Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

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Bilinearization and Casorati Determinant Solutions to Non-autonomous 1 + 1 Dimensional Discrete Soliton Equations

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Abstract

Some techniques of bilinearization of the non-autonomous 1 + 1 dimensional discrete soliton equations is discussed by taking the discrete KdV equation, the discrete Toda lattice equation, and the discrete Lotka-Volterra equation as examples. Casorati determinant solutions to those equations are also constructed explicitly.

1 Introduction

The Hirota-Miwa equation, or the discrete KP equation is the bilinear difference equation of Hirota type given by

\[ a(b - c)\tau(l + 1, m, n)\tau(l, m + 1, n + 1) + b(c - a)\tau(l, m + 1, n)\tau(l + 1, m, n + 1) + c(a - b)\tau(l, m, n + 1)\tau(l + 1, m + 1, n) = 0. \]  

Eq.(1) is well-known as one of the most important integrable systems[3, 11, 16]. Here, \( a, b, c \) are arbitrary constants playing a role of lattice intervals of discrete independent variables \( l, m, n \), respectively. The Casorati determinant solution to eq.(1) is given by

\[
\tau(l, m, n) = \begin{vmatrix}
\varphi_r^{(i)}(l, m, n) & \varphi_r^{(i+1)}(l, m, n) & \cdots & \varphi_r^{(i+N-1)}(l, m, n) \\
\varphi_r^{(i+1)}(l, m, n) & \varphi_r^{(i+2)}(l, m, n) & \cdots & \varphi_r^{(i+N)}(l, m, n) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_r^{(i+N-1)}(l, m, n) & \varphi_r^{(i+N)}(l, m, n) & \cdots & \varphi_r^{(i+2N-1)}(l, m, n)
\end{vmatrix},
\]  

where \( \varphi_r^{(i)}(l, m, n) \) \( (r = 1, \ldots, N) \) are arbitrary functions satisfying the linear relations

\[
\begin{align*}
\frac{\varphi_r^{(i+1)}(l, m, n) - \varphi_r^{(i)}(l, m, n)}{a} &= \varphi_r^{(i+1)}(l, m, n), \\
\frac{\varphi_r^{(i+1)}(k, l + 1, m) - \varphi_r^{(i)}(k, l + 1, m)}{b} &= \varphi_r^{(i+1)}(l, m, n), \\
\frac{\varphi_r^{(i+1)}(l, m, n + 1) - \varphi_r^{(i)}(l, m, n)}{c} &= \varphi_r^{(i+1)}(l, m, n).
\end{align*}
\]  

For example, choosing \( \varphi_r^{(i)} \) to be exponential type function as

\[ \varphi_r^{(i)}(l, m, n) = \alpha_r p^i(1 + \alpha p)^m(1 + \beta p)^n + \beta_r q^i(1 + \alpha q)^m(1 + \beta q)^n, \]  

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where $\alpha, \beta, p_r, q_r$ are arbitrary constants and $p_r, q_r$ are parameters, then it gives the $N$-soliton solution.

Eq.(1) is known to yield various discrete and continuous soliton equations by the reductions and limiting procedure. For example, let us impose the condition

$$\tau(l + 1, m + 1, n) = \tau(l, m, n), \tag{5}$$

where $\equiv$ denotes the equivalence up to multiple of gauge functions which leaves the bilinear equation invariant. Then using eq.(5) to suppress $l$-dependence and taking $a = -b$, eq.(1) yields

$$(b - c) \tau_{n+1}^{m+1} \tau_{n+1}^{m+1} - (b + c) \tau_n^{m+1} \tau_{n+1}^{m-1} + 2c \tau_n^{m+1} \tau_n^{m} = 0, \tag{6}$$

where $\tau_n^m = \tau(l, m, n)$. Eq.(6) is transformed to

$$v_n^m v_n^{m+1} = \frac{b - c}{b + c} \left( \frac{1}{v_{n+1}^{m+1}} - \frac{1}{v_n^m} \right), \tag{7}$$

by the dependent variable transformation

$$v_n^m = \frac{\tau_n^{m+1} \tau_{n+1}^m}{v_{n+1}^{m+1} v_n^m}. \tag{8}$$

Eq.(6) or eq.(7) are called the discrete KdV equation[2, 16]. The condition (5) is realized by choosing $q = -p$ on the level of $\varphi^{(3)}(l, m, n)$ in eq.(4). Therefore choosing the entries of the determinant as

$$\varphi^{(3)}(m, n) = \alpha, p_r(1 + bp_r)p^{\alpha}(1 + cp_r)^p + \beta,(-p_r)^q(1 - bp_r)q^{\alpha}(1 + cp_r)^p, \tag{9}$$

it gives the $N$-soliton solution of the discrete KdV equation. Hence, if a given equation turns out to be derived by the reduction or other procedure from the Hirota-Miwa equation, it is possible to construct wide class of solutions in this manner.

On the other hand, it has been pointed out that generalization of the Hirota-Miwa equation is possible in such a way that the lattice intervals are arbitrary functions of the corresponding independent variables[7, 23]. Such generalization to “inhomogeneous lattice” or to non-autonomous equation is regarded as an important problem in the context of ultradiscretization or the box and ball systems, since it corresponds to a generalization such that the capacity of the boxes changes according to the lattice sites[14]. Moreover, many discrete soliton equations are shown to describe discrete surfaces or curves in various settings of the discrete differential geometry. In this context, such inhomogeneity of lattices corresponds to the scaling freedom of the parametrization of geometric objects and therefore it is geometrically natural[1].

The generalization to non-autonomous equation is technically straightforward for the “generic” equation such as Hirota-Miwa equation because of its gauge invariance. However, when we consider the reduction to $1 + 1$ dimensional system such as the discrete KdV equation, the reduction procedure does not work consistently because of the non-autonomous property on the level of both bilinear equation and solution. Therefore the $1 + 1$ dimensional non-autonomous discrete soliton equations have not been studied well.

Recently, Tsujimoto and Mukaihara have considered the non-autonomous discrete Toda lattice (1DTL) equation on semi-infinite lattice from the standpoint of $R_1$ and $R_2$ type bi-orthogonal functions[12, 13]. By introducing certain auxiliary $\tau$ function which does not appear in the expression of the solution, they succeeded in bilinearization of the equation and constructing molecular type solution. Then it has been shown that the non-autonomous discrete 1DTL equation on infinite lattice also admits similar bilinearization, and the soliton type solutions have been constructed[8]. Moreover, three different bilinearizations of different origins have been presented for the non-autonomous discrete KdV equation in [9], each of which requires some auxiliary $\tau$ function, respectively. The techniques developed in recent researches may enable systematic study of non-autonomous discrete soliton equations.

