Critical Layers in Shear Flows

By S. A. Maslowe *

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Abstract

The normal mode approach to investigating the stability of a parallel shear flow involves the superposition of a small wavelike perturbation on the basic flow. Its evolution in space and/or time is then determined. In the linear inviscid theory, if $\bar{u}(y)$ is the basic velocity profile, then a singularity occurs at critical points y_c, where $\bar{u} = c$, the perturbation phase speed. This is plausible intuitively because energy can be exchanged most efficiently where the wave and mean flow are travelling at the same speed. The problem is of the singular perturbation type; when viscosity or nonlinearity, for example, are restored to the governing equations, the singularity is removed. In this lecture, the classical viscous theory is first outlined before presenting a newer perturbation approach using a nonlinear critical layer (i.e., nonlinear terms are restored within a thin layer). The application to the case of a density stratified shear flow is discussed and, finally, the results are compared qualitatively with radar observations and also with recent numerical simulations of the full equations.

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1 Introduction

In the classical approach to investigating the stability of a parallel shear flow $\bar{u}(y)$, a small perturbation is superimposed on the mean flow and the equations governing this perturbation are then linearized. If the flow is two dimensional and incompressible, it is convenient to employ a stream function $\psi(x, y)$ related to the horizontal and vertical velocity components by $(u, v) = (\psi_y, -\psi_x)$. The mean and fluctuating part of the stream function are separated by writing

$$\psi(x, y, t) = \bar{\psi}(y) + \varepsilon\hat{\psi}(x, y, t)\,, \tag{1}$$

where $\varepsilon << 1$ is a small dimensionless amplitude parameter.

The basic equation describing the evolution of the flow is the vorticity equation which can be written

$$\omega_t + \psi_y\,\omega_x - \psi_x\,\omega_y = Re^{-1}\,\nabla^2\omega, \tag{2}$$

where the vorticity $\omega = -\nabla^2\psi$ and Re is the Reynolds number. Substituting (1) into (2) leads to the PDE governing the evolution of the perturbation $\hat{\psi}$, namely,

$$\hat{\omega}_t + \bar{u}\,\hat{\omega}_x + \bar{u}''\hat{\psi}_x + \varepsilon(\hat{\psi}_y\,\hat{\omega}_x - \hat{\psi}_x\,\hat{\omega}_y) = Re^{-1}\,\nabla^2\hat{\omega}\,, \tag{3}$$

where $\hat{\omega} = -\nabla^2\hat{\psi}$. In the normal mode approach, the variables are separated by writing $\hat{\psi} = \phi(y)\exp\{i\alpha(x - ct)\}$ and ϕ satisfies the Orr-Sommerfeld equation

$$(\bar{u} - c)(\phi'' - \alpha^2\phi) - \bar{u}''\phi = \frac{1}{i\alpha Re}\,(\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi)\,. \tag{4}$$

In the classical theory, the wavenumber α is real, whereas c is complex and $\alpha\,c_i$ is the amplification factor of an unstable perturbation. On a solid boundary, both ϕ and ϕ' must vanish, whereas exponential decay is usually imposed if the flow is unbounded.

For a bounded flow, such as Poiseuille flow in a channel, the modal solutions are complete and the linear problem is solved. However, in other cases, there is also a continuous spectrum, so we will say a few words on that topic. First, let us suppose that $Re >> 1$, as it is in most important applications, and we can then neglect the viscous terms on the right-hand side of (4). The result of doing this is the Rayleigh equation and, for many problems (e.g., an unbounded mixing layer), the Rayleigh equation yields the most important features of the stability problem. However, for flows with no inflection point in the velocity profile, such as Couette flow or Poiseuille flow, there are no inviscid modes and a more general approach is required. (The case of Couette flow is discussed in the first lecture of Prof. Llewellyn Smith.)

The most general approach to linear stabiliy would be to solve (3) with $\varepsilon = 0$ by taking a Fourier transform in x and a Laplace transform in t. However, the essential features are associated with the Laplace transform inversion, so we may write $\hat{\psi} = \exp(i\alpha x)\Phi(y, t)$ and substitute this into (3). The equation for Φ can be solved approximately by first taking the Laplace transform in time and then solving the resulting ODE to determine the variation in y. Finally, asymptotic methods can be used to invert the transform and it is found typically that $\hat{\psi} \sim O(t^{-2})$ if there are no normal modes. This algebraic decay is the outcome of a branch cut emanating from a singular point analogous to the normal mode critical point to be discussed below.

There is also in the case of a boundary layer, for example, a continuous spectrum associated with the Orr-Sommerfeld Eq. (4). Such solutions are required to be bounded in the free stream. They, in fact, turn out to be oscillatory rather than to decay exponentially like normal modes. As a consequence, their magnitude is greater near the edge of the boundary layer and this property has led to suggestions that they play a role in subcritical transition (i.e., transition to turbulence at Reynolds numbers below critical). It has long been known that turbulence in the free stream can induce boundary layer transition and Zaki & Durbin (2006) have shown in numerical simulations how the continuous spectrum can be used to model this free-stream turbulence.

2 Asymptotic solution of the Orr-Sommerfeld eq.

In this section, the Orr-Sommerfeld theory for high Reynolds numbers is reviewed briefly in order to gain some historical perspective. At the same time, we can set the stage for presenting below the newer, nonlinear critical layer approach and its application to stratified shear flows. To begin, we suppose that the solution of (4) can be expressed as a power series in powers of $\delta = (\alpha Re)^{-1}$. The lowest-order term in the expansion, $\phi^{(0)}$, satisfies the Rayleigh equation, i.e., (4) with the right hand side equal to zero. The Rayleigh equation provides an adequate representation of the solution everywhere except near a solid boundary or at a critical point y_c, where $\bar{u} = c$. The method of Frobenius can be used to express the solution of $\phi^{(0)}$ as a linear combination of the two power series

$$\phi_A = (y - y_c) + \frac{\bar{u}_c''}{2\bar{u}_c'}(y - y_c)^2 + \cdots \quad \text{and} \quad \phi_B = 1 + \cdots + \frac{\bar{u}_c''}{\bar{u}_c'}\,\phi_A \log(y - y_c) + \cdots \tag{5}$$

