf{x},\alpha) = 0$. From which the lemma follows.

Lemma 3. *Let the notations be as above, and $\Lambda_{4,1} = \Lambda_{4,1}(\mathcal{L})$ be a subset of 4-vectors introduced in the section 4. Then it holds that*

$$\sum_{\mathbf{y}\in\Lambda_{4,1}}(\alpha,\mathbf{y})^{2\nu} = 8s(s-1)+4s(s-1)\cdot2^{2\nu}+24s(n-s) \tag{6.3}$$

Proof. Each 4-vector $\mathbf{y}$ in $\Lambda_{4,1}$ has the form

$$\mathbf{y} = \mathbf{x}_{j_1} + \mathbf{x}_{j_2}, \quad (1 \le j_1 < j_2 \le n),$$

where $\mathbf{x}_{j_1}$ and $\mathbf{x}_{j_2}$ are 2-vectors belonging to different components A_{j_1} and A_{j_2} respectively of M_n (cf. (4.31)). By elementary combinatorial countings, we can verify that there are $\binom{s}{2}\cdot 2^2$ combinations of $\mathbf{x}_{j_1}$ and $\mathbf{x}_{j_2}$ such that $(\mathbf{y},\alpha) = 2$, there are $\binom{s}{2}\cdot2\cdot2\cdot2^2$ combinations of $\mathbf{x}_{j_1}$ and $\mathbf{x}_{j_2}$ such that $(\mathbf{y},\alpha) = \pm1$ with $\{j_1, j_2\} \subset \{i_1, i_2, \cdots, i_s\}$, there are $24s(n-s)$ combinations of $\mathbf{x}_{j_1}$ and $\mathbf{x}_{j_2}$ such that $|\ \{j_1, j_2\} \cap \{i_1, i_2, \cdots, i_s\}\ | = 1$ and $(\mathbf{y},\alpha) = \pm1$. Other types of 4-vectors do not contribute to the left-hand side of (6.3). From these the lemma follows. Q.E.D.

Lemma 4. *Let the notations be the same as above. Then the sums*

$$\sum_{\mathbf{x}\in\Lambda_2}(\alpha,\mathbf{x})^{2\nu}$$

and

$$\sum_{\mathbf{y}\in\Lambda_{4,1}}(\alpha,\mathbf{y})^{2\nu}$$

are invariant if α is replaced by

$$\alpha' = \tilde{\varphi}^{(i_1)} + \tilde{\varphi}^{(i_2)} + \cdots + \tilde{\varphi}^{(i_s)},$$

where $\tilde{\varphi}^{(i_t)}$ is one of six vectors, which are isomorphic copies of (4.2) in the t-th component $A_2^{t\#}/A_2^t$.

Proof. Let $\xi = \xi_{i_t}$ be 2-vectors in the i_t-th component A_{i_t}. The reflections σ_ξ with respect to 2-vector ξ and $(-1)id$, where id is the identity mapping on the lattice, act transitively on $A_{i_1}^{\#}$. The automorphisms of $\mathcal{L}(n, A_2)$ generated by these mappings can transform any α into α' preserving the sums in the lemma. Q.E.D.

As a consequence of Lemma 4, we can prove another lemma :

Lemma 5. *Let the notations be as above. Then we have (i) When $L = \mathcal{L}(12, A_2)$ and $\Lambda_{4,2} = \Lambda_{4,2}(L)$, then the sum $\sum_{\mathbf{y}\in\Lambda_{4,2}} (\alpha, \mathbf{y})^{2\nu} \quad (1 \le \nu \le 5)$ is invariant if α is replaced by*

$$\alpha' = \tilde{\varphi}^{(i_1)} + \tilde{\varphi}^{(i_2)} + \cdots + \tilde{\varphi}^{(i_s)},$$

where α' has the same meaning as the preceding lemma, (ii) The similar statement to (i) holds for $L = \mathcal{L}(16, A_2)$ with $1 \le \nu \le 3$ and for $L = \mathcal{L}(20, A_2)$ with $\nu = 1$.

Proof of (i). This is easily shown by Lemma 4 and the formulas (5.19),(5.20),(5.21),(5.22) and (5.23), because for each ν the sum in question is expressed by other terms, which are invariant under the change of α to α'.
(ii) is also shown in the same way. Q.E.D.

We need to transform the sum $\sum_{\mathbf{y}\in\Lambda_{4,2}} (\alpha, \mathbf{y})^{2\nu} \quad (1 \le \nu \le 5)$.

Proposition 2. *Let α be a vector of the form (6.1) and* **a**= *t-supp(α). Suppose, for a fixed sextet* **u** *in $\mathcal{K}_6$, $(\mathbf{a}\natural\mathbf{u}) = (k_1, k_2)$, where $k_1 = (\mathbf{a}\natural\mathbf{u})_1$ and , $k_2 = (\mathbf{u}\natural\mathbf{a})_2$, then*

$$\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4} (\mathbf{y}, \alpha)^{2\nu}$$
$$= 3^{6-k_1-k_2}\left\{\sum_{i=0}^{k_1}\sum_{j=0}^{k_2}\left(\frac{2}{3}k_1 - \frac{2}{3}k_2 - i + j\right)^{2\nu} \cdot \binom{k_1}{i} 2^i \binom{k_2}{j} 2^j\right\}$$

Proof. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n) \in \mathcal{K}_6$. A general element $\mathbf{y}$ of $\Phi^{-1}(\mathbf{u}) \cap \Lambda_4$ has the form (4.4)

$$\mathbf{y} = \sum_{g=1}^{6} \varphi_t^{(i_g)},$$

where $1 \le i_1 < i_2 < \cdots < i_6 \le n$ and i_g's are non-zero coordinate positions of $\mathbf{u}$. A basic fact is that the vectors in (4.2) satisfy

$$\begin{aligned}(\varphi_i, \varphi_i) &= \frac{2}{3} \quad 1 \le i \le 6,\\ (\varphi_1, \varphi_j) &= -1 \quad (j = 2, 3).\end{aligned}$$

By this fact, we can count combinatorially that the number of $\mathbf{y} \in \Phi^{-1}(\mathbf{u}) \cap \Lambda_4$ such that

$$(\mathbf{y}, \alpha) = \frac{2}{3}k_1 - i + \frac{1}{3}k_2 - j \quad (0 \le i \le k_1,\ 0 \le j \le k_2)$$

is $3^{6-k_1-k_2}\binom{k_1}{i} 2^i \binom{k_2}{j} 2^j$. Thus

$$\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4} (\mathbf{y}, \alpha)^{2\nu}$$
$$= 3^{6-k_1-k_2}\left\{\sum_{i=0}^{k_1}\sum_{j=0}^{k_2}\left(\frac{2}{3}k_1 - \frac{2}{3}k_2 - i + j\right)^{2\nu} \cdot \binom{k_1}{i} 2^i \binom{k_2}{j} 2^j\right\}$$

Q.E.D.