The purpose of this paper is to give a review and some new results on bilinearization of non-autonomous $1 + 1$ dimensional discrete soliton equations and construction of their Casorati determinant solutions. This paper is organized as follows. In Section 2, we give a brief review of non-autonomous discrete KP hierarchy and its solutions. In Section 3, we discuss three bilinearizations of the non-autonomous discrete KdV equation. Section 4 deals with two bilinearizations of the non-autonomous discrete 1DTL equation. We discuss in Section 5 the case of discrete Lotka-Volterra equation, where direct reduction from the discrete two-dimensional Toda lattice equation works without auxiliary $\tau$ function.
2 Non-autonomous Discrete KP Hierarchy

2.1 \( \tau \) Function and Bilinear Equations

We define the \( \tau \) function \( \tau_N(s; l, m; x, y) \) depending on infinitely many independent variables \( N \in \mathbb{Z}, ~ l = (l_1, l_2, \ldots), ~ m = (m_1, m_2, \ldots), ~ x = (x_1, x_2, \ldots), \) and \( y = (y_1, y_2, \ldots) \) by

\[
\tau_N(s; l, m; x, y) = \begin{vmatrix}
\varphi_1^{(s)} & \varphi_1^{(s+1)} & \cdots & \varphi_1^{(s+N-1)} \\
\varphi_2^{(s)} & \varphi_2^{(s+1)} & \cdots & \varphi_2^{(s+N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N^{(s)} & \varphi_N^{(s+1)} & \cdots & \varphi_N^{(s+N-1)} \\
\end{vmatrix},
\]

where \( \varphi_r^{(s)} = \varphi_r^{(s)}(l, m; x, y) \) \( (r = 1, \ldots, N) \) satisfy the linear equations

\[
\frac{\varphi_r^{(s)}(l_r + 1) - \varphi_r^{(s)}(l_r)}{a_r(l_r)} = \varphi_r^{(s+1)}(l_r),
\]

\[
\frac{\varphi_r^{(s)}(m_r + 1) - \varphi_r^{(s)}(m_r)}{b_r(m_r)} = \varphi_r^{(s-1)}(m_r),
\]

\[
\frac{\partial}{\partial x_r} \varphi_r^{(s)} = \varphi_r^{(s+1)},
\]

\[
\frac{\partial}{\partial y_r} \varphi_r^{(s)} = \varphi_r^{(s-1)},
\]

for \( r = 1, 2, \ldots \). Here the lattice intervals \( a_r \) and \( b_r \) \( (r = 1, 2, \ldots) \) are arbitrary functions with respect to the indicated variables. We note that in the following, we indicate only the relevant independent variables for notational simplicity, as in eqs.(11)-(14). For example, the \( N \)-soliton solution is obtained by choosing \( \varphi_r^{(s)} \) as

\[
\varphi_r^{(s)} = \alpha_r p_r^s \prod_{i=1}^{l_r} (1 + a_r(i)p_r) \prod_{j=1}^{m_r} (1 + b_r(j)q_r) \sum_{l=1}^{\infty} x_r l^l q_r^l + \sum_{s=1}^{\infty} y_r s^s q_r^s,
\]

where \( \alpha_r, \beta_r \) are arbitrary constants and \( p_r, q_r \) are parameters.

It is known that \( \tau_N(s; l, m; x, y) \) satisfies infinitely many difference, differential and difference-differential bilinear equations of Hirota type (for autonomous case, see for example [21]). We call this hierarchy of equations non-autonomous discrete KP hierarchy. We give a list of some typical examples included in the hierarchy:

**KP equation** \( (x = x_1, y = x_2, t = x_3) \)

\[
(D_{x_1}^4 - 4D_{x_1}D_{x_2} + 3D_{x_2}^2) \tau \cdot \tau = 0.
\]

**Two-dimensional Toda lattice (2DTL) equation** \( (x = x_1, y = y_1, n = s) \)

\[
\frac{1}{2} D_{x_1} D_{x_2} \tau_n \cdot \tau_n = \tau_{n+1} \tau_{n-1}.
\]

**Non-autonomous discrete KP equation** \( (l = l_i, m = l_j, n = l_k, \ a_i = a_i(l_i), \ b_m = a_j(l_j), \ c_m = a_k(l_k), \ [i, j, k] \subset \{1, 2, 3, \cdots\}) \)

\[
\begin{align*}
& a_i(b_m - c_n) \tau(l + 1, m, n) \tau(l, m + 1, n + 1) \\
& + b_m(c_n - a_i) \tau(l, m + 1, n) \tau(l + 1, m, n + 1) \\
& + c_n(a_m - b_n) \tau(l, m, n + 1) \tau(l, m + 1, n + 1) = 0.
\end{align*}
\]
Non-autonomous discrete 2DTL equation \((l = l_i, m = m_j, n = s, a_l = a_i(l), b_m = b_j(m), [i, j] \subset \{1, 2, \cdots\})\)

\[
(1 - a_l b_m) \tau_n(l + 1, m + 1) \tau_n(l, m) - \tau_n(l + 1, m) \tau_n(l, m + 1) + a_l b_m \tau_{n+1}(l, m + 1) \tau_{n+1}(l + 1, m) = 0.
\] (19)

Bäcklund transformation (BT) of 2DTL equation \((x = x_i, m = m_i, n = s, b_m = b_i(m), i \in \{1, 2, \cdots\})\)

\[
(D_x - b_m) \tau_n(m) \cdot \tau_n(m + 1) + b_m \tau_{n+1}(m) \tau_{n+1}(m + 1) = 0.
\] (20)

BT of non-autonomous discrete KP(2DTL) equation \((l = l_i, m = m_j, n = s, a_l = a_i(l), b_m = a_j(l), [i, j] \subset \{1, 2, \cdots\})\)

\[
a_l \tau_{n+1}(l, m + 1) \tau_n(l + 1, m) - b_m \tau_{n+1}(l + 1, m) \tau_n(l, m + 1) + (a_l - b_m) \tau_{n+1}(l, m) \tau_n(l + 1, m + 1) = 0.
\] (21)

2.2 Casorati Technique

In order to prove that the \(\tau\) function given in the form of Casorati determinant satisfies the bilinear equations, the Casorati technique is quite useful[16, 17]. We demonstrate the outline of the technique by taking eq.(20) as an example.

Under the setting of eq.(20), the \(\tau\) function (10) reads

\[
\tau_n(m) = \begin{vmatrix}
\varphi_1^{(n)}(m) & \varphi_1^{(n+1)}(m) & \cdots & \varphi_1^{(n+N-1)}(m) \\
\varphi_2^{(n)}(m) & \varphi_2^{(n+1)}(m) & \cdots & \varphi_2^{(n+N-1)}(m) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N^{(n)}(m) & \varphi_N^{(n+1)}(m) & \cdots & \varphi_N^{(n+N-1)}(m)
\end{vmatrix},
\] (22)

where \(\varphi_k^{(n)}(m) (k = 1, \ldots, N)\) satisfy the linear relations

\[
\partial_x \varphi_k^{(n)}(m) = \varphi_k^{(n+1)}(m),
\] (23)

\[
\varphi_k^{(n)}(m) - \varphi_k^{(n)}(m - 1) = b_{m-1} \varphi_k^{(n-1)}(m - 1).
\] (24)

For instance, if we choose \(\varphi_k^{(n)}(m)\) as

\[
\varphi_k^{(n)}(m) = a_k p_k^{n} \prod_{j=0}^{m-1} (1 + b_j p_k^{-1}) e^{b_j x} + b_k q_k^{n} \prod_{j=0}^{m-1} (1 + b_j q_k^{-1}) e^{b_j x},
\] (25)

we obtain the \(N\)-soliton solution.

The bilinear equation (20) is reduced to the Plücker relation, which is the quadratic identity among the determinants whose columns are properly shifted. Therefore, we first construct difference/differential formulas which express determinants whose columns are shifted by \(\tau_n(m)\).