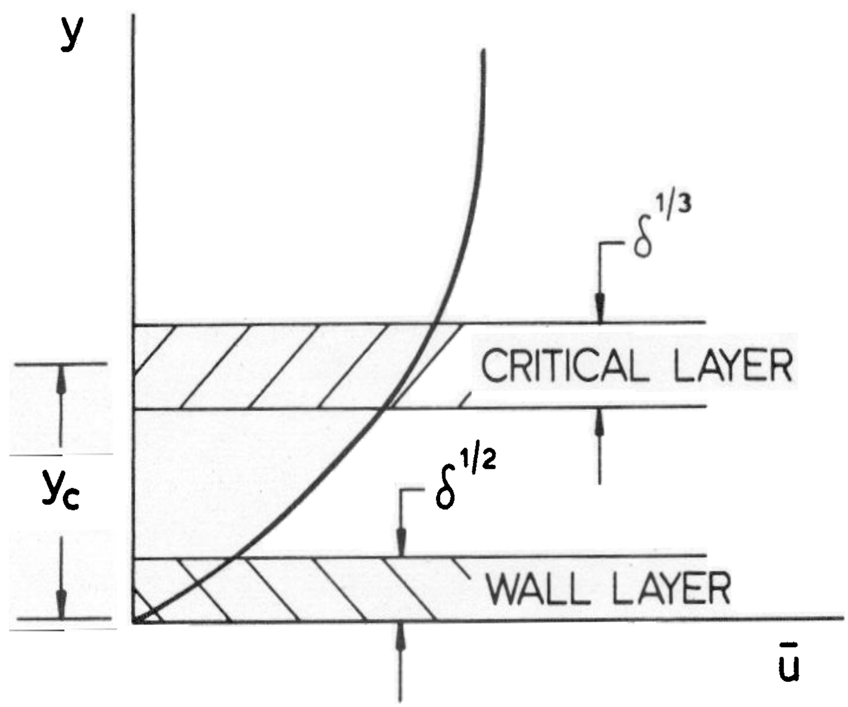

Figure 1: Boundary layer profile showing location of viscous layers.

The logarithmic singularity in ϕ_B leads to two difficulties in the case of a neutral or nearly-neutral mode. (Note that these series solutions are valid even for $c_i \neq 0$, in which case, the critical point is off the real axis.) First, the horizontal perturbation velocity is proportional to ϕ', which becomes unbounded as $y \to y_c$. Secondly, the eigenvalue problem associated with Rayleigh's equation cannot be solved until it is decided how to write the log term in ϕ_B when $y < y_c$. An asymptotic analysis of (4) employing a viscous critical layer (see Fig. 1) shows that for $y < y_c$, we must write $\log(y - y_c) = \log|y - y_c| - i\,\pi$ (if $\bar{u}'_c > 0$). One says, in that case, that there is a "$-\pi$ phase change" across the critical layer. This causes a jump in the Reynolds stress $\tau \equiv -\varrho\overline{u'v'}$ that leads to the celebrated Tollmien-Schlichting mechanism of instability. Miles (1957) employed this same mechanism in his theory for the generation of water waves by wind.

3 Stability of stratified shear flows

A stratified shear flow can be thought of, in mathematical terms, as the flow of an incompressible fluid of variable density. The inviscid governing equations are the vorticity equation and a second equation requiring that the density of an individual fluid particle remains constant. These equations can be written

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{u} + \frac{1}{\rho^2}(\nabla\rho \times \nabla p) \quad \text{and} \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla\rho = 0 \; . \qquad (6).$$

Denoting the stream function and density perturbations $\hat{\psi}$ and $\hat{\rho}$, respectively, the two-dimensional linearized vorticity equation can be written

$$\nabla^2\hat{\psi}_t + \bar{u}\,\nabla^2\hat{\psi}_x - \bar{u}''\hat{\psi}_x - \frac{g}{\bar{\rho}}\,\hat{\rho}_x = 0\,, \qquad (7)$$

where $\bar{u}(y)$ and $\bar{\rho}(y)$ are the velocity and density profiles of the mean flow. An approximation similar to the Boussinesq approximation has been made in deriving (7) from the momentum equations. Specifically, derivatives of the density ρ have been neglected except in that term where g, the gravitational constant, appears. Separating variables now, we again let $\hat{\psi} = \phi(y)\exp\{i\alpha(x - ct)\}$ and, in addition, $\hat{\rho} = P(y)\exp\{i\alpha(x - ct)\}$. From the second of eqs. (6), after linearizing and employing normal modes, we obtain

$$P = \frac{\bar{\rho}'}{(\bar{u} - c)}\,\phi \qquad (8)$$

and, after substituting into (7), ϕ satisfies the Taylor-Goldstein equation

$$\frac{d^2\phi}{d\,y^2} - \left[\alpha^2 + \frac{\bar{u}\,''}{(\bar{u}-c)} - \frac{\bar{r}\,'J_0}{(\bar{u}-c)^2}\right]\phi = 0\ . \tag{9}$$

The overall Richardson number is defined by $J_0 = gL/V^2$ and $\bar{r}\,' = -d(log\bar{\rho})/dy$.

The Miles-Howard theorem is the best known result of the linear stability theory, i.e., the theory associated with (9). Specifically, Miles(1961) demonstrated that a necessary condition for instability is that the local Richardson number $J(y) = g\bar{r}\,'/\bar{u}\,'^2$ be somewhere less than 1/4. His proof was limited to monotonic velocity profiles, but was generalized by Howard (1961) to include non monotonic profiles such as jets.

Miles used Frobenius expansions near the critical point to derive a number of important results, including the Richardson number 1/4 theorem. Following his approach and notation, all variable coefficients in (9) are expanded around the critical point y_c to obtain a solution valid locally having the form

$$\phi(y) = A\,\phi_+(y) + B\,\phi_-(y)\,, \tag{10}$$

where

$$\phi_\pm(y) = (y-y_c)^{\frac{1}{2}\,(1\pm\nu)}\,w_\pm(y) \tag{11}$$

and the functions $w_\pm(y)$ are regular in the neighborhood of y_c; the parameter ν in (11) is related to J_c by $\nu = (1-4\,J_c)^{1/2}$.

Using arguments based on the variation of the Reynolds stress, Miles proved a number of useful results that apply to singular neutral modes. For example, within the framework of linear theory, a neutral mode comprising part of a stability boundary must be proportional to one or the other of the Frobenius solutions. With the exception of profiles that are specially constructed to avoid dealing with critical points, there is a $-\pi$ phase change as y_c is crossed and this is true whether the initial-value approach is used or diffusive effects are restored within a critical layer.