We treat the double sum in the above proposition in a unified way as follows.

Put

$$f_h(k) = \sum_{i=0}^{k}\left(\frac{2}{3}k - i\right)^h \binom{k}{i} 2^i,$$

where k is temporarily an integer valued variable. By using binomial expansion of $(1 + x)^h$ and its derivatives we can show the following proposition.

Proposition 3. *For the above defined quantities $f_h(k)$, we have*

$$\begin{aligned}
f_0(k) &= 3^k\\
f_1(k) &= 0\\
f_2(k) &= 2k \cdot 3^{k-2}\\
f_3(k) &= 2k \cdot 3^{k-3}\\
f_4(k) &= 3^{k-4}(12k^2 - 6k)\\
f_5(k) &= 3^{k-5}(40k^2 - 30k)\\
f_6(k) &= 3^{k-6}(120k^3 - 140k^2 + 42k)\\
f_7(k) &= 3^{k-7}(840k^3 - 1680k^2 + 882k)\\
f_8(k) &= 3^{k-8}(1680k^4 - 2800k^3 + 252k^2 + 954k)\\
f_9(k) &= 3^{k-9}(20160k^4 - 73360k^3 + 93240k^2 - 39870k)\\
f_{10}(k) &= 3^{k-10}(30240k^5 - 50400k^4 - 133560k^3\\
&\quad +358020k^2 - 203958k)\\
f_{11}(k) &= 3^{k-11}(554400k^5 - 3080000k^4 + 6625080k^3\\
&\quad -6399360k^2 + 2300562k)\\
f_{12}(k) &= 3^{k-12}(665280k^6 - 554400k^5 - 14223440k^4\\
&\quad +51876000k^3 - 67439988k^2 + 29677914k)\\
f_{13}(k) &= 3^{k-13}(17297280k^6 - 132132000k^5\\
&\quad +405525120k^4 - 609729120k^3 + 439999560k^2\\
&\quad -120958110k)\\
f_{14}(k) &= 3^{k-14}(17297280k^7 + 20180160k^6 - 1105424320k^5\\
&\quad +5501496000k^4 - 11926834920k^3 + 12150995220k^2\\
&\quad -4657703958k)
\end{aligned}$$

Next we set

$$\begin{aligned}
&g_r(k_1, k_2)\\
&= \sum_{i=0}^{k_1}\sum_{j=0}^{k_2}\left(\frac{2}{3}k_1 - \frac{2}{3}k_2 - i + j\right)^r \binom{k_1}{i} 2^i \binom{k_2}{j} 2^j\\
&= \sum_{t=0}^{r}\binom{r}{t}(-1)^t f_{r-t}(k_1) f_t(k_2),
\end{aligned}$$

where k_1 and k_2 are temporarily integer valued variables (but later we may take k_1 (resp. k_2) to be $(\mathbf{u}\natural\mathbf{a})_1$ (resp. $(\mathbf{u}\natural\mathbf{a})_2$) . Computation shows the following proposition :

Proposition 4. *Let the notations be as above. Putting $k = k_1 + k_2$, we get*

$$\begin{aligned}
g_2(k_1,k_2) &= f_2(k),\\
g_4(k_1,k_2) &= f_4(k),\\
g_6(k_1,k_2) &= f_6(k) - 160\cdot 3^{k-6}k_1k_2\\
g_8(k_1,k_2) &= f_8(k) - 3^{k-8}\,[8960\cdot k_1k_2k - 13440k_1k_2]\\
g_{10}(k_1,k_2) &= f_{10}(k) - 3^{k-10}k_1k_2\,[403200k^2\\
&\quad -1411200k + 1300320]\\
g_{12}(k_1,k_2) &= f_{12}(k) - 3^{k-12}k_1k_2\,[17740800k^3\\
&\quad -104473600k^2 - 153996480\\
&\quad -3942400k_1k_2 + 217768320k]\\
g_{14}(k_1,k_2) &= f_{14}(k) - 3^{k-14}k_1k_2\,[807206400k^4\\
&\quad -6906099200k^3 - 717516800k_1k_2k\\
&\quad +23543520000k^2 - 37073836800k\\
&\quad +2152550400k_1k_2 + 21617275680]
\end{aligned}$$

6.2. Main Results

Since the ternary Golay code case is most typical and exibits all difficulties to prove the results, we concentrate ourselves to this case. For other cases such as the ternary [16,8,6] and [20,10,6] codes, we only give the results. Those proofs are very similar to or rather easier than the ternary Golay code case.

(I) Ternary Golay [12,6,6] code.

Theorem 1. *Let $\mathcal{K}_6$ be the set of sextets in ternary Golay [12,6,6] code, α be any vector of the form (6.1). We put* $\mathbf{a} = t\text{-}supp(\alpha)$, $wt(\mathbf{a}) = s$ *, then we have*

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a}) = 132s, \tag{6.4}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^2 = 60s^2 + 72s, \tag{6.5}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^3 = 24s^3 + 108s^2, \tag{6.6}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^4 = 8s^4 + 96s^3 + 76s^2 - 48s, \tag{6.7}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^5 = 2s^5 + 60s^4 + 190s^3 - 120s^2, \tag{6.8}$$

$$\begin{aligned}
&\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^7 + \left(\frac{7}{6} - \frac{7}{4}s\right)\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u} * \mathbf{a})^6\\
&-\frac{140}{3}\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u} * \mathbf{a})^4\\
&\left(\frac{10780}{27} + \frac{140}{3}s\right)\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u} * \mathbf{a})^3\\
&+\frac{1120}{27}\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u} * \mathbf{a})\\
&-\left(\frac{1120}{9} + \frac{280}{27}s\right)\sum_{\mathbf{u}\in\mathcal{K}_6} ((\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2)^2\\
&= -\frac{1}{2}s^7 - \frac{56}{3}s^6 + \frac{17395}{54}s^5 + \frac{44170}{9}s^4\\
&+\frac{223069}{27}s^3 - \frac{154574}{9}s^2 + \frac{11272}{3}s,
\end{aligned} \tag{6.9}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2 = 15s^2 - 15s, \tag{6.10}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u} * \mathbf{a}) = 6s^3 + 12s^2 - 18s, \tag{6.11}$$

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u} * \mathbf{a})^2 = 2s^4 + 18s^3 - 8s^2 - 12s. \tag{6.12}$$

Throughout the proofs, we set : a ternary vector $\mathbf{a} = (a_1, a_2, \cdots, a_{12}) \in \mathbf{F}_3^{12}$ of weight s, whose non-zero coordinate positions are $i_1, i_2, \cdots, i_s$. For the sake of convenience, we sometimes use the notations $k = (\mathbf{u} * \mathbf{a}) = k_1 + k_2, k_1 = (\mathbf{u}\natural\mathbf{a})_1, k_2 = (\mathbf{u}\natural\mathbf{a})_2$ interchangeably.