**Lemma 2.1** The following formulas hold:

\[
\tau_n(m) = |0, \cdots, N - 2, N - 1|,
\] (26)

\[
\tau_n(m - 1) = |m_{n-1}, 1, \cdots, N - 2, N - 1|,
\] (27)

\[
-b_{m-1} \tau_n(m - 1) = |m_{n-1}, 1, \cdots, N - 2, N - 1|,
\] (28)

\[
\partial_x \tau_n(m) = |0, \cdots, N - 2, N|,
\] (29)

\[
(\partial_x + b_{m-1}) \tau_n(m - 1) = |0_{m-1}, \cdots, N - 2, N|.
\] (30)
where “$\mathbf{j}_m$” is the column vector

$$
\mathbf{j}_m = \begin{pmatrix}
q^{(m+j)}_1(m) \\
q^{(m+j)}_2(m) \\
\vdots \\
q^{(m+j)}_N(m)
\end{pmatrix},
$$

and the subscript is shown only when $m$ is shifted.

**Proof.** Eq.(26) follows by definition, and eq.(29) is derived from the differential rule of determinant. Using eq.(24) to the $i$-th column of $\tau_n(m-1)$ for $i = N, N-1, \ldots, 2$, we have

$$
\tau_n(m-1) = \begin{vmatrix}
0_{m-1}, & 1_{m-1}, & \cdots, & N-2_{m-1}, & N-1_{m-1}
\end{vmatrix}
= \begin{vmatrix}
0_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N-1
\end{vmatrix},
$$

which is eq.(27). Multiplying $-b_{m-1}$ to the first column of the right hand side of eq.(27) and using eq.(24) we have eq.(28),

$$
-b_{m-1}\tau_n(m-1) = \begin{vmatrix}
-b_{m-1} \cdot 0_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N-1
\end{vmatrix}
= \begin{vmatrix}
1_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N-1
\end{vmatrix}.
$$

Differentiating eq.(27), we have

$$
\partial_x \tau_n(m-1) = \begin{vmatrix}
1_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N-1
\end{vmatrix} + \begin{vmatrix}
0_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N
\end{vmatrix}
= -b_{m-1}\tau_n(m-1) + \begin{vmatrix}
0_{m-1}, & 1_{m-1}, & \cdots, & N-2, & N
\end{vmatrix},
$$

from which we obtain eq.(30). This completes the proof. \qed

Finally, eq.(20) is derived by applying Lemma 2.1 to the Plücker relation

$$
\begin{align*}
0 &= \begin{vmatrix}
0_{m-1}, & 0, & \cdots, & 1, & \cdots, & N-2, & N-1, & N
\end{vmatrix} \\
&= \begin{vmatrix}
0, & 1, & \cdots, & N-2, & N-1
\end{vmatrix} \times \begin{vmatrix}
0_{m-1}, & 1, & \cdots, & N-2, & N
\end{vmatrix} \\
&+ \begin{vmatrix}
0, & 1, & \cdots, & N-2, & N
\end{vmatrix} \times \begin{vmatrix}
0_{m-1}, & 1, & \cdots, & N-2, & N-1
\end{vmatrix}
\end{align*}
$$

(32)

Therefore we have shown that the $\tau$ function (22) actually satisfies the bilinear equation (20). Other equations are derived in a similar manner. We refer to [16, 17] for further details of the technique.

### 3 Non-autonomous Discrete KdV Equation

#### 3.1 Casorati Determinant Solution

In this section we consider the following difference equation[10]

$$
\begin{align*}
\frac{1}{a_m} + \frac{1}{b_{n+1}} y_{m+1}^n &= \frac{1}{a_{m+1}} + \frac{1}{b_n} y_m^n \\
= \frac{1}{a_m} \frac{1}{b_n} y_m^n - \frac{1}{a_{m+1}} \frac{1}{b_{n+1}} y_{m+1}^n
\end{align*}
$$

(33)

where $a_m$, $b_n$ are arbitrary functions of $m$ and $n$, respectively. If $a_m$ and $b_n$ are constants, eq.(33) is equivalent to the discrete KdV equation (7). We call eq.(33) the non-autonomous discrete KdV equation.

The $N$-soliton solutions to eq.(33) can be expressed by Casorati determinants as follows:
Theorem 3.1 For each $N \in \mathbb{N}$, we define an $N \times N$ determinant $\tau^m_n$ by

$$
\tau^m_n = \begin{vmatrix}
\varphi^{(i)}_1(m, n) & \varphi^{(i+1)}_1(m, n) & \cdots & \varphi^{(i+N-1)}_1(m, n) \\
\varphi^{(i)}_2(m, n) & \varphi^{(i+1)}_2(m, n) & \cdots & \varphi^{(i+N-1)}_2(m, n) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(i)}_N(m, n) & \varphi^{(i+1)}_N(m, n) & \cdots & \varphi^{(i+N-1)}_N(m, n)
\end{vmatrix},
$$

(34)

Let

$$
\varphi^{(i)}_j(m, n) = \alpha_j b_j \prod_{i=0}^{m-1} \left(1 + a_j p_i\right) \prod_{j=0}^{n-1} \left(1 + b_j p_j\right)
$$

(35)

and

$$
v^m_n = \frac{\tau^m_{n+1} \tau^m_{n+1}}{\tau^m_{n+1}}.
$$

(36)

satisfies eq.(33).

Unlike the autonomous case, eq.(33) cannot be put into the bilinear equation directly in terms of single $\tau$ function $\tau^m_n$ because of non-autonomous property. This difficulty is overcome by introduction of auxiliary $\tau$ function. In the following, we discuss three different bilinearizations.

3.2 Bilinearization (I)

Proposition 3.2 Let $\tau^m_n$ and $\sigma^m_n$ be functions satisfying the bilinear equations

$$
- \epsilon (a_m - b_n) \tau^m_n \tau^m_{n+1} + a_m (b_n + \epsilon) \tau^m_n \tau^m_{n+1} - b_n (\epsilon + a_m) \tau^m_{n+1} \sigma^m_n = 0,
$$

(37)

$$
\epsilon (a_m - b_n) \sigma^m_n \tau^m_{n+1} + a_m (b_n - \epsilon) \tau^m_{n+1} \sigma^m_n + b_n (\epsilon - a_m) \tau^m_n \sigma^m_{n+1} = 0,
$$

(38)

respectively, where $\epsilon$ is a constant. Then

$$
\Psi^m_n = \frac{\sigma^m_n}{\tau^m_n}, \quad v^m_n = \frac{\tau^m_{n+1} \tau^m_{n+1}}{\tau^m_{n+1} \tau^m_{n+1}}.
$$

(39)

satisfy

$$
\left(\frac{1}{b_n} - \frac{1}{a_m}\right) \frac{\Psi^{m+1}_n}{\Psi^m_n} + \left(\frac{1}{\epsilon} + \frac{1}{b_n}\right) \Psi^m_n + \left(\frac{1}{\epsilon} + \frac{1}{a_m}\right) \Psi^{m+1}_n = 0,
$$

(40)

$$
\left(\frac{1}{b_n} - \frac{1}{a_m}\right) \frac{\Psi^m_n}{\Psi^{m+1}_n} + \left(\frac{1}{\epsilon} - \frac{1}{b_n}\right) \Psi^{m+1}_n + \left(\frac{1}{\epsilon} - \frac{1}{a_m}\right) \Psi^m_n = 0,
$$