A closed form neutral solution that illustrates many of the theorems proved by Miles was found by Hølmboe (unpublished lecture notes) for the velocity and density profiles

$\bar{u} = \tanh y$ and $\bar{\rho} = e^{-\beta \tanh y}$. His solution for the eigenvalue relation has $c = 0$ and $J_0 = \alpha(1-\alpha)$. Instability occurs beneath this parabola in the (J, α) plane, whose maximum is at $J_0 = J_c = \frac{1}{4}$ and $\alpha = \frac{1}{2}$. The corresponding eigenfunction consistent with a linear critical layer would be

$$\phi(y) = \begin{cases} (\operatorname{sech} y)^{\alpha} (\tanh y)^{1-\alpha}, & y > 0 \\ (\operatorname{sech} y)^{\alpha} |\tanh y|^{1-\alpha} \, e^{-i\pi(1-\alpha)}, & y < 0. \end{cases}$$

The critical layer branch point at $y_c = 0$ is evident and it can be easily determined by comparison with (11) that ϕ is proportional to ϕ_+ for $0 \leq \alpha \leq \frac{1}{2}$ and to ϕ_- for $\frac{1}{2} \leq \alpha \leq 1$.

4 Nonlinear critical layers

From the basic equations in §1, it can be seen that the Rayleigh equation results when in Eq. (3) the two small parameters ε and $\delta = (\alpha \, Re)^{-1}$ are set to zero and normal modes are then used to separate variables. The large Reynolds number asymptotic theory is obtained by first setting $\varepsilon = 0$ in (3) and then separating variables to obtain the Orr-Sommerfeld equation. A generalization that we mention, in passing, is to employ a *weakly nonlinear* theory. In that approach, $\hat{\psi}$ is expanded in powers of ε and the perturbation amplitude satisfies a nonlinear evolution equation. Some of the deficiencies of linear theory (such as the outcome being independent of the initial perturbation amplitude) can be remedied by such an approach. Again, viscosity is employed to deal with critical point singularities that arise at each order. It will be seen below that this probably explains why weakly nonlinear analyses are less successful in treating flows where there are critical layers than they are in dealing with problems having no critical layer, such as Bénard convection.

In this section, we present a very different treatment of the critical layer by noting that even if the viscous terms on the right side of (3) are neglected, there will be no singularity provided that the nonlinear terms multiplied by ε are retained. An asymptotic normal mode approach based on this observation was first formulated by Benney & Bergeron (1969).

Using matched asymptotic expansions, it develops that an inviscid nonlinear critical layer of thickness $O(\varepsilon^{1/2})$ is appropriate and, because the approach is nonlinear, it is convenient to introduce a total stream function

$$\psi = \int_{y_c}^{y} (\bar{u} - c)\, dy + \varepsilon\, \hat{\psi}(\xi, y)\,, \tag{12}$$

where c is the phase speed, $\xi = \alpha x$ and the flow is steady in a coordinate system travelling at speed c. Expanding $(\bar{u} - c)$ in a Taylor series near y_c and noting that according to (5), $\hat{\psi} \sim O(1)$ as $y \to y_c$, we see that the mean flow and perturbation are both $O(\varepsilon)$. It is therefore appropriate to define inner variables Y and Ψ as follows:

$$y - y_c = \varepsilon^{1/2} Y \quad \text{and} \quad \psi(\xi, y) = \varepsilon\, \bar{u}_c' \Psi(\xi, Y)\,.$$

Employing these variables now in the vorticity equation (2), the governing equation in the critical layer takes the form

$$\Psi_Y \Psi_{YY\xi} - \Psi_\xi \Psi_{YYY} + O(\varepsilon) = \lambda\, \Psi_{YYYY}\ , \tag{13}$$

where $\lambda \equiv 1/(\alpha\, Re\, \varepsilon^{3/2})$. The parameter λ is seen to be a measure of the ratio of the two critical layer thicknesses, i.e., $\lambda^{1/3} = \delta_{visc}/\delta_{NL}$ and we are interested here in the case $\lambda \ll 1$.

Although the details of the nonlinear critical layer theory are too involved for presentation here, we can still outline the analysis and state the most significant results. The most successful applications of this theory have been to geophysical shear flows because the Reynolds numbers are so large. For example, in the context of clear air turbulence, a typical value for Re is of order 10^6, so it is clear that unless ε is truly infinitesimal, the parameter λ is in the nonlinear critical layer regime $\lambda \ll 1$. In engineering applications, on the other hand, λ is typically $O(1)$ so the value of the theory is more in the insights that it provides. Nonetheless, the analysis for the case of a homogeneous shear flow will be outlined below both for these insights and because it is tractable. The results for the stratified case can then be, at least understood and appreciated, after comparing with those for the homogeneous flow.

To begin, we observe that to lowest order in ε, the solution to (13) satisfying the matching condition to the outer expansion is simply

$$\Psi^{(0)} = \frac{Y^2}{2} + \cos\xi \ . \tag{14}$$

Remarkably, the solution (14) applies even when $\lambda \sim O(1)$, i.e., the case where both viscosity and nonlinearity are significant. The streamline pattern associated with (14) is known as the Kelvin cat's-eye configuration and it is illustrated in Fig. 2.

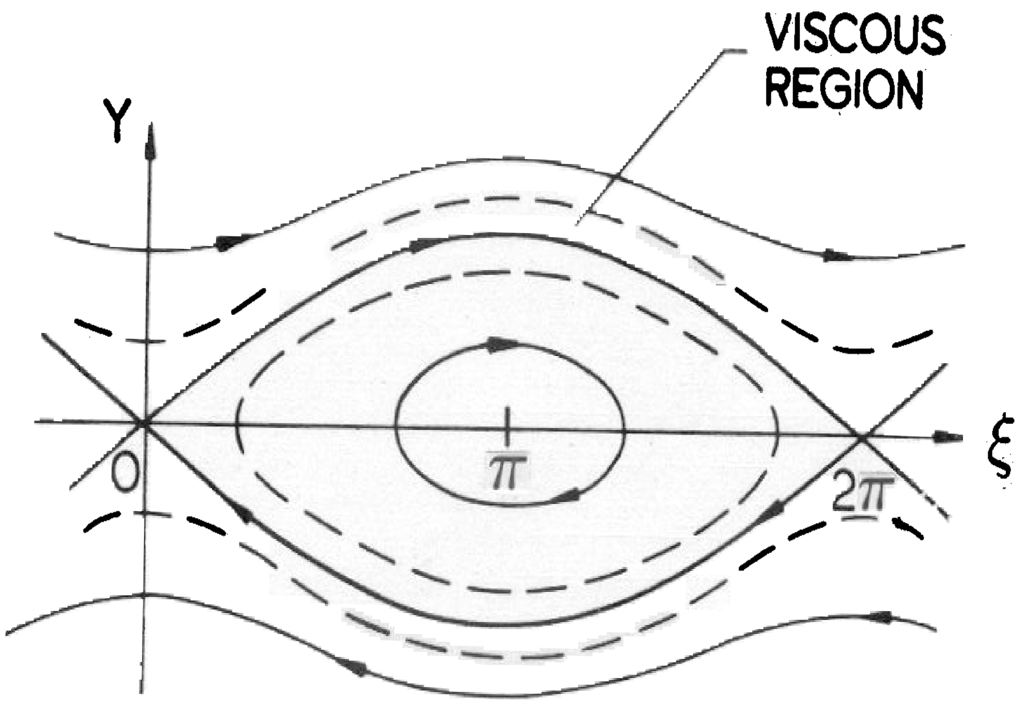

Figure 2: Streamline pattern in the nonlinear critical layer.