Proof of (6.4). Take a ternary vector $\mathbf{a}= (a_1, a_2, \cdots, a_{12})$ of weight s with $a_{i_1} = a_{i_2} = \cdots = a_{i_s} = 1$. Associated with this $\mathbf{a}$, we set

$$\alpha = \varphi_1^{(i_1)} + \varphi_1^{(i_2)} + \cdots + \varphi_1^{(i_s)}.$$

Substituting this α into the formula (5.19), we get

$$\sum_{\mathbf{y}\in\Lambda_{4,1}} (\mathbf{y}, \alpha)^2 + \sum_{\mathbf{y}\in\Lambda_{4,2}} (\mathbf{y}, \alpha)^2 = 21648s.$$

By Lemma 3,

$$\sum_{\mathbf{y}\in\Lambda_{4,1}} (\mathbf{y}, \alpha)^2 = 264s.$$

Thus we get

$$\sum_{\mathbf{y}\in\Lambda_{4,2}} (\mathbf{y}, \alpha)^2 = 21384s.$$

We know that

$$\sum_{\mathbf{y}\in\Lambda_{4,2}} (\mathbf{y}, \alpha)^2 = \sum_{\mathbf{u}\in\mathcal{K}_6} \sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4} (\mathbf{y}, \alpha)^2.$$

By Propositions 2 and 4, we obtain

$$\begin{aligned}
\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4} (\mathbf{y}, \alpha)^2 &= 2k\cdot 3^{k-2}3^{6-k}\\
&= 2\cdot 3^4 k.
\end{aligned}$$

Therefore we get

$$\sum_{\mathbf{u}\in\mathcal{K}_6}\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4}(\mathbf{y},\alpha)^2 = \sum_{\mathbf{u}\in\mathcal{K}_6} 2\cdot 3^4(\mathbf{u}*\mathbf{a}) = 21384s,$$

and

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a}) = 132s.$$

By Remark 1, this implies that $\mathcal{K}_6$ forms a 1-design with parameters $(v,k,\lambda_1) = (12,6,66)$.

Proof of (6.5). Take the same α as in the proof of (6.4). This time we start from the formula (5.20). Using Lemma 2, the right-hand side of (5.20) becomes

$$6528s^2 + 864s.$$

The left-hand side is decomposed into

$$48s^2 + 216s + \sum_{\mathbf{y}\in\Lambda_{4,2}}(\mathbf{y},\alpha)^4,$$

so that we have

$$\sum_{\mathbf{y}\in\Lambda_{4,2}}(\mathbf{y},\alpha)^4 = \sum_{\mathbf{u}\in\mathcal{K}_6}\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4}(\mathbf{y},\alpha)^4 = 6480s^2 + 648s.$$

On the other hand, by Proposition 2

$$\sum_{\mathbf{u}\in\mathcal{K}_6}\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4}(\mathbf{y},\alpha)^4 = 3^{6-k}3^{k-4}(12k^2-6k).$$

Therefore we have

$$3^2 6\sum_{\mathbf{u}\in\mathcal{K}_6}[2(\mathbf{u}*\mathbf{a})^2 - (\mathbf{u}*\mathbf{a})] = 6480s^2 + 648s.$$

By using (6.4) proved above, we have

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^2 = 60s^2 + 72s.$$

The conditions (6.4) and (6.5) so far proved suffice to show that $\overline{\mathcal{K}_6}$ forms a 2-design with parameters $(v,k,\lambda_2) = (12,6,30)$ (cf. also the later arguement to derive (6.14)).

Proof of (6.6). So far we have easily proceeded. But a difficulty awaits us afterward. Starting from the formula (5.21), and treating the both sides of it in a similar way as in the preceding proofs, we get

$$\begin{aligned}&3^{k-6}3^{6-k}\sum_{\mathbf{u}\in\mathcal{K}_6}[120(\mathbf{u}*\mathbf{a})^3 - 140(\mathbf{u}*\mathbf{a})^2 + 42(\mathbf{u}*\mathbf{a})\\ &-160(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2]\\ =\ &2880s^3 + 2160s^2 - 2136s.\end{aligned}$$

With the help of (6.4) and (6.5), we obtain

$$\sum_{\mathbf{u}\in\mathcal{K}_6}[120(\mathbf{u}*\mathbf{a})^3 - 160(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2] = 2880s^3 + 10560s^2 + 2400s,$$

or

(6.13)
$$\sum_{\mathbf{u}\in\mathcal{K}_6}[3(\mathbf{u}*\mathbf{a})^3 - 4(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2] = 72s^3 + 264s^2 + 60s.$$

We want to determine the value of $\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2$ in the case of $wt(\mathbf{a}) = s = 3$. For this purpose we introduce a family of symbols $\lambda(s;\mathbf{a};\tau_1,\tau_2,\cdots,\tau_s)$. Here $\tau_1,\tau_2,\cdots,\tau_s$ belong to $\mathbf{F}_3 = \{0,1,2\}$. $\lambda(s;\mathbf{a};\tau_1,\tau_2,\cdots,\tau_s)$ is the number of codewords $\mathbf{u} = (u_1,u_2,\cdots,u_{12}) \in \mathcal{K}_6$ such that $\tau_1 = u_{i_1}, \tau_2 = u_{i_2},\cdots,\tau_s = u_{i_s}$.
It is useful to prove the following elementary lemma :

Lemma 6. *Let $\lambda(s;\mathbf{a};\tau_1,\tau_2,\cdots,\tau_s)$ be the symbol introduced above, and $\tau_1',\tau_2',\cdots,\tau_s'$ the elements in $\mathbf{F}_3$ obtained from $\tau_1,\tau_2,\cdots,\tau_s$ by multiplying with $2\in\mathbf{F}_3$ respectively in the same order. Then it holds that*

$$\lambda(s;\mathbf{a};\tau_1,\tau_2,\cdots,\tau_s) = \lambda(s;\mathbf{a};\tau_1',\tau_2',\cdots,\tau_s').$$

Proof of the Lemma. Consider two subsets of $\mathcal{K}_6$ defined by

$$T(\mathbf{a};\tau_1,\tau_2,\cdots,\tau_s) = \{\mathbf{u}\in\mathcal{K}_6 \mid \tau_1 = u_{i_1},\cdots,\tau_s = u_{i_s}\}$$

and

$$T(\mathbf{a};\tau_1',\tau_2',\cdots,\tau_s') = \{\mathbf{u}\in\mathcal{K}_6 \mid \tau_1' = u_{i_1},\cdots,\tau_s' = u_{i_s}\}.$$

The mapping ,which maps each $\mathbf{u}$ in the former set to $2\mathbf{u}$ (a scaler multiplication by $2\in\mathbf{F}_3$) in the latter set, is a bijection. The cardinalities of these sets are the both sides of the equation in the lemma. Q.E.D.