(41)

and non-autonomous discrete KdV equation (33). In particular, eq.(34) and

$$
\sigma^m_n = \begin{vmatrix}
\varphi^{(i)}_1(m, n) & \varphi^{(i+1)}_1(m, n) & \cdots & \varphi^{(i+N-1)}_1(m, n) \\
\varphi^{(i)}_2(m, n) & \varphi^{(i+1)}_2(m, n) & \cdots & \varphi^{(i+N-1)}_2(m, n) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(i)}_N(m, n) & \varphi^{(i+1)}_N(m, n) & \cdots & \varphi^{(i+N-1)}_N(m, n)
\end{vmatrix},
$$

(42)
where

$$\varphi_{ij}^q (m, n) = a_{ij} (1 + \epsilon p_i) \prod_{i=m_0}^{m-1} (1 + a_i p_i) \prod_{j=n_0}^{n-1} (1 + b_j p_j)$$

(43)

$$+ \beta_i (-p_i)^q (1 - \epsilon p_i) \prod_{i=m_0}^{m-1} (1 - a_i p_i) \prod_{j=n_0}^{n-1} (1 - b_j p_j).$$

solve the bilinear equations (37) and (38).

The bilinearization described in Proposition 3.2 is derived from the discrete KP hierarchy. The key idea is to introduce auxiliary “autonomous” independent variables (corresponding lattice intervals are constants) simultaneously, and to apply the reduction procedure through those autonomous variables. Let us take $k = l_1$, $l = l_2$, $m = l_3$ and $n = l_4$, and choose the corresponding lattice intervals as $a_1(l_1) = \delta$, $a_2(l_2) = \epsilon$, $a_3(l_3) = a_m$, $a_4(l_4) = b_n$, where $\delta$ and $\epsilon$ are constants. The variables $k$ and $l$ are the autonomous variables mentioned above.

We now consider the discrete KP equation (18) with respect to the variables $(k, m, n)$

$$\delta (a_m - b_n) \tau(k + 1, l, m, n) \tau(k, l, m + 1, n + 1)$$

(44)

$$+ a_m (b_n - \delta) \tau(k, l, m + 1, n) \tau(k + 1, l, m, n + 1)$$

$$+ b_n (\delta - a_m) \tau(k, l, m + 1, n) \tau(k + 1, l, m + 1, n) = 0,$$

and the same equation with respect to the variables $(l, m, n)$

$$\epsilon (a_m - b_n) \tau(k, l + 1, m, n) \tau(k, l, m + 1, n + 1)$$

(45)

$$+ a_m (b_n - \epsilon) \tau(k, l + 1, m, n) \tau(k + 1, l, m, n + 1)$$

$$+ b_n (\epsilon - a_m) \tau(k, l, m + 1, n) \tau(k + 1, l + 1, m, n) = 0.$$

Under this setting, the $\tau$ function (10) is written as

$$\tau(k, l, m, n) = \begin{vmatrix}
\varphi_1^{(i)} & \varphi_1^{(i+1)} & \cdots & \varphi_1^{(i+N-1)} \\
\varphi_2^{(i)} & \varphi_2^{(i+1)} & \cdots & \varphi_2^{(i+N-1)} \\
\vdots & \vdots & \cdots & \vdots \\
\varphi_N^{(i)} & \varphi_N^{(i+1)} & \cdots & \varphi_N^{(i+N-1)}
\end{vmatrix},$$

(46)

$$\varphi_{ij}^q (k, l, m, n) = a_{ij} (1 + \delta p_i)^q (1 + \epsilon p_i)^q \prod_{i=m_0}^{m-1} (1 + a_i p_i) \prod_{j=n_0}^{n-1} (1 + b_j p_j)$$

(47)

$$+ \beta_i q_i^q (1 + \delta q_i)^q (1 + \epsilon q_i)^q \prod_{i=m_0}^{m-1} (1 + a_i q_i) \prod_{j=n_0}^{n-1} (1 + b_j q_i).$$

We next impose the reduction condition on the autonomous independent variables $k$, $l$ as

$$\tau(k + 1, l + 1, m, n) = \tau(k, l, m, n).$$

(48)

This is achieved by imposing the condition on $\varphi_{ij}^q (r = 1, \ldots, N)$ as

$$\varphi_{ij}^q(k + 1, l + 1, m, n) = \varphi_{ij}^q(k, l, m, n).$$

(49)

In order to realize eq.(49), one may take

$$q_r = -p_r, \quad \delta = -\epsilon,$$

(50)
so that
\[ \varphi^{(i)}(k+1,l+1,m,n) = (1 - \epsilon^2 p_j^2) \varphi^{(i)}(k,l,m,n), \]
\[ \tau(k+1,l+1,m,n) = \prod_{j=1}^{N} (1 - \epsilon^2 p_j^2) \tau(k,l,m,n). \] 

(51)

Then, suppressing the \( k \)-dependence by using eq.(48), the bilinear equations (44) and (45) are reduced to
\[- \epsilon(a_m - b_m) \tau(l,m,n) \tau(l+1,m+1,n+1) + a_m(b_m + \epsilon) \tau(l+1,m+1,n) \tau(l,m,n+1)
- b_m(\epsilon + a_m) \tau(l+1,m+n+1) \tau(l,m,n+1) = 0,
\]
\[ \epsilon(a_m - b_m) \tau(l+1,m,n) \tau(l,m+1,n+1) + a_m(b_m - \epsilon) \tau(l,m+1,n) \tau(l+1,m,n+1)
+ b_m(\epsilon - a_m) \tau(l,m,n+1) \tau(l+1,m+1,n) = 0, \]
respectively. By putting
\[ \tau_m^n = \tau(l,m,n), \quad \sigma_m^n = \tau(l+1,m,n), \] 
(52)

we obtain the bilinear equations (37) and (38). Then an easy calculation shows that \( \Psi_m^n \) and \( \nu_m^n \) satisfy eqs.(40) and (41).

We finally show that \( \nu_m^n \) satisfies eq.(33). Eq.(33) is derived from the cubic equation in terms of \( \tau_m^n \) which is obtained by eliminating \( \sigma_m^n \) from the bilinear equations (37) and (38). However, this procedure can be done more systematically in the following manner. Introducing a vector
\[ \Phi_m^n = \begin{pmatrix} \Psi_m^{n+1} \\ \Psi_m^n \end{pmatrix}, \] 
(53)
eqs.(40) and (41) can be rewritten as the following linear system:
\[ \Phi_{n+1}^m = L_n^m \Phi_n^m, \quad \Phi_n^{m+1} = M_n^m \Phi_n^m, \] 
(54)

\[ L_n^m = \frac{1}{b_n - \frac{1}{\epsilon}} \begin{pmatrix} \frac{1}{b_n} + \frac{1}{a_m} & \frac{1}{a_m + \frac{1}{\epsilon}} \\ \frac{1}{a_m} - \frac{1}{\epsilon} & \frac{1}{b_n} - \frac{1}{a_m} \end{pmatrix}, \] 
(55)
\[ M_n^m = \frac{1}{b_{n+1} - \frac{1}{\epsilon}} \begin{pmatrix} \frac{1}{b_n} + \frac{1}{a_m} & \frac{1}{a_m - \frac{1}{\epsilon}} \nu_m^n \nu_n^{m+1} & - \frac{1}{a_m + 1} \\ \frac{1}{a_m} - \frac{1}{\epsilon} & \frac{1}{b_n} - \frac{1}{a_{m+1}} & 0 \end{pmatrix}. \] 
(56)

Then the compatibility condition of the linear system
\[ L_n^{m+1} M_n^m = M_{n+1}^m L_n^m \] 
gives eq.(33). This completes the proof of Proposition 3.2.