The phase change across the critical layer is determined at $O(\varepsilon^{1/2})$ by matching the outer solution to $\Psi^{(1/2)}$, the $O(\varepsilon^{1/2})$ term in the expansion of Ψ. This can be seen by writing the log term in (5) as $\log|y - y_c| + i\,\theta_R$ for $y < y_c$, where θ_R is termed the phase change. Although the PDE satisfied by $\Psi^{(1/2)}$ is linear, finding a solution continuous throughout the critical layer (i.e., as $|Y| \to \pm\infty$) proves to be a formidable task. First, all harmonics of the fundamental perturbation become of the same order of magnitude. Solutions outside of the closed streamline region can be found as integrals, but these cannot be matched to the solution inside where, according to the Prandtl-Batchelor theorem, the vorticity must be a constant. To smooth out discontinuities in vorticity along the critical streamline $\Psi^{(0)} = 1$, viscous shear layers of thickness $O(\lambda^{1/2})$ must be included, as indicated in Fig. 2.

Once a solution having both continuous vorticity and velocity has been found, matching to the linear, inviscid outer flow leads to the conclusion that the only solutions compatible with a nonlinear critical layer must have *zero* phase change. As a result, new solutions to the Rayleigh equation exist and these were computed for various flows by Benney & Bergeron. These neutral mode solutions often can be found in regions of parameter space where linear modes would be damped. This property may make them especially pertinent in geophysical applications, as discussed below.

To conclude this outline of the nonlinear critical layer theory, we say a few words about extensions of the idea to stratified shear flows. What makes the analysis more difficult in the case of a stratified flow is that, according to (11), the branch point singularity in ϕ is algebraic rather than logarithmic. Moreover, the density(see (8)) and horizontal velocity perturbations are even more singular, behaving, for example, as $(y - y_c)^{-\frac{1}{2}}$ when $J_c = 1/4$. One consequence of this is that in the critical layer all the harmonics are the same order of magnitude as the fundamental disturbance mode.

Fortunately, it is still possible to make some progress analytically even though the results are less complete than those for the homogeneous case. Utilizing a von Mises transformation, whereby ξ is replaced by Ψ as an independent variable, the nonlinear critical layer equations at zeroth order can be integrated to obtain

$$\Theta = F(\Psi) \quad \text{and} \quad \Psi_{YY} = J_c F' Y + G(\Psi)\,, \tag{14}$$

where Θ is the scaled temperature (or, equivalently, the density in the Boussinesq approximation). The critical layer thickness in the stratified case is ε^p, where $p = \frac{2}{3}$ if $J_c \geq \frac{1}{4}$ and p decreases from $\frac{1}{2}$ to $\frac{2}{3}$, as J_c increases from 0 to $\frac{1}{4}$; the scaling for the stream function and temperature is, respectively, $\psi = \varepsilon^{2p} \bar{u}'_c \Psi$ and $T - \bar{T}_c = \varepsilon^p \bar{T}'_c \Theta$.

The basic flow structure turns out to be similar to that illustrated in Fig. 2, but certain features are more striking. First, the streamline pattern closely resembles the cat's-eye configuration except that there are cusps at the corners, where the critical streamlines meet. Inside, where there are closed streamlines, the temperature, as well as the vorticity must be constant for a steady, stratified flow. Again, thin diffusive layers along the critical streamlines must be added, where viscosity and heat-conduction are included. Although discontinuities in velocity and temperature are smoothed out in these layers, the local Richardson number can be very small and small-scale instabilities may result. There is radar evidence, however (see Fig. 3 below), that the large scale coherence of the wave can still be maintained despite the presence of localized turbulence.

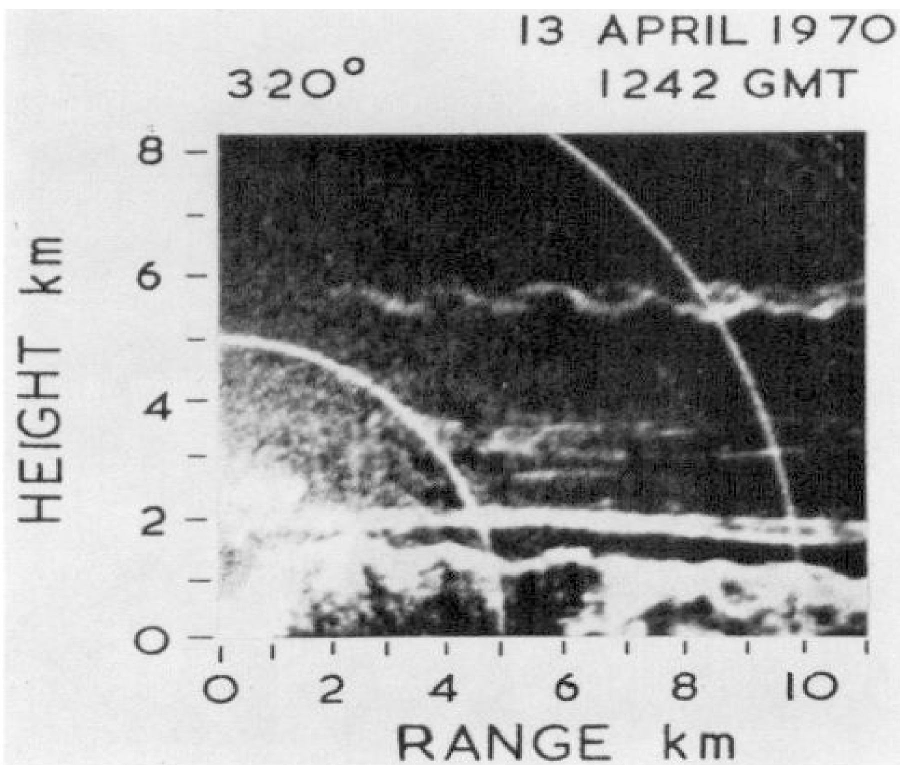

Figure 3: Radar observation of a Kelvin-Helmholtz billow at 5.6 km altitude.