When $wt(\mathbf{a}) = s = 2$ and i_1, i_2 are the non-zero coordinate positions of $\mathbf{a}$, there arise nine symbols such as $\lambda(2;\mathbf{a};1,1), \lambda(2;\mathbf{a};1,2)$ e.t.c..
First we take such $\mathbf{a}$ with $a_{i_1} = a_{i_2} = 1$:

$$\mathbf{a} = (0,\cdots,\overset{i_1}{1},0,\cdots,0,\overset{i_2}{1},0,\cdots).$$

We put

$$\begin{aligned}\lambda_2(\mathbf{a}) &= \lambda(2;\mathbf{a};1,1) + \lambda(2;\mathbf{a};2,2),\\ \lambda_{1,1}(\mathbf{a}) &= \lambda(2;\mathbf{a};1,2) + \lambda(2;\mathbf{a};2,1),\\ \lambda_2(2;\mathbf{a}) &= \lambda_2(\mathbf{a}) + \lambda_{1,1}(\mathbf{a}),\\ \lambda_1(2;\mathbf{a}) &= \lambda(2;\mathbf{a};0,1) + \lambda(2;\mathbf{a};0,2) + \lambda(2;\mathbf{a};1,0)\\ &\quad+\lambda(2;\mathbf{a};2,0),\\ \lambda_0(2;\mathbf{a}) &= \lambda(2;\mathbf{a};0,0).\end{aligned}$$

We see that the relations

$$\lambda_j(2;\mathbf{a}) = |\{\mathbf{u}\in\mathcal{K}_6 \mid \mathbf{u}*\mathbf{a} = j\}| \quad \text{for } j = 0,1,2,$$

hold. From the equations (6.4) and (6.5), we get

$$\begin{aligned}\lambda_1(2;\mathbf{a}) + 2\lambda_2(2;\mathbf{a}) &= 264,\\ \lambda_1(2;\mathbf{a}) + 4\lambda_2(2;\mathbf{a}) &= 384,\end{aligned}$$

and

$$\lambda_1(2;\mathbf{a}) = 144, \quad \lambda_2(2;\mathbf{a}) = 60. \tag{6.14}$$

For the above assigned ternary vector $\mathbf{a}$, we easily observe that

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2 = \lambda_{1,1}(\mathbf{a}). \tag{6.15}$$

Next we take another $\mathbf{a}'$ with $a'_{i_1} = 1, a'_{i_2} = 2$, that is

$$\mathbf{a}' = (0,\cdots,\overset{i_1}{1},0,\cdots,0,\overset{i_2}{2},0,\cdots).$$

Lemma 5 is reinterpreted to be a statement on a codeword property via Propositions 2 and 4. Namely the left-hand side of (6.10)

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2$$

is invariant under the change

$$\begin{aligned} \mathbf{a} &= (0,\cdots,\overset{i_1}{1},0,\cdots,0,\overset{i_2}{1},0,\cdots) \\ &\to \mathbf{a}' = (0,\cdots,\overset{i_1}{1},0,\cdots,0,\overset{i_2}{2},0,\cdots). \end{aligned}$$

Therefore the relation

$$\begin{aligned} \lambda_{1,1}(\mathbf{a}) &= \sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2 \\ &= \sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a}')_1 \cdot (\mathbf{u}\natural\mathbf{a}')_2 \\ &= \lambda_{1,1}(\mathbf{a}') \end{aligned}$$

must hold. By definition

$$\lambda_{1,1}(\mathbf{a}) = \lambda(2;\mathbf{a};1,2) + \lambda(2;\mathbf{a};2,1)$$

and

$$\lambda_{1,1}(\mathbf{a}') = \lambda(2;\mathbf{a}';1,2) + \lambda(2;\mathbf{a}';2,1).$$

By the shape of $\mathbf{a}'$, we see that

$$\lambda(2;\mathbf{a}';1,2) = \lambda(2;\mathbf{a};1,1)$$

and

$$\lambda(2;\mathbf{a}';2,1) = \lambda(2;\mathbf{a};2,2).$$

Lemma 6 tells us that

$$\lambda(2;\mathbf{a};1,1) = \lambda(2;\mathbf{a};2,2) \quad \lambda(2;\mathbf{a};1,2) = \lambda(2;\mathbf{a};2,1).$$

Combining the above relations together with (6.14), we get

$$\begin{aligned} &\lambda(2;\mathbf{a};1,1) = \lambda(2;\mathbf{a};2,2) = 15, \\ &\lambda(2;\mathbf{a};1,2) = \lambda(2;\mathbf{a};2,1) = 15. \end{aligned} \tag{6.16}$$

When $wt(\mathbf{a}) = s = 3$ and i_1, i_2, i_3 are the non-zero coordinate positions of $\mathbf{a}$, as such $\mathbf{a}$ we take initially

$$\mathbf{a} = (\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\,\overset{i_3}{1},\cdot),$$

where the dots denote the zeros. Our current purpose is to determine the value of

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2.$$

We put

$$\lambda_{2,1}(3;\mathbf{a}) = \lambda(3;\mathbf{a};1,1,2) + \lambda(3;\mathbf{a};1,2,1) + \lambda(3;\mathbf{a};2,1,1),$$

and

$$\lambda_{1,2}(3;\mathbf{a}) = \lambda(3;\mathbf{a};1,2,2) + \lambda(3;\mathbf{a};2,1,2) + \lambda(3;\mathbf{a};2,2,1),$$

$$\begin{aligned} \lambda_{1,1}(3;\mathbf{a},i_3) &= \lambda(3;\mathbf{a};1,2,0) + \lambda(3;\mathbf{a};2,1,0), \\ \lambda_{1,1}(3;\mathbf{a},i_2) &= \lambda(3;\mathbf{a};1,0,2) + \lambda(3;\mathbf{a};2,0,1), \\ \lambda_{1,1}(3;\mathbf{a},i_1) &= \lambda(3;\mathbf{a};0,1,2) + \lambda(3;\mathbf{a};0,2,1). \end{aligned}$$

Considering the meaning of the symbols introduced, we see that

$$\begin{aligned} &\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2 \\ &= 2\lambda_{2,1}(3;\mathbf{a}) + 2\lambda_{1,2}(3;\mathbf{a}) \\ &+\lambda_{1,1}(3;\mathbf{a},i_3) + \lambda_{1,1}(3;\mathbf{a},i_2) + \lambda_{1,1}(3;\mathbf{a},i_1). \end{aligned} \tag{6.17}$$