**Remark 1**
1. The linear system (54)-(56) is the auxiliary linear system of eq.(33) and the matrices \( L_n^m, M_n^m \) are the Lax pair, where the lattice interval \( \epsilon \) plays a role of the spectral parameter. In this sense, the bilinearization in this section can be regarded as that for the auxiliary linear system.
2. If we eliminate \( \nu_m^n \) from eqs.(40) and (41), and put \( w_n^m = \Psi_m^n \), we obtain the non-autonomous potential modified KdV equation
\[ w_{n+1}^{m+1} = \frac{w_n^m w_{n+1}^m - w_{m+1}^n}{\gamma_n^m w_n^{m+1}}, \quad \gamma_n^m = \frac{b_n}{a_m}, \] 
(58)
by taking $\epsilon \to \infty$. The solution of eq.(58) admits several expressions. For example, let us use the internal variable $s$ in eq.(34) explicitly and write $\tau^{m}_n = \tau^{m}_n(s)$. Then it is shown that $w^{m}_n = \frac{\tau^{(m+1)}_n}{\tau^{(m)}_n}$ satisfies eq.(58).

Similarly, writing $\tau^{m}_n$ with determinant size $N$ as $\tau^{m}_n = \tau^{m}_{n}(N)$, then it is also shown that $w^{m}_n = \frac{\tau^{(m+1)}_{n}}{\tau^{(m)}_{n}}$ satisfies eq.(58).

### 3.3 Bilinearization (II)

The non-autonomous discrete KdV equation (33) admits an alternate bilinearization involving auxiliary $\tau$ function which does not appear in the expression of the solution.

**Proposition 3.3** Let $\tau^{m}_n$ and $\kappa^{m}_n$ be functions satisfying the bilinear equations

\begin{align*}
&b_{n}(a_{m-1} + a_{n})\kappa^{m}_n - a_{m-1}(a_{m} + b_{n})\tau^{m+1}_{n+1} + a_{n}(a_{m-1} - b_{n})\tau^{m+1}_{n+1} = 0, \tag{59} \\
&b_{n}(a_{m-1} - a_{n})\tau^{m+1}_{n+1} - a_{m-1}(a_{m} - b_{n})\tau^{m+1}_{n+1} + a_{n}(a_{m-1} - b_{n})\tau^{m+1}_{n+1} = 0. \tag{60}
\end{align*}

Then $\tau^{m}_n$ defined in eq.(36) satisfies eq.(33). In particular, $\tau^{m}_n$ in eq.(34) and

\[ k^{m}_n = \begin{vmatrix} 
\psi^{(1)}_1(m, n) & \psi^{(1+1)}_1(m, n) & \cdots & \psi^{(1+n-1)}_1(m, n) \\
\psi^{(1)}_2(m, n) & \psi^{(1+1)}_2(m, n) & \cdots & \psi^{(1+n-1)}_2(m, n) \\
\vdots & \vdots & \ddots & \vdots \\
\psi^{(1)}_N(m, n) & \psi^{(1+1)}_N(m, n) & \cdots & \psi^{(1+n-1)}_N(m, n) 
\end{vmatrix}, \tag{61}
\]

\[ \psi^{(1)}_j(m, n) = \alpha_j p^{j}_r(1 + a_{m} p_r) \prod_{j=0}^{m-2} \prod_{k=0}^{n-1} \prod_{k=m}^{1} (1 + b_{k} p_r) + \beta_r(-p_r)^{j}(1 - a_{m} p_r) \prod_{j=0}^{m-2} \prod_{k=0}^{n-1} (1 - b_{k} p_r), \tag{62} \]

solve eqs. (59) and (60).

**Remark 2** In the autonomous case, namely if $a_{m}$ and $b_{n}$ are constants, the auxiliary $\tau$ function $\kappa^{m}_n$ reduces to $\tau^{m}_n$, the bilinear equation (59) yields the equation which is equivalent to eq.(6), and eq.(60) becomes trivial, respectively.

Proposition 3.3 is proved by applying the Casorati technique based on the linear relations among the entries of the determinants

\begin{align*}
\varphi^{(1)}_r(m + 1, n) - \varphi^{(1)}_r(m, n) &= a_{m} \varphi^{(1+1)}_r(m, n), \tag{63} \\
\varphi^{(1)}_r(m + 1, n) + a_{m} \varphi^{(1+1)}_r(m - 1, n) &= \psi^{(1)}_r(m, n), \tag{64} \\
\psi^{(1)}_r(m, n) - a_{m} \psi^{(1+1)}_r(m, n) &= (1 - a_{m}^2 p^2_r) \varphi^{(1)}_r(m - 1, n), \tag{65} \\
\varphi^{(1)}_r(m, n + 1) - \varphi^{(1)}_r(m, n) &= b_{n} \varphi^{(1+1)}_r(m, n). \tag{66}
\end{align*}

We refer to [9] for further details of the proof.

### 3.4 Bilinearization (III)

The non-autonomous discrete KdV equation (33) admits the third bilinearization through non-autonomous version of the potential discrete KdV equation[15]

\[ u^{m+1}_{n+1} - u^{m}_n = \left( \frac{1}{a_{m}^2} - \frac{1}{b_{n}^2} \right) \frac{1}{u^{m+1}_{n+1} - u^{m}_n}, \tag{67} \]
or
\[
\left[ \begin{array}{c} u_{n+m+1}^m - u_n^m \\ 1 \end{array} \right] = \left( \frac{1}{b_m} + \frac{1}{c_m} \right) \left[ \begin{array}{c} u_{n+m}^m - u_{n+1}^m \\ 1 \end{array} \right] = \frac{1}{b_m^2} - \frac{1}{c_m^2},
\]
(68)
where \( u_n^m \) and \( \tilde{u}_n^m \) are related as
\[
\tilde{u}_n^m = u_n^m + \sum_{i=m}^{n-1} \frac{1}{a_i} + \sum_{j=m}^{n-1} b_j.
\]
(69)
We note that \( u_n^m \) is related to \( v_n^m \) in eq.(33) as
\[
\left( \frac{1}{a_m} - \frac{1}{b_n} \right) v_n^m = u_{n+1}^m - u_n^{m+1}.
\]
(70)