Interestingly, it is the thermal boundary layers, required by the asymptotic matching, that render these "Kelvin-Helmholtz billows" observable to sensitive radars. The greatest utility of the foregoing theory, however, is arguably in numerical simulations where structural details first revealed by the critical layer analysis did not appear in actual computations until the Ph. D. thesis of Patnaik (1973). These numerical simulations illustrating the fine-scale diffusive structure were published in Patnaik, Sherman & Corcos (1976), although the comparisons with theory contained in Patnaik's thesis were omitted.

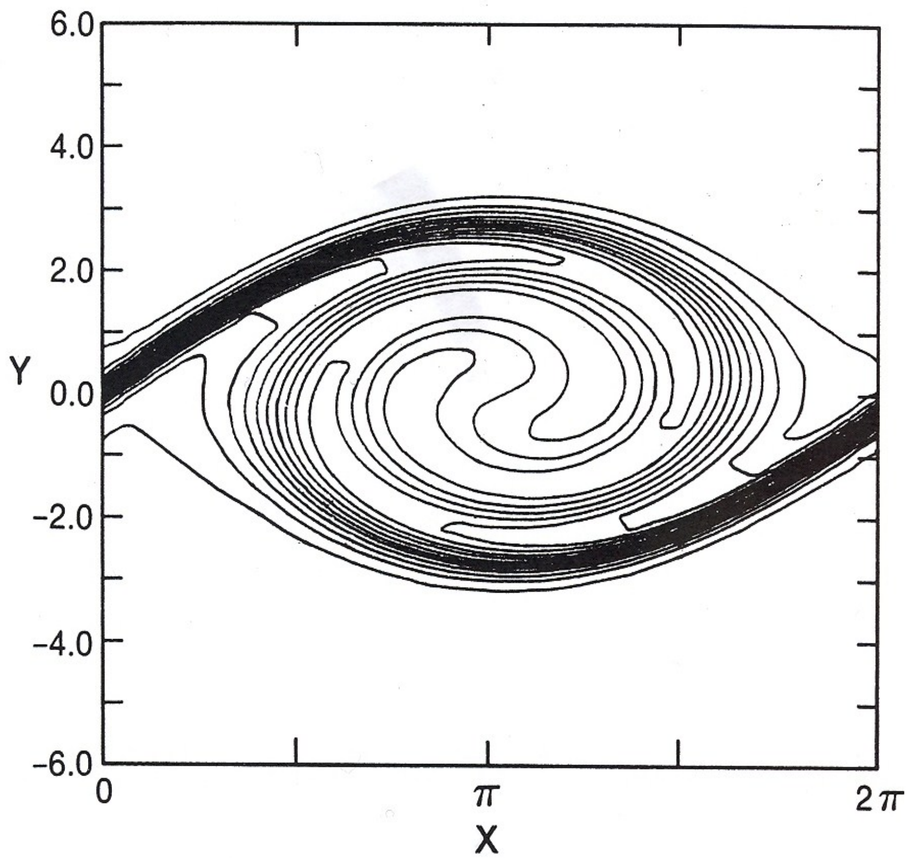

Figure 4: Pseudospectral simulation of a Kelvin-Helmholtz billow with $J_0 = 0.10$ and $Re = 200$; the contours shown are isopycnics (i.e., constant-density contours).

The radar observations and the simulations of Patnaik et al. generated interest in the question of localized instabilities within the critical layer. Striking examples of these "braid instabilities" are illustrated in the high Reynolds number simulations reported by Staquet (1995) done at $J_0 = 0.167$; Sec. 4.6 of her paper discusses the relationship between the computed structures (which evolve in time) and the steady nonlinear critical layer theory. Both convective and shear instabilities were observed in Staquet's simulations, with the initial conditions determining the outcome. From the nonlinear analysis, it is clear that many Fourier modes (at least 64) are required in pseudospectral simulations and in the vertical coordinate a critical layer whose thickness can be as small as $O(\varepsilon^{2/3})$must be adequately resolved. Indeed, as many as 1536 modes were employed by Staquet (1995), enabling instabilities to be observed that were absent in earlier simulations performed by other researchers at lower Reynolds number.

References

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Resonant interactions in shear flows

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Abstract

The theory of weakly nonlinear resonant wave interactions was applied in the 1960s to water waves. It is not often recognized that much more dramatic instabilities can occur in the presence of a shear flow, because all modes can amplify by extracting energy from the basic mean flow. In this talk, the foregoing idea will be employed to propose a mechanism for generating subcritical nonlinear critical layer modes; i.e., neutral modes that cannot be explained by linear theory because their viscous counterparts would be damped. The problem of Rossby waves propagating in a mixing layer with velocity profile $\bar{u}(y)$ will be utilized to illustrate the theory. The beta parameter, which is a measure of the stabilizing Coriolis force, is taken to be large enough so that linear instability cannot occur. Then, full numerical simulations are carried out to illustrate how nonlinear critical layer modes can be generated by resonant interaction with ordinary Rossby waves, even when the singular mode is absent initially.

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1 Introduction

Weakly nonlinear theories can be formulated systematically by employing a perturbation scheme in which the dependent variables are expanded in powers of ε, a small dimensionless amplitude parameter. We suppose that the system of governing equations can be expressed in the form

$$\mathcal{L}\mathbf{u} = \varepsilon \mathcal{N}\mathbf{u} \ , \tag{1}$$

and the linearized problem is obtained by setting $\varepsilon = 0$. If the linear problem admits dispersive wave solutions, these will be proportional to $A_i \exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\}$, where the frequency and wavenumber are related through the dispersion relation $\omega = W(\mathbf{k})$.

Let us now consider the interaction of a set of three such wavetrains by writing

$$u = \sum_{n=1}^{3} A_i(X,T) \exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\} + A_i^*(X,T) \exp\{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)\} \, , \tag{2}$$

where $X = \varepsilon x$ and $T = \varepsilon t$ are slow space and time scales. The nonlinear terms $\mathcal{N}$ in (1) are usually quadratic so that a sum or difference between two of the waves in (2) may be equal to the third member of the triad. In that case, resonance is possible; the resonance conditions are often written

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0 \ \text{ and } \ \omega_1 \pm \omega_2 \pm \omega_3 = 0 \ . \tag{3}$$

O. M. Philips(1960) is usually credited with first formulating a resonant interaction theory along the foregoing lines and during the next 20 years, this was a very active area of research. A nice survey of this work can be found in Philips(1981). However, the above necessary conditions for resonance are not sufficient for the case of water waves, the application that had motivated Philips. There, four waves are necessary, the development must be carried out to higher order and the time scale for the interaction is $\varepsilon^2 t$. Because we are most interested in the application to waves in shear flows, the work of L. G. McGoldrick, who had been a student of Philips, turns out to be most pertinent. McGoldrick's research was on capillary-gravity waves and it develops that interacting triads are obtained when the effect of surface tension is included.