Take a ternary vector $\mathbf{a}_1$ of weight 2 of the form

$$\mathbf{a}_1 = (\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\,\overset{i_3}{0},\cdot),$$

then it holds that (both sides count the same sextets)

$$\lambda(2;\mathbf{a}_1;1,2) = \lambda(3;\mathbf{a};1,2,1)+\lambda(3;\mathbf{a};1,2,2)+\lambda(3;\mathbf{a};1,2,0), \tag{6.18}$$

Similarly by using

$$\mathbf{a}_2 = (\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{0},\cdot\cdot\,\overset{i_3}{1},\cdot),$$

we obtain

$$\lambda(2;\mathbf{a}_2;1,2) = \lambda(3;\mathbf{a};1,1,2)+\lambda(3;\mathbf{a};1,2,2)+\lambda(3;\mathbf{a};1,0,2), \tag{6.19}$$

and from

$$\mathbf{a}_3 = (\cdot\cdot,\overset{i_1}{0},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\,\overset{i_3}{1},\cdot),$$

we have

$$\lambda(2;\mathbf{a}_3;1,2) = \lambda(3;\mathbf{a};1,1,2)+\lambda(3;\mathbf{a};2,1,2)+\lambda(3;\mathbf{a};0,1,2). \tag{6.20}$$

Substituting the formulas (6.18),(6.19),(6.20) into the equation (6.17), and transforming it by means of (6.16) and Lemma 6, we have

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}\natural\mathbf{a})_1 \cdot (\mathbf{u}\natural\mathbf{a})_2 = 90. \tag{6.21}$$

Thus, for $wt(\mathbf{a}) = 3$, (see the formula (6.13))

$$\sum_{\mathbf{u}\in\mathcal{K}_6} (\mathbf{u}*\mathbf{a})^3 = 1620. \tag{6.22}$$

If we set, for $\mathbf{a}$ with $wt(\mathbf{a})=3$,

$$\lambda_j(3;\mathbf{a}) = \mid \{\mathbf{u}\in\mathcal{K}_6 \mid \mathbf{u}*\mathbf{a}=j\} \mid \quad \text{for } j=1,2,3,$$

the equations (6.4),(6.5) and (6.22) yield

$$\begin{aligned}\lambda_1(3;\mathbf{a})+2\lambda_2(3;\mathbf{a})+3\lambda_3(3;\mathbf{a}) &= 396,\\ \lambda_1(3;\mathbf{a})+4\lambda_2(3;\mathbf{a})+9\lambda_3(3;\mathbf{a}) &= 756,\\ \lambda_1(3;\mathbf{a})+8\lambda_2(3;\mathbf{a})+27\lambda_3(3;\mathbf{a}) &= 1620,\end{aligned}$$

from which we get

(6.23) $\lambda_1(3;\mathbf{a})=108,\ \lambda_2(3;\mathbf{a})=108,\ \lambda_3(3;\mathbf{a})=24.$

The last equation in (6.23) implies that the number of $\mathbf{u}\in\mathcal{K}_6$ which intersect with $\mathbf{a}$ in three positions, is 24 dependent only on $wt(\mathbf{a})=\underline{3}$, and this implies the quotient set of the set of sextets $\overline{\mathcal{K}_6}$ forms a 3-design with design parameters $(v,k,\lambda_3)=(12,6,12)$. The other parameters $\lambda_2=30$ and $\lambda_1=66$ are already determined. Therefore by Lemma 1, up to the formulas (3.4) , we can conclude

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^3 = 24s^3+108s^2$$

and consequently (owing to (6.6),(6.12))

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2 = 15s^2-15s.$$

We finished the proofs of (6.6) and (6.10).
For the rest of the proofs, we give rather sketchy ones.
We start from (5.22). Using Lemmas 2 and 3 and Proposition 2, we get

$$\begin{aligned}&\sum_{\mathbf{u}\in\mathcal{K}_6}\sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4}(\mathbf{y},\alpha)^8\\ &= 3^{-2}\sum_{\mathbf{u}\in\mathcal{K}_6}[1680k^4-2800k^3+252k^2+954k\\ &\quad -8960kk_1k_2+13440k_1k_2]\\ &= \frac{4480}{3}s^4+4480s^3-7280s^2+2568s.\end{aligned}$$

Substituting the formulas (6.4),(6.5),(6.6) and (6.10) in the above equation, we get

(6.24) $$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^4-\frac{16}{3}\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2 = 8s^4+64s^3+12s^2+48s$$

Putting

$$\mathbf{a}=(\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\ \overset{i_3}{1},\cdot),$$

we consider the sum

$$\begin{aligned}&\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\\ &= 3\cdot 2[\lambda(3;\mathbf{a};1,1,2)+\lambda(3;\mathbf{a};1,2,1)+\lambda(3;\mathbf{a};2,1,1)\\ &\quad+\lambda(3;\mathbf{a};1,2,2)+\lambda(3;\mathbf{a};2,1,2)+\lambda(3;\mathbf{a};2,2,1)]\\ &\quad+2[\lambda(3;\mathbf{a};0,1,2)+\lambda(3;\mathbf{a};1,0,2)+\lambda(3;\mathbf{a};0,2,1)\\ &\quad+\lambda(3;\mathbf{a};1,2,0)+\lambda(3;\mathbf{a};2,0,1)+\lambda(3;\mathbf{a};2,1,0)]\end{aligned}$$

By Lemma 5, this sum is invariant under the change

$$\mathbf{a}\to\mathbf{a}',$$

where $\mathbf{a}'$ is another ternary vector obtained from $\mathbf{a}$ by replacing some 1's in the coordinates of $\mathbf{a}$ by 2's. Similarly to the arguements in the proof of (6.6) and (6.10), the above fact yields a more precise knowledge on $\lambda(3;\mathbf{a};\tau_1,\tau_2,\tau_3)$. That is,

(6.25) $$\lambda(3;\mathbf{a};\tau_1,\tau_2,\tau_3)=3,$$

where the triple $\tau_i's$, $1\le i\le 3$ run over all combinations of 1's and 2's. (There arise 8 triples.)
To get informations on the case when $wt(\mathbf{a})=4$, we need to determine the values of

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\ ,$$

for $\mathbf{a}$ with $wt(\mathbf{a})=4$. Before doing so, we propose a lemma :