**Proposition 3.4** Let \( \tau_n^m \) and \( \rho_n^m \) be functions satisfying the bilinear equations
\[
\rho_n^{m+1} \tau_n^{m+1} - \rho_n^m \tau_n^m = \left( \frac{1}{a_m} - \frac{1}{b_n} \right) \left( \tau_n^{m+1} \tau_n^0 - \tau_n^{m+1} \tau_n^0 \right),
\]
(71)
\[
\rho_n^{m+1} \tau_n^m = \rho_n^m \tau_n^{m+1} = \left( \frac{1}{a_m} - \frac{1}{b_n} \right) \left( \tau_n^{m+1} \tau_n^0 - \tau_n^{m+1} \tau_n^0 \right).
\]
(72)
Then \( v_n^m \) defined by eq.(36) and
\[
u_n^m = \rho_n^m \tau_n^m - \sum_{i=m}^{n-1} \frac{1}{a_i} - \sum_{j=m}^{n-1} b_j,
\]
satisfy eq.(33) and eq.(67), respectively. In particular, \( \tau_n^m \) defined by eq.(34) and
\[
\rho_n^m = \left[ \begin{array}{cccc} \varphi_1^{(m,n)}(m,n) & \cdots & \varphi_{1+N-2}^{(m,n)}(m,n) & \varphi_{1+N}^{(m,n)}(m,n) \\ \varphi_2^{(m,n)}(m,n) & \cdots & \varphi_{2+N-2}^{(m,n)}(m,n) & \varphi_{2+N}^{(m,n)}(m,n) \\ \vdots & \cdots & \vdots & \vdots \\ \varphi_N^{(m,n)} & \cdots & \varphi_{N+N-2}^{(m,n)}(m,n) & \varphi_{N+N}^{(m,n)}(m,n) \end{array} \right],
\]
(74)
where \( \varphi_r^{(m,n)}(r=1, \ldots, N) \) are given by eq.(35) solve eqs.(71) and (72).

Proof of Proposition 3.4 is given by the Casorati technique by using the linear relations (63) and (66)[9].

**Remark 3** 1. Eq.(70) follows immediately from eq.(71) by dividing the both sides by \( \tau_n^{m+1} \).

2. If we introduce the continuous independent variables \( x_1, x_2, \ldots \) through \( \varphi_{r+1}(m,n) \) as
\[
\varphi_{r+1}(m,n) = \alpha_r(p_r) \prod_{j=m}^{n-1} (1 + b_j p_r) \prod_{k=m}^{n-1} (1 + c_k p_r) e^{\beta_r x_1 + \beta_r^* x_2 + \cdots}
\]
(75)
\[
+ \beta_r (-p_r) \prod_{j=m}^{n-1} (1 - b_j p_r) \prod_{k=m}^{n-1} (1 - c_k p_r) e^{-\beta_r x_1 - \beta_r^* x_2 + \cdots},
\]
then \( \tau_n^m \) becomes the \( \tau \) function of the KdV hierarchy. In this case, \( \rho_n^m \) and \( \tilde{u}_n^m \) can be expressed as
\[
\rho_n^m = \frac{\partial \tau_n^m}{\partial x_1}, \quad \tilde{u}_n^m = \frac{\partial}{\partial x_1} \log \tau_n^m,
\]
(76)
respectively. Accordingly, \( \tilde{u}_n^m \) satisfies (68) and the potential KdV equation
\[
\frac{\partial^2 \tilde{u}_n^m}{\partial x_1^2} - \frac{3}{2} \left( \frac{\partial \tilde{u}_n^m}{\partial x_1} \right)^2 - \frac{1}{4} \frac{\partial^3 \tilde{u}_n^m}{\partial x_1^3} = 0,
\]
(77)
simultaneously. This is consistent with the fact that the potential discrete KdV equation is derived as the Bäcklund transformation of the potential KdV equation[15].
4 Non-autonomous Discrete Toda Lattice Equation

4.1 Casorati Determinant Solution

The non-autonomous discrete 1DTL equation is given by[4, 8, 12, 13, 22]
\[ A_{n+1} + B_{n+1}^+ = A_n^+ + B_{n+1}^+ + \lambda_t, \]
\[ A_{n+1}^t B_{n+1} = A_n^t B_n^t, \] (78)
where \( \lambda_t \) is an arbitrary function in \( t \). The \( N \)-soliton solution is expressed by Casorati determinants as follows[8, 22]:

**Theorem 4.1** For each \( N \in \mathbb{N} \), we define an \( N \times N \) determinant \( \tau_n^t \) by
\[
\tau_n^t = \begin{vmatrix}
\varphi_1^{(n)}(t) & \varphi_1^{(n+1)}(t) & \cdots & \varphi_1^{(n+N-1)}(t) \\
\varphi_2^{(n)}(t) & \varphi_2^{(n+1)}(t) & \cdots & \varphi_2^{(n+N-1)}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N^{(n)}(t) & \varphi_N^{(n+1)}(t) & \cdots & \varphi_N^{(n+N-1)}(t)
\end{vmatrix},
\] (79)
where \( \varphi_i^{(n)}(t) = \alpha_i p_n^0 \prod_{j=0}^{n-1} (1 - p_j \mu_j) + \beta_i p_n^n \prod_{j=0}^{n-1} (1 - p_j^{-1} \mu_j), \) (80)
\( \Psi_n^t = \frac{\theta_n^t}{\tau_n^t} \), \( A_n^t = -\mu_i^{-1} \frac{\varphi_i^{(n+1)} t_{n+1}^{p+1}}{\tau_{n+1}^{p+1} t_n^{p+1}}, \) \( B_n^t = -\mu_i^{-1} \frac{\varphi_i^{(n+1)} t_{n+1}^{p+1}}{\tau_{n+1}^{p+1} t_n^{p+1}}, \) \( \lambda_t = \mu_i + \mu_i^{-1} \), (81)
satisfy (78).

4.2 Bilinearization (I)

**Proposition 4.2** Let \( \tau_n^t \) and \( \theta_n^t \) be functions satisfying bilinear equations
\[
(1 - \delta \mu_i) \tau_n^{t+1} \theta_n^t - \tau_n^t \theta_n^{t+1} + \delta \mu_i \tau_n^{t+1} \theta_n^{t+1} = 0,
\] (82)
\[
\mu_i \tau_n^{t+1} \theta_n^{t+1} - \delta \tau_n^{t+1} \theta_n^t = (\mu_i - \delta) \tau_n^{t+1} \theta_n^{t+1}.
\] (83)
Then
\[
\Psi_n^t = \frac{\theta_n^t}{\tau_n^t}, \quad A_n^t = -\mu_i^{-1} \frac{\varphi_i^{(n+1)} t_{n+1}^{p+1}}{\tau_{n+1}^{p+1} t_n^{p+1}}, \quad B_n^t = -\mu_i^{-1} \frac{\varphi_i^{(n+1)} t_{n+1}^{p+1}}{\tau_{n+1}^{p+1} t_n^{p+1}},
\] (84)
satisfy
\[
(1 - \delta \mu_i) \Psi_n^{t+1} - \Psi_n^{t+1} - \delta B_n^t \Psi_n^{t+1} = 0,
\] (85)
\[
\mu_i \Psi_n^{t+1} + \delta \mu_i A_n^t \Psi_n^t = (\mu_i - \delta) \Psi_n^{t+1},
\] (86)
and the non-autonomous discrete 1DTL equation (78). In particular, eq.(79) and
\[
\theta_n^t = \begin{vmatrix}
\varphi_1^{(n)}(t) & \varphi_1^{(n+1)}(t) & \cdots & \varphi_1^{(n+N-1)}(t) \\
\varphi_2^{(n)}(t) & \varphi_2^{(n+1)}(t) & \cdots & \varphi_2^{(n+N-1)}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N^{(n)}(t) & \varphi_N^{(n+1)}(t) & \cdots & \varphi_N^{(n+N-1)}(t)
\end{vmatrix},
\] (87)
where

\[
\begin{align*}
\varphi_r^{(n)}(n, t) &= \alpha_r p_r^n (1 - \delta p_r) \prod_{j=t_0}^{t-1} (1 - p_r \mu_j) + \beta_r q_r^n (1 - \delta q_r) \prod_{j=t_0}^{t-1} (1 - q_r \mu_j).
\end{align*}
\]

(88)

solve the bilinear equations (82) and (83).