A special case, that is important, can be best used to illustrate the theory and this case is termed second harmonic resonance. It has all the features of triad resonance, but is simpler because only two waves are involved. McGoldrick (1972) used the following model equation to illustrate the theory:

$$L\{u\} = u_{tt} - u_{xx} + u + \frac{1}{4}\, u_{xxxx} = 3\,\varepsilon\, u^2\ . \tag{4}$$

Suppose that we try to find a solution of (4) by a straightforward power series in ε. At zeroth order, the solution of the linear problem can be written

$$u^{(0)} = A\ \exp\{i(k\,x - \omega t)\} + A^*\ \exp\{-i(k\,x - \omega t)\},\ \text{where}\ \omega(k) = \pm(1 + \frac{1}{2}\,k^2)$$

is the dispersion relation. The $O(\varepsilon)$ term in the expansion must satisfy $L\{u^{(1)}\} = 3\,(u^{(0)})^2$, whose solution can be written

$$u^{(1)} = \frac{3}{\{[\omega(2k)]^2 - 2[\omega(k)]^2\}}\{A^2 e^{2i(k\,x-\omega t)} + (A^*)^2 e^{-2i(k\,x-\omega t)}\} + 6\,A\,A^*\,.$$

Clearly, resonance occurs when $\omega(2k) = 2\,\omega(k)$.

Whether treating capillary-gravity waves or shear flows, the method of multiple scales provides a systematic framework to analyze resonant interactions. The relevant slow scales are $X = \varepsilon x$ and $T = \varepsilon t$, so that in the governing equations we transform derivatives according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\,\frac{\partial}{\partial \tau}\ \ \text{and}\ \ \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\,\frac{\partial}{\partial X}\ .$$

To deal with the case of second harmonic resonance, the basic perturbation must include both modes, so we write

$$u^{(0)} = \sum_{n=1}^{2} A_n(X,T)\,e^{i\,n(kx-\omega t)} + A_n^*(X,T)\,e^{-i\,n(kx-\omega t)}\ . \tag{5}$$

To separate variables now in the $O(\varepsilon)$ problem, it is found that $u^{(1)}$ must be of the form

$$u^{(1)} = A_2 A_1^*\,e^{i\,(kx-\omega t)} + A_1^2\,e^{2i(kx-\omega t)} + \text{complex conjugates} + \text{nonsecular terms}\ .$$

In order for the expansion of u to be well ordered, the so-called secular terms must be eliminated and this requires the amplitudes to satisfy the following evolution equations:

$$\frac{\partial\,A_1}{\partial \tau} + \omega'\frac{\partial\,A_1}{\partial X} = i\,\gamma_1\,A_2 A_1^*\quad\text{and}\quad \frac{\partial\,A_2}{\partial \tau} + \omega'\frac{\partial\,A_2}{\partial X} = i\,\gamma_2\,A_1^2\,. \tag{6}$$

It is informative to derive an energy integral from Eqs. (6) by neglecting the variation in X and forming expressions for $\frac{d|A_i|^2}{d\tau}$. These expressions, for $i = 1$ and 2, can be combined and integrated with respect to τ to obtain

$$|A_1|^2 + \frac{\gamma_1}{\gamma_2}\,|A_2|^2 = E\,. \tag{7}$$

In a conservative system, γ_1 and γ_2 will both have the same sign, so that the total energy is shared by the two waves. For example, in the model Eq. (4), $\gamma_1 = \frac{3}{\omega}$ and $\gamma_2 = \frac{3}{4\,\omega}$.

What is significant (and not generally realized) in shear flow stability problems is that it is possible for γ_1 and γ_2 to have opposite signs, in which case, *both* waves can amplify. While this seems counter-intuitive, the explanation is simply that both waves can amplify by extracting energy from the mean flow. To take into account the mean flow energy, it is necessary to go one step further in the perturbation expansion. The monograph by Craik (1985) treats this in detail (see, in particular, Sections 17.2 and 26.1, where it is explained that the energy is transferred to the perturbations in the vicinity of the critical layer).

Much of Craik's own research dealt with a special triad configuration that is possible in both boundary layers and mixing layers. Specifically, a plane wave is employed along with a pair of subharmonic oblique waves inclined at equal and opposite angles (slightly less than 60°) to the flow direction. The frequency of the oblique waves is half that of the plane wave and, because the conditions for resonance are satisfied exactly, all modes share a common critical layer. This sort of triad was found by Liu & Maslowe (1999) to be extremely effective in the case of an adverse pressure gradient boundary layer.

2 Resonance of two modes in a stratified mixing layer

As an application of the foregoing theory, we return to Hølmboe's mixing layer model. The stability boundary is given by $J_0 = \alpha(1 - \alpha)$ and because of the symmetry of the profiles, $c = 0$ along this boundary. The conditions for second harmonic resonance are satisfied at a Richardson number $J_0 = \frac{2}{9}$ for the two modes with wavenumbers $\alpha = \frac{1}{3}$ and $\alpha = \frac{2}{3}$.

