A set of variant Hermite tetrahedral elements for three-dimensional problems
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Abstract.
We present a set of variant Hermite tetrahedral elements of degree three for three-dimensional problems. A finite element space constructed from these elements has advantages that the degrees of freedom are much smaller than those of the Lagrange element and that it is easily applicable to problems subject to Dirichlet boundary conditions. Applying it to Poisson problems, we prove best possible a priori error estimates. Two numerical examples reflect the theoretical results.

Keywords. variant Hermite elements, tetrahedral elements, a priori error estimates

1. Introduction
The finite element method is one of the most powerful techniques for numerical solutions of partial differential equations. Many kinds of elements have been developed for various problems [2], [3]. Here we focus on the Lagrange element and the Hermite element of degree three. Both elements have a high approximation property. While the former is easily applied to problems subject to Dirichlet boundary conditions, the latter has an advantage that the degrees of freedom (DOF) are much smaller than those of the former. In order to gain both advantages we have presented a set of variant triangular Hermite elements for two-dimensional problems [6]. Those elements connect the Lagrange element and the Hermite element.

In this paper we extend the result to three-dimensional problems and present a set of variant Hermite tetrahedral elements. After discussing the interpolation property, we apply it to Poisson problems and prove best possible a priori error estimates in $H^1$- and $L^2$-norms.

The contents of this paper are as follows. In Section 2 we prepare some definitions and assumptions. In Section 3 we present a set of variant Hermite tetrahedral elements and discuss the interpolation error. In Section 4, applying it to Poisson problem, we give a priori error estimates. Numerical results are shown in Section 5 and some remarks are given in Section 6.

2. Preliminaries
Let $\Omega$ be a convex polyhedral domain in $\mathbb{R}^3$ with boundary $\Gamma$. Let us consider a simplicial decomposition of $\Omega$ composed of elements $\{K\}$. Each $K$ is an image of the reference element $\hat{K} = [\hat{a}_1, \cdots, \hat{a}_4] = [e_1, e_2, e_3, 0]$, shown in Fig. 1, by an affine mapping $F_K : \hat{K} \rightarrow \mathbb{R}^3$. We set $h_K = \text{diam} \hat{K}$, $\rho_K = \sup\{\text{diam} S; S$ is a ball contained in $K\}$, and $h = \max_K h_K$. We deal with a family of simplicial decompositions of $\Omega$, $\{T_h\}_{h \downarrow 0}$. Let us assume the followings:

(T1) All nodes on the boundary of the polyhedron $\Omega_h \equiv (\cup_{K \in T_h} K)^2$ are on $\Gamma$ and $\Omega = \Omega_h$.
(T2) There exists a constant $\sigma_1 > 0$ such that for all $h$ and all $K \in T_h$,
\begin{equation}
    h_K \leq \sigma_1 \rho_K.
\end{equation}
(T3) There exists a constant $\sigma_2 > 0$ such that for all $h$ and all $K \in T_h$,
\begin{equation}
    h_K \geq \sigma_2 h.
\end{equation}

Let $L^2(\Omega)$ and $H^m(\Omega)$, $m \geq 1$, be the Sobolev spaces with norms denoted by $\| \cdot \|_{0,\Omega}$ and $\| \cdot \|_{m,\Omega}$, respectively. We also use the semi-norms $| \cdot |_{m,\Omega}$. The symbol $(\cdot, \cdot)$ is used for the inner product in $L^2(\Omega)$. Similarly the Sobolev spaces $L^2(\Gamma)$ and $H^m(\Gamma)$, $m \geq 1$, and their norms and semi-