Proposition 4.2 can be proved by the similar technique to that in Section 3.2. Let us take \( k = l_1, t = l_2, l = m_1 \) and \( n = s \), and choose the corresponding lattice intervals as \( a_1(l_1) = -\delta, a_2(l_2) = -\mu, b_1(m_1) = \epsilon \), where \( \delta \) and \( \epsilon \) are constants. The variables \( k \) and \( l \) are the autonomous variables. We consider the discrete 2DTL equation (19) with respect to the variables \((n, l, t)\)

\[
(1 + \epsilon \mu_t)\tau_n(k, l + 1, t + 1)\tau_n(k, l, t) - \tau_n(k, l, t + 1)\tau_n(k, l + 1, t)
\]

\[= -\epsilon \mu_t \tau_{n+1}(k, l + 1, t)\tau_{n-1}(k, l, t + 1) = 0,
\]

and its Bäcklund transformation (21) with respect to the variable \((n, k, t)\)

\[
\mu_t \tau(k + 1, l, t)\tau_n(k, l, t + 1) + \delta \tau_{n+1}(k, l, t + 1)\tau_n(k + 1, l, t)
\]

\[= -(\mu_t - \delta) \tau_{n+1}(k, l, t)\tau_n(k + 1, l, t + 1) = 0.
\]

Under this setting, \( \tau \) function (10) is now written as

\[
\tau_n(k, l, t) = \begin{vmatrix}
\varphi_1^{(n)} & \varphi_1^{(n+1)} & \cdots & \varphi_1^{(n+N-1)} \\
\varphi_2^{(n)} & \varphi_2^{(n+1)} & \cdots & \varphi_2^{(n+N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N^{(n)} & \varphi_N^{(n+1)} & \cdots & \varphi_N^{(n+N-1)} \\
\end{vmatrix},
\]

(91)

\[
\varphi_r^{(n)}(k, l, t) = \alpha_r p_r^n (1 - \delta p_r)^\gamma (1 + \epsilon p_r^{-1})^\gamma \prod_{i=0}^{t-1} (1 - \mu_i p_r)
\]

\[= \beta_r q_r^n (1 - \delta q_r)^\gamma (1 + \epsilon q_r^{-1})^\gamma \prod_{i=0}^{t-1} (1 - \mu_i q_r).
\]

(92)

We impose the reduction condition on \( k, l \) as

\[
\tau_n(k + 1, l + 1, t) = \tau_n(k, l, t).
\]

(93)

This is realized by choosing the parameters of the solutions as

\[
\epsilon = -\delta, \quad q_r = \frac{1}{p_r}
\]

(94)

so that

\[
\varphi_r^{(n)}(k + 1, l + 1, t) = (1 - \delta p_r)(1 - \delta p_r^{-1})\varphi_r^{(n)}(k, l, t),
\]

(95)

\[
\tau_n(k + 1, l + 1, t) = \prod_{r=1}^{N} (1 - \delta p_r)(1 - \delta p_r^{-1}) \tau_n(k, l, t).
\]

Suppressing \( l \)-dependence by using eq. (93), and denoting

\[
\tau_n(k, l, t) = \tau'_n, \quad \tau_n(k + 1, l, t) = \theta'_n,
\]

the bilinear equations (89) and (90) are reduced to eqs. (82) and (83), respectively.
Proposition 4.3
Let autonomous discrete KdV equation discussed in Section 3.3.

4.3 Bilinearization (II)
Then the compatibility condition follows from the bilinear equations (82) and (83), respectively, through the dependent variable transformation (84). In order to obtain eq.(78), we introduce
\[
\Phi'_n = \begin{pmatrix} \Psi_{n+1} \\ \Psi_n \end{pmatrix}.
\]
After some manipulation, the linear equations (85) and (86) can be rewritten as
\[
\Phi'_{n+1} = L_n \Phi'_n, \quad \Phi'_{n} = M'_n \Phi'_n,
\]
\[
L_n = \begin{pmatrix} \delta \left[ A_{n+1} + B_{n+1} + (1 - \delta \mu) \left( \frac{1}{\mu} - \frac{1}{\delta} \right) \right] & -\delta^2 A'_n B'_{n+1} \\ -\delta A'_n & \delta^2 A'_n \end{pmatrix},
\]
\[
M'_n = \frac{1}{\mu^2 - \delta^2} \begin{pmatrix} B'_{n+1} + (1 - \delta \mu) \left( \frac{1}{\mu} - \frac{1}{\delta} \right) & \delta A'_n B'_{n+1} \\ -\delta A'_n & -A'_n \end{pmatrix}.
\]
Then the compatibility condition \( L_n^{n+1} M'_n = M'_n L_n^{n+1} \) gives eq.(78). This completes the proof of Proposition 4.2.

4.3 Bilinearization (II)

There is an alternate bilinearization for eq.(78), which is similar to the second bilinearization of the non-autonomous discrete KdV equation discussed in Section 3.3.

Proposition 4.3 Let \( \tau'_n \) and \( \eta'_n \) be functions satisfying the bilinear equations
\[
\tau'_{n+1} \tau'_{n-1} - \tau'_{n} \eta'_n = \mu_\tau \tau_{n-1} \left( \tau'_{n-1} \tau'_{n+1} - \tau'_{n} \eta'_n \right),
\]
\[
\mu_\eta \eta'_n \tau'^{n+1}_{n+1} - \mu_\tau \eta'_n \tau'^{n+1}_{n} = (\mu_\tau - \mu_\mu) \tau'^{n+1}_{n} \tau'^{n+1}_{n+1}.
\]
Then \( A'_n \) and \( B'_n \) defined in eq.(84) satisfy eq.(78). In particular, eq.(79) and
\[
\eta'_n = \begin{pmatrix} \psi_0^{(n)}(t) & \psi_1^{(n+1)}(t) & \cdots & \psi_{N}^{(n+N-1)}(t) \\ \psi_0^{(n)}(t) & \psi_2^{(n+1)}(t) & \cdots & \psi_{2N}^{(n+N-1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0^{(n)}(t) & \psi_{N}^{(n+1)}(t) & \cdots & \psi_{N}^{(n+N-1)}(t) \end{pmatrix},
\]
\[
\psi^{(n)}_r(t) = \alpha_r \beta_r^{(n)}(1 - p_r \mu) \prod_{j=0}^n (1 - p_r \mu) + \beta_r \beta_r^{(n)}(1 - p_r^{-1} \mu) \prod_{j=0}^{n-1} (1 - p_r^{-1} \mu),
\]
solve eqs.(101) and (102).

Eq. (78) is derived from the bilinear equations (101) and (102) as follows: multiplying \( (1 - \frac{1}{\mu_\tau \mu_\eta}) \tau'^{n+1}_{n} \tau'^{n+1}_{n+1} \) to eq.(102) and using eq.(101) we have
\[
(\tau'^{n+1}_{n+1} \tau'^{n+1}_{n+1} - \mu_\tau \tau'_{n-1} \tau'^{n+1}_{n} \tau'^{n+1}_{n+1} - \mu_\tau \tau'_{n} \tau'^{n+1}_{n} \tau'^{n+1}_{n+1})
= (\lambda - \lambda_{n-1}) \tau'^{n+1}_{n+1} \tau'^{n+1}_{n+1} \tau'^{n+1}_{n+1}.
\]
Dividing equation (105) by \( \tau'^{n+1}_{n+1} \tau'^{n+1}_{n+1} \tau'^{n+1}_{n+1} \), we obtain the first equation of eq. (78). The second equation is an identity under the variable transformation (84).