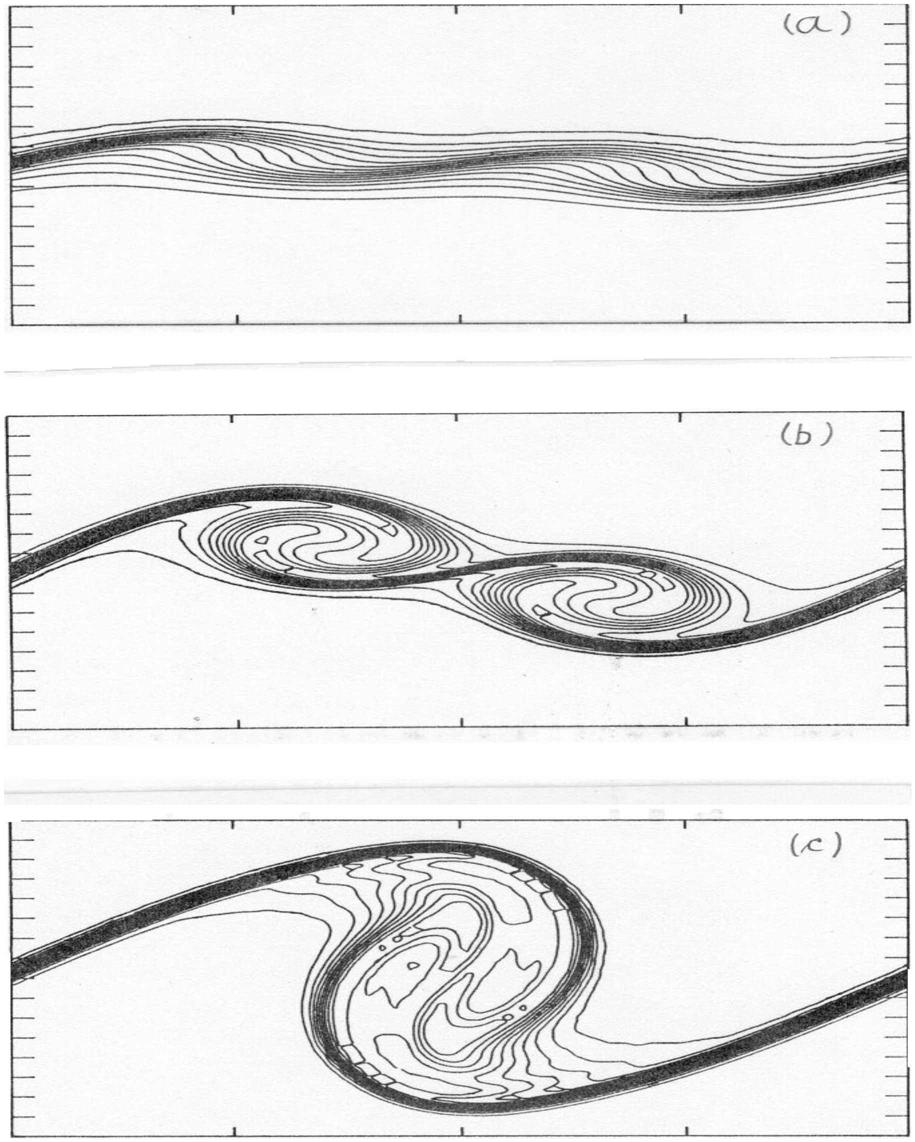

Figure 1: Evolution of constant-density contours for two waves, $\alpha = 0.215$ and 0.43, with $J_0 = 0.07$ and $Re = 200$: the times are (a) $t = 16$, (b) $t = 32$ and (c) $t = 48$.

The results shown in Fig. 1 above are from the paper by Collins & Maslowe (1988). As discussed therein, the amplitude equations (6) can be generalized to include weakly amplified modes by adding a term proportional to A on the right hand side. Because the phase speed $c_r = 0$, even for unstable modes, any two waves for which $\alpha_2 = 2\,\alpha_1$ will interact resonantly. In the paper by Collins and the author, results are reported primarily for $0.07 \leq J_c \leq 0.174$. For Richardson numbers in the lower part of this range, the streamlines can be said to depict the phenomenon of vortex pairing, familiar in homogeneous mixing layers. As J_c increases, the pairing is less energetic and at $J_c = 0.14$ a sort of limiting case is reached in which the vortex on the left rises only slightly and the one on the right descends a small amount. There is nonetheless a strong interaction because the equilibrium amplitude is 35 times as large as that attained by the single most amplified wave when the subharmonic is absent.

3 Resonant interactions in zonal shear flows

Atmospheric observations of the instability of zonal currents have motivated numerous studies of the barotropic stability characteristics of such flows. The theory is relevant to the oceans, as well, with a number of investigations motivated by Gulf Stream phenomena. We consider here perturbations to the zonal shear flow $\bar{u} = \tanh y$. The basic flow is to the east (x-direction), y is the north-south coordinate, and variations in the vertical are neglected. Kuo (1973) in a comprehensive survey article presents observational data for wind profiles over both the Atlantic and Pacific oceans which are well approximated by the hyperbolic tangent function.

The governing equation of the linear, inviscid theory is the Rayleigh-Kuo equation

$$(\bar{u} - c)(\phi'' - \alpha^2 \phi) + (\beta - \bar{u}'')\phi = 0 \quad . \tag{8}$$

This equation is the analogue of the Taylor-Goldstein equation in the sense that it is Rayleigh's equation with an additional term representing a stabilizing influence. Here, it is the Coriolis force resulting from planetary rotation rather than buoyancy; this influence is modelled in (8) by a linearization about some mean latitude and β is the derivative of the Coriolis parameter (assumed constant). The properties of (8) are well-known, the most significant being the generalization of Rayleigh's inflection point theorem stating that the quantity $(\beta - \bar{u}'')$ must change sign at some value of y for instability to occur.

Neutral mode solutions of (8) are usually regular due to the vanishing of the absolute vorticity $(\beta - \bar{u}'')$ at the critical point y_c, where $\bar{u} = c$. However, unlike the case $\beta = 0$, it is possible to have neutral "radiating modes"; for an unbounded, monotonic velocity profile such modes are singular and they decay exponentially on one side of the shear layer, but are oscillatory on the other side, where a boundary condition is imposed on the group velocity to ensure outward energy propagation.

We restrict attention here though to the more conventional "trapped modes" whose eigenfunctions decay exponentially to zero as $|y| \to \infty$. The linear, neutral solution was obtained in closed form by Howard & Drazin (1964). The eigenvalue condition relating the phase speed, wavenumber and beta parameter is given by

$$c^2 = 1 - \alpha^2 \quad \text{and} \quad \beta = -2c(1 - c^2). \tag{9}$$

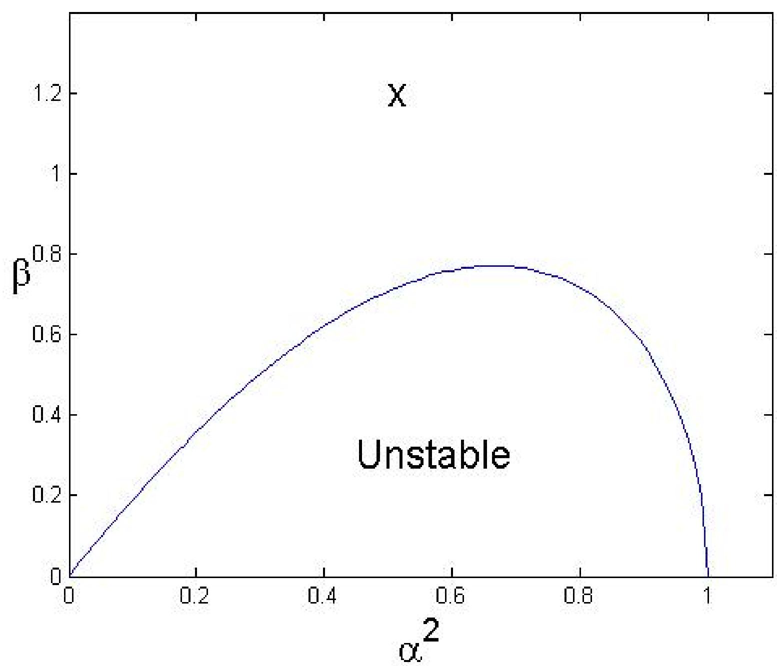

Figure 2: Stability diagram for the zonal shear flow $\bar{u} = \tanh y$.