Lemma 7. *We consider within the ternary Golay code $\mathcal{G}_{12}$. In counting the quantities $\lambda(4;\mathbf{a};\tau_1,\tau_2,\tau_3,\tau_4)$ with $\tau_i\in\mathbf{F}_3-\{0\}$ $1\le i\le 4$), they are divided into two subsets. Any two members in the same subset are co-existent and any two members in the opposite subsets are non co-existent. One subset consists of*

(1) $\lambda(4;\mathbf{a};1,1,1,1),\lambda(4;\mathbf{a};1,1,2,2),\lambda(4;\mathbf{a};2,2,1,1),$
$\lambda(4;\mathbf{a};1,2,1,2),\lambda(4;\mathbf{a};2,1,1,2),\lambda(4;\mathbf{a};1,2,2,1),$
$\lambda(4;\mathbf{a};2,1,2,1),\lambda(4;\mathbf{a};2,2,2,2)$

and the other consists of

(2) $\lambda(4;\mathbf{a};1,1,1,2),\lambda(4;\mathbf{a};1,1,2,1),\lambda(4;\mathbf{a};1,2,1,1),$
$\lambda(4;\mathbf{a};2,1,1,1),\lambda(4;\mathbf{a};2,2,2,1),\lambda(4;\mathbf{a};2,2,1,2),$
$\lambda(4;\mathbf{a};2,1,2,2),\lambda(4;\mathbf{a};1,2,2,2).$

Proof. We may suppose that $\mathbf{a}$ takes of the form

$$\mathbf{a}=(\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\ \overset{i_3}{1},\cdots,\overset{i_4}{1},\cdots).$$

If, for example, it holds that both the numbers $\lambda(4;\mathbf{a};1,1,1,1)$ and $\lambda(4;\mathbf{a};1,1,1,2)$ are positive, then there are two sextets $\mathbf{u}=(u_1,\cdots,u_{12})$ and $\mathbf{v}=(v_1,\cdots,v_{12})$ in $\mathcal{G}_{12}$ with the condition

$$u_{i_1}=u_{i_2}=u_{i_3}=u_{i_4}=1$$

and

$$v_{i_1}=v_{i_2}=v_{i_3}=1,\ \text{and}\ v_{i_4}=2.$$

But this implies that

$$0<wt(\mathbf{u}-\mathbf{v})\le 5$$

which contradicts the fact that the minimal non-zero weight of $\mathcal{G}_{12}$ is 6. Thus this does not happen. For any two members of the same subset, we could not observe such kind of contradiction. Q.E.D.

When we treat the subset of type (1), we obtain a relation :

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2$$
$$\begin{aligned}&= 4\cdot 4[\lambda(4;\mathbf{a};1,1,2,2)+\lambda(4;\mathbf{a};2,2,1,1)+\lambda(4;\mathbf{a};1,2,1,2)\\&+\lambda(4;\mathbf{a};2,1,1,2)+\lambda(4;\mathbf{a};1,2,2,1)+\lambda(4;\mathbf{a};2,1,2,1)]+\\&3\cdot 2[\lambda(4;\mathbf{a};1,1,2,0)+\lambda(4;\mathbf{a};1,1,0,2)+\lambda(4;\mathbf{a};1,0,2,2)\\&\lambda(4;\mathbf{a};0,1,2,2)+\lambda(4;\mathbf{a};0,2,1,1)+\lambda(4;\mathbf{a};2,0,1,1)\\&+\lambda(4;\mathbf{a};2,2,0,1)+\lambda(4;\mathbf{a};2,2,1,0)+\lambda(4;\mathbf{a};0,2,1,2)\\&+\lambda(4;\mathbf{a};1,0,1,2)+\lambda(4;\mathbf{a};1,2,0,2)+\lambda(4;\mathbf{a};1,2,1,0)\\&+\lambda(4;\mathbf{a};0,1,1,2)+\lambda(4;\mathbf{a};2,0,1,2)+\lambda(4;\mathbf{a};2,1,0,2)\\&+\lambda(4;\mathbf{a};2,1,1,0)+\lambda(4;\mathbf{a};0,2,2,1)+\lambda(4;\mathbf{a};1,0,2,1)\\&+\lambda(4;\mathbf{a};1,2,0,1)+\lambda(4;\mathbf{a};1,2,2,0)+\lambda(4;\mathbf{a};0,1,2,1)\\&+\lambda(4;\mathbf{a};2,0,2,1)+\lambda(4;\mathbf{a};2,1,0,1)+\lambda(4;\mathbf{a};2,1,2,0)]\\&+2[\lambda(4;\mathbf{a};0,0,2,1)+\lambda(4;\mathbf{a};0,0,1,2)+\lambda(4;\mathbf{a};0,1,0,2)\\&+\lambda(4;\mathbf{a};0,2,0,1)+\lambda(4;\mathbf{a};0,1,2,0)+\lambda(4;\mathbf{a};0,2,1,0)\\&+\lambda(4;\mathbf{a};1,0,0,2)+\lambda(4;\mathbf{a};2,0,0,1)+\lambda(4;\mathbf{a};1,0,2,0)\\&+\lambda(4;\mathbf{a};2,0,1,0)+\lambda(4;\mathbf{a};1,2,0,0)+\lambda(4;\mathbf{a};2,1,0,0)]\end{aligned}$$

There are some relations between the terms of the above equation. The first kind is such as

(6.26) $$\begin{aligned}&\lambda(4;\mathbf{a};0,0,2,1)+\lambda(4;\mathbf{a};2,0,2,1)+\lambda(4;\mathbf{a};0,1,2,1)\\&+\lambda(4;\mathbf{a};1,0,2,1)+\lambda(4;\mathbf{a};0,2,2,1)\\&+\lambda(4;\mathbf{a};2,1,2,1)+\lambda(4;\mathbf{a};1,2,2,1)=\lambda(2;\mathbf{b};2,1),\end{aligned}$$

where $\mathbf{b}$ is of weight 2 and is given by

$$\mathbf{b}=(\cdot\cdot,\overset{i_1}{0},\cdot\cdot,\overset{i_2}{0},\cdot\cdot\overset{i_3}{1},\cdots,\overset{i_4}{1},\cdots).$$

But we know from (6.16) that

$$\lambda(2;\mathbf{b};2,1)=15.$$

There are 12 relations of the type (6.26). The second kind of relation is such as

(6.27) $\lambda(4;\mathbf{a};1,1,2,0)+\lambda(4;\mathbf{a};1,1,2,2)=\lambda(3;\mathbf{c};1,1,2),$

where $\mathbf{c}$ is of weight 3 and is given by

$$\mathbf{c}=(\cdot\cdot,\overset{i_1}{1},\cdot\cdot,\overset{i_2}{1},\cdot\cdot\overset{i_3}{1},\cdots,\overset{i_4}{0},\cdots).$$

We know from (6.25) that

$$\lambda(3;\mathbf{c};1,1,2)=3.$$

There are 24 relations of the type (6.27). These two kinds of relations suffice to get the value

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2=504,$$

for the vector $\mathbf{a}$ of weight 4.