Proposition 4.3 is proved by applying the Casoratian technique based on the linear relations among the entries of the determinant
\[
\psi^{(n)}_r(t + 1) = \phi^{(n)}_r(t) - \mu \psi^{(n+1)}_r(t),
\]
\[
\psi^{(n)}_r(t) = \phi^{(n)}_r(t - 1) - \mu \psi^{(n+1)}_r(t - 1),
\]
\[
(1 - p_r \mu)(1 - p_r^{-1} \mu) \psi^{(n)}_r(t - 1) = \psi^{(n)}_r(t) - \mu \psi^{(n+1)}_r(t).
\]
We refer to [8] for further details of the proof.
Remark 4 Recently Tsujimoto has presented a theoretical background of appearance of the auxiliary $\tau$ function $\eta$ in this section by considering the two-dimensional chain of the Darboux transformations[22].

5 Non-autonomous discrete Lotka-Volterra Equation

5.1 Lotka-Volterra Equation

The Lotka-Volterra equation

$$\frac{d}{dt} \log u_n = u_{n+1} - u_{n-1}, \quad (109)$$

can be transformed to the bilinear equation

$$(D_x + 1)\tau_{n+1} \cdot \tau_{n} = \tau_{n-1} \tau_{n+2}, \quad (110)$$

through the dependent variable transformation

$$u_n = \frac{\tau_{n-1} \tau_{n+2}}{\tau_{n+1} \tau_n} = \frac{d}{dt} \log \frac{\tau_{n+1}}{\tau_n} + 1. \quad (111)$$

The $N$-soliton solution to eq. (109) is given by

$$\phi^{(a)}_{k} = \alpha_{k} (1 + r_{k})^{n} e^{(1+r_{k})^{n}} + \beta_{k} \left(1 + \frac{1}{r_{k}}\right)^{n} e^{1 + \frac{1}{r_{k}}^{n}}. \quad (113)$$

where $\alpha_{k}, \beta_{k}$ are arbitrary constants and $r_{k}$ are parameters ($k = 1, \ldots, N$).

The Lotka-Volterra equation is reduced from the Bäcklund transformation of the 2DTL equation (20). Let us impose a reduction condition for $\tau_{n}(m)$ and $\phi^{(a)}_{k}(m)$ given in eqs.(22) and (25), respectively:

$$\tau_{n}(m+1) = \tau_{n+1}(m), \quad \phi^{(a)}_{k}(m+1) = \phi^{(a+1)}_{k}(m), \quad (114)$$

The condition (114) is achieved by putting

$$b_{m} = -b, \quad q_{k} = -\frac{p_{k}}{1 - \frac{p_{k}}{b}}. \quad (115)$$

or

$$p_{k} = b(1 + r_{k}), \quad q_{k} = b \left(1 + \frac{1}{r_{k}}\right). \quad (116)$$

Then eq.(20) is rewritten as

$$(D_x + b) \tau_{n} \cdot \tau_{n+1} = b \tau_{n+2} \tau_{n-1}. \quad (117)$$

Noticing that $b$ can be normalized to be 1 without loss of generality, we obtain the bilinear equation (110) and its Casorati determinant solution (112) and (113).
\section*{5.2 Non-autonomous Discrete Lotka-Volterra Equation}

The discrete Lotka-Volterra equation\cite{5, 6} can be derived by discretizing the independent variable $x$ in the Bäcklund transformation of 2DTL equation, which implies that it can be formulated as the reduction from the discrete 2DTL equation itself. The reduction procedure works well also for non-autonomous case without auxiliary $	au$ function, as shown below.

We consider the non-autonomous discrete 2DTL equation (89) with $l = l_1, m = m_1, n = s, a_l = a_1(l_1), b_m = -b_1(m_1)$:

\begin{equation}
(1 + a_l b_m)\tau_n(l + 1, m + 1)\tau_n(l, m) - \tau_n(l + 1, m)\tau_n(l, m + 1) = a_l b_m \tau_{n+1}(l, m + 1)\tau_{n-1}(l + 1, m),
\end{equation}

where the $\tau$ function is given by

\begin{equation}
\tau^m_n = \begin{bmatrix}
\varphi^{(n)}_1 & \varphi^{(n+1)}_1 & \cdots & \varphi^{(n+N-1)}_1 \\
\varphi^{(n)}_2 & \varphi^{(n+1)}_2 & \cdots & \varphi^{(n+N-1)}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(n)}_N & \varphi^{(n+1)}_N & \cdots & \varphi^{(n+N-1)}_N
\end{bmatrix}.
\end{equation}

Then \( \varphi^{(n)}_k(l, m) = \alpha_k p_k^n \prod_{i=l_0}^{l-1} (1 + a_l p_k) \prod_{j=m_0}^{m-1} (1 - b_j q_k^{-1}) \)

\begin{equation}
+ \beta_k q_k^n \prod_{i=l_0}^{l-1} (1 + a_l q_k) \prod_{j=m_0}^{m-1} (1 - b_j q_k^{-1}).
\end{equation}

We impose the reduction condition,

\begin{equation}
\tau_n(l, m + 1) = \tau_{n+1}(l, m), \quad \varphi^{(n)}_k(l, m + 1) = \varphi^{(n+1)}_k(l, m),
\end{equation}

which is achieved by putting

\begin{equation}
b_m = 1, \quad p_k = 1 + r_k, \quad q_k = 1 + \frac{1}{r_k}.
\end{equation}

Then \( \varphi^{(n)}_k \) is written as

\begin{equation}
\varphi^{(n)}_k = \alpha_k (1 + q_k) \prod_{i=l_0}^{l-1} (1 + a_l + a_l r_k) + \beta_k \left( 1 + \frac{1}{r_k} \right) \prod_{i=l_0}^{l-1} (1 + a_l + a_l r_k).
\end{equation}

Now suppressing $m$-dependence and writing \( \tau_n(l, m) = \tau^*_n \), the bilinear equation (118) is reduced to

\begin{equation}
(1 + a_l) \tau^*_{n+1} = \tau^*_{n} - \tau^*_{n+1} = a_l \tau^*_{n+2} \tau^*_{n-1},
\end{equation}

from which we obtain the non-autonomous discrete Lotka-Volterra equation

\begin{equation}
\frac{1}{1 + a_l} \frac{\tau^*_{n+1}}{\tau^*_{n}} = \frac{1}{1 + a_l} \frac{\tau^*_{n}}{\tau^*_{n+1}},
\end{equation}

through the dependent variable transformation

\begin{equation}
\frac{\tau^*_{n}}{\tau^*_{n+1}} = \frac{\tau^*_{n+1}}{\tau^*_{n}}.
\end{equation}

Eq.(124) is equivalent to the generalization of the discrete Lotka-Volterra equation in \cite{4, 18, 19, 20}. Further generalization of eq.(124) is proposed in \cite{18} which corresponds to de-autonomization of $n$. 

15
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