As shown in Fig. 2, the primary effect of rotation is stabilizing and the range of unstable wavenumbers decreases with increasing β until the critical value $\beta = 4/3^{3/2}$ is reached;

above this value the flow is stable on a linear basis. We should add that the solution (9) applies also to the velocity profile $\bar{u} = -\tanh y$; in that case, which corresponds to the observations cited by Kuo, the positive root would be taken for c and β would be negative.

We return now to the subject of subcritical neutral modes with nonlinear critical layers. Within the framework of linear theory, such singular modes cannot exist for $\beta > 4/3^{3/2}$ (i.e., at a point such as X in Fig. 2). The reason is that the boundary conditions are not compatible with a jump in the Reynolds stress that would be the outcome of a phase change across the critical point y_c. It was shown, however, by Maslowe & Clarke (2002) that singular neutral modes can be obtained when there is no phase change and dispersion curves were computed for $\beta = 3$. The critical point singularity in the Rayleigh-Kuo equation (8) is of the same form as that for the Rayleigh equation. Therefore, in solving the eigenvalue problem, we simply write $\log(y - y_c) = \log|y - y_c|$ and integrate numerically from a small distance on either side of y_c to the boundaries.

Having shown that nonlinear neutral modes are possible mathematically, a mechanism for generating them must be found if they are to be of physical significance. The mechanism we employ here is that of resonant interaction with Rossby waves. The latter are nonsingular and they are modified only quantitatively by the shear; dispersion curves for each mode are computed easily by solving (8) numerically. The simulation is initiated using only the two Rossby waves, each of which in the case $\bar{u} = \tanh y$ belongs to a different mode. The frequencies and wavenumbers are chosen so that the triad resonance conditions (3) are satisfied with the third member being a singular neutral mode. Dispersion curves with the resonant values of ω and α are illustrated in Fig. 1 of Maslowe & Clarke.

One motivation for this approach is the proof by Becker & Grimshaw (1993) that in the similar problem of internal gravity wave interaction in a stratified flow, a necessary condition for explosive instability is that at least one mode must have a critical layer. This idea has been pursued in the present context by Vanneste (1998), who formulated a finite amplitude theory. In his analysis, the singular mode is represented by the continuous spectrum, a procedure quite different from that used in the numerical simulations of Maslowe & Clarke.

In fig. 3, the total vorticity contours are shown for a numerical simulation of the inviscid, nonlinear barotropic vorticity equation. Initially, only two Rossby waves are present, but at later times the nonlinear critical layer mode is generated, as is clear from the propagating (blue) cats-eyes pattern.

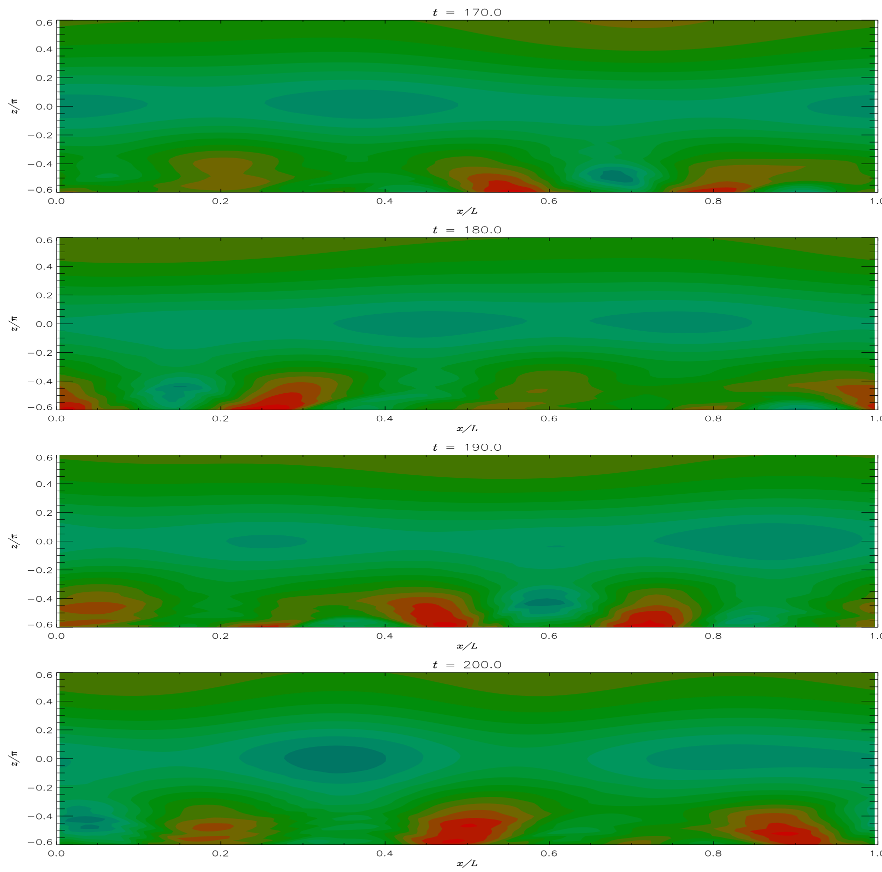

Figure 3: Formation of a nonlinear singular mode with $\alpha = 1.2$ as a result of resonant interaction with Rossby modes having wavenumbers $\alpha = 0.6$ and $\alpha = 1.8$. The basic zonal shear flow is the mixing layer $\bar{u} = \tanh y$ and the Coriolis parameter $\beta = 3$.

One important effect that is absent in the theory, but was very noticeable in the numerical simulations is a strong variation in the mean flow during the triad evolution. Very rapid oscillations were observed in the $\alpha = 0$ component of our pseudospectral computation. The linear shear profile $\bar{u} = y$ was also investigated, but the nonlinear critical layer mode generation was less evident for that case.

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