When we treat the subset of type (2), we obtain a relation :

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2$$
$$\begin{aligned}&= 4\cdot 3[\lambda(4;\mathbf{a};1,1,1,2)+\lambda(4;\mathbf{a};1,1,2,1)+\lambda(4;\mathbf{a};1,2,1,1)\\&+\lambda(4;\mathbf{a};2,1,1,1)+\lambda(4;\mathbf{a};2,2,2,1)+\lambda(4;\mathbf{a};2,2,1,2)\\&+\lambda(4;\mathbf{a};2,1,2,2)+\lambda(4;\mathbf{a};1,2,2,2)]\\&3\cdot 2[\lambda(4;\mathbf{a};0,1,1,2)+\lambda(4;\mathbf{a};1,0,1,2)+\lambda(4;\mathbf{a};1,1,0,2)\\&\lambda(4;\mathbf{a};0,1,2,1)+\lambda(4;\mathbf{a};1,0,2,1)+\lambda(4;\mathbf{a};1,1,2,0)\\&+\lambda(4;\mathbf{a};0,2,1,1)+\lambda(4;\mathbf{a};1,2,0,1)+\lambda(4;\mathbf{a};1,2,1,0)\\&+\lambda(4;\mathbf{a};2,0,1,1)+\lambda(4;\mathbf{a};2,1,0,1)+\lambda(4;\mathbf{a};2,1,1,0)\\&+\lambda(4;\mathbf{a};0,2,2,1)+\lambda(4;\mathbf{a};2,0,2,1)+\lambda(4;\mathbf{a};2,2,0,1)\\&+\lambda(4;\mathbf{a};0,2,1,2)+\lambda(4;\mathbf{a};2,0,1,2)+\lambda(4;\mathbf{a};2,2,1,0)\\&+\lambda(4;\mathbf{a};0,1,2,2)+\lambda(4;\mathbf{a};2,1,0,2)+\lambda(4;\mathbf{a};2,1,2,0)\\&+\lambda(4;\mathbf{a};1,0,2,2)+\lambda(4;\mathbf{a};1,2,0,2)+\lambda(4;\mathbf{a};1,2,2,0)]\\&+2[\lambda(4;\mathbf{a};0,0,2,1)+\lambda(4;\mathbf{a};0,0,1,2)+\lambda(4;\mathbf{a};0,1,0,2)\\&+\lambda(4;\mathbf{a};0,2,0,1)+\lambda(4;\mathbf{a};0,1,2,0)+\lambda(4;\mathbf{a};0,2,1,0)\\&+\lambda(4;\mathbf{a};1,0,0,2)+\lambda(4;\mathbf{a};2,0,0,1)+\lambda(4;\mathbf{a};1,0,2,0)\\&+\lambda(4;\mathbf{a};2,0,1,0)+\lambda(4;\mathbf{a};1,2,0,0)+\lambda(4;\mathbf{a};2,1,0,0)].\end{aligned}$$

In a similar manner to the former case we can determine that

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2=504.$$

The quantities

$$\lambda_j(4;\mathbf{a})=|\ \{\mathbf{u}\in\mathcal{K}_6\mid \mathbf{u}*\mathbf{a}=j\}\ |\quad \text{for } j=1,2,3,4,$$

where $wt(\mathbf{a})=4$, are completely determined by the equations (6.4),(6.5),(6.6),
and (6.7), and they are independent of the non-zero coordinate positions assigned to $\mathbf{a}$. Actually we have $\lambda_1(4;\mathbf{a})=64$, $\lambda_2(4;\mathbf{a})=120$, $\lambda_3(4;\mathbf{a})=64$,and $\lambda_4(4;\mathbf{a})=8$. This implies that $\overline{\mathcal{K}_6}$ holds a 4-design with parameters $(v,k,\lambda_4)=(12,6,4)$. The other parameters $\lambda_1,\lambda_2,\lambda_3$ of the section 3 are already determined. Hence by Lemma 1, the equality

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^4=8s^4+96s^3+76s^2-48s$$

holds for any ternary vector $\mathbf{a}$ with $wt(\mathbf{a})=s$ $(1\le s\le 12)$. And consequently (with (6.24) in mind)

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u}*\mathbf{a})=6s^3+12s^2-18s.$$

This is a rough description of the proofs of (6.7) and (6.11).

The proofs of (6.8) and (6.12) are roughly sketched as follows.
We begin with (5.23). Using Lemmas 2 and 3, Propositions

2 and 4, we arrive at

$$\begin{aligned}
&\sum_{\mathbf{u}\in\mathcal{K}_6}\ \sum_{\mathbf{y}\in\Phi^{-1}(\mathbf{u})\cap\Lambda_4}(\mathbf{y},\alpha)^{10}\\
&= 3^{-4}\sum_{\mathbf{u}\in\mathcal{K}_6}\Big\{30240(\mathbf{u}*\mathbf{a})^5-50400(\mathbf{u}*\mathbf{a})^4\\
&\quad -133560(\mathbf{u}*\mathbf{a})^3+358020(\mathbf{u}*\mathbf{a})^2-203958(\mathbf{u}*\mathbf{a})\\
&\quad -(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\Big[403200(\mathbf{u}*\mathbf{a})^2\\
&\quad -1411200(\mathbf{u}*\mathbf{a})+1300320\Big]\Big\}\\
&= \frac{2240}{3}s^5+\frac{22400}{3}s^4-13440s^3+3120s^2+2664s.
\end{aligned}$$

Substituting the formulas (6.4),(6.5),(6.6),(6.7), (6.10) and (6.11) in the above equation, we get

$$\text{(6.28)}\quad \sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^5-\frac{40}{3}\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^2\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2$$

$$=2s^5+\frac{100}{3}s^4-50s^3-\frac{40}{s}^2+160s.$$

As before we express the sum

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^2\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\quad\text{for } \mathbf{a} \text{ with } wt(\mathbf{a})=4$$

in terms of the quantities $\lambda(4;\mathbf{a};\tau_1,\tau_2,\tau_3,\tau_4)$. Then the invariance of this sum under the change

$$\mathbf{a}\to\mathbf{a}'$$

gives the precise knowledge about $\lambda(4;\mathbf{a};\tau_1,\tau_2,\tau_3,\tau_4)$. And this knowledge suffices to determine the value

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^2\cdot(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2=2440\quad\text{for } \mathbf{a} \text{ with } wt(\mathbf{a})=5$$

giving

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^5=64500.$$

The last condition together with the conditions (6.4),(6.5),(6.6), and (6.7) for $wt(\mathbf{a})=s=5$ guarantees that $\overline{\mathcal{K}_6}$ forms a 5-design with parameters $(v,k,\lambda_5)=(12,6,1)$. Then by Lemma 1, we conclude that

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}*\mathbf{a})^5=2s^5+60s^4+190s^3-120s^2$$

holds for any ternary vector $\mathbf{a}$ with $wt(\mathbf{a})=s$, and by (6.28) we obtain

$$\sum_{\mathbf{u}\in\mathcal{K}_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u}*\mathbf{a})^2=2s^4+18s^3-8s^2-12s.$$

The formula (6.9) is derivable from (5.24), and we omit the detail.

(II) Result on ternary [16,8,6] code.

Theorem 2. *Let K_6 be the set of sextets in ternary [16,8,6] code. α be any vector of the form (6.1) with $1\le s\le 16$. We put $\mathbf{a}=t\text{-}supp(\alpha)$, $wt(\mathbf{a})=s$, then we have*

$$\text{(6.29)}\quad \sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})=84s.$$

$$\text{(6.30)}\quad \sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})^2=28s^2+56s,$$

$$\text{(6.31)}\quad \sum_{u\in K_6}(\mathbf{u}*\mathbf{a})^3=8s^3+60s^2+16s,$$

$$\begin{aligned}
\text{(6.32)}\quad \sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})^5 &= \frac{(20+15s)}{4}\sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})^4\\
&+\frac{40}{3}\sum_{\mathbf{u}\in K_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u}*\mathbf{a})^2\\
&-\frac{(140+20s)}{3}\sum_{\mathbf{u}\in K_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\cdot(\mathbf{u}*\mathbf{a})\\
&-2s^5-\frac{80}{3}s^4-\frac{5}{3}s^3+\frac{560}{3}s^2-\frac{952}{3}s,
\end{aligned}$$

$$\text{(6.33)}\quad \sum_{u\in K_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2=7s^2-7s.$$

(III) Ternary [20,10,6] code.

Theorem 3. *Let K_6 be the set of sextets in ternary [20,10,6] code. α be any vector of the form (6.1) with $1\le s\le 20$. We put $\mathbf{a}=t\text{-}supp(\alpha)$, $wt(\mathbf{a})=s$, then we have*

$$\text{(6.34)}\quad \sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})=36s,$$

$$\begin{aligned}
\text{(6.35)}\quad \sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})^3 &= \Big(\frac{7}{6}+\frac{3}{4}s\Big)\sum_{\mathbf{u}\in K_6}(\mathbf{u}*\mathbf{a})^2\\
&+\frac{4}{3}\sum_{\mathbf{u}\in K_6}(\mathbf{u}\natural\mathbf{a})_1\cdot(\mathbf{u}\natural\mathbf{a})_2\\
&-5s^3-120s^2-16s.
\end{aligned}$$

Remark 3. It should be noted that our present method is also applicable to (i) the case of the subset $\mathcal{K}_m$ with $m\ge 9$ in the ternary [12,6,6] (or [16,8,6] or [20,10,6] resp.) codes and (ii) the case of ternary extremal codes of larger lengths. The computation in each case will be more complicated , but the methods of proofs are no more complicated.

Appendices

A. Generator matrices of some ternary self-dual codes

1. A generator matrix of the ternary Golay code.

Let $\mathcal{G}_{12}$ be the ternary Golay code of length 12. A generator matrix for $\mathcal{G}_{12}$ is given by

$$(I_6 H_6)\,,$$

where I_6 is the unit matrix of order 6, and H_6 is the matrix given by

$$H_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

2. A generator matrix of the unique ternary [16,8,6] code $2f_8$.

A generator matrix of the code is given by

$$(I_8 H_8)\,,$$

where I_8 is the unit matrix of order 8, and H_8 is the matrix given by

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

3. A generator matrix of one of six non-equivalent ternary [20,10,6] codes.
Since the intersection properties which these six codes have are the same ones , we only give a generator matrix of one such code, which is denoted by $10f_2$ in [25] :

$$\begin{pmatrix}
1&1&1&1&1&2&&&&&&&&&&&&&& \\
&&&&1&1&1&1&1&2&&&&&&&&&& \\
&&&&&&&&1&1&1&1&1&2&&&&&& \\
&&&&&&&&&&&&1&1&1&1&1&2&& \\
1&2&&&&&&&&&&&&&&&1&1&1&1 \\
1&1&2&2&&&&&&&1&2&&&&&&&& \\
&&&&1&1&2&2&&&&&&&1&2&&&& \\
&&&&&&&&1&1&2&2&&&&&&&1&2 \\
&&1&2&&&&&&&&&1&1&2&2&&&& \\
&&&&&&1&2&&&&&&&&&1&1&2&2
\end{pmatrix}$$

B. Gegenbauer polynomials of degrees from 10 to 14

$$\begin{aligned}
H_{10}(u) &= u^{10} - \frac{45u^8}{(8k+16)} + \frac{630u^6}{(8k+16)(8k+14)} \\
&\quad - \frac{3150u^4}{(8k+16)(8k+14)(8k+12)} \\
&\quad + \frac{4725u^2}{(8k+16)(8k+14)(8k+12)(8k+10)} \\
&\quad - \frac{945}{\prod_{m=4}^{7}(8k+2m)} \\
H_{12}(u) &= u^{12} - \frac{66u^{10}}{(8k+20)} + \frac{1485u^8}{(8k+20)(8k+18)} \\
&\quad - \frac{13860u^6}{(8k+20)(8k+18)(8k+16)} \\
&\quad + \frac{51975u^4}{(8k+20)(8k+18)(8k+16)(8k+14)} \\
&\quad - \frac{62370u^2}{\prod_{m=6}^{10}(8k+2m)} \\
&\quad + \frac{10395}{\prod_{m=5}^{10}(8k+2m)} \\
H_{14}(u) &= u^{14} - \frac{91u^{12}}{(8k+24)} + \frac{3003u^{10}}{(8k+24)(8k+22)} \\
&\quad - \frac{45045u^8}{(8k+24)(8k+22)(8k+20)} \\
&\quad + \frac{315315u^6}{(8k+24)(8k+22)(8k+20)(8k+18)} \\
&\quad - \frac{945945u^4}{\prod_{m=8}^{12}(8k+2m)} + \\
&\quad + \frac{945945u^2}{\prod_{m=7}^{12}(8k+2m)} \\
&\quad - \frac{135135}{\prod_{m=6}^{12}(8k+2m)}
\end{aligned}$$

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Michio Ozeki
Emeritus Professor at the Department of Mathematics, Faculty of Science, Yamagata University, Yamagata 990-8560, Japan.
E-mail: ozeki.mitio(at)ruby.plala.or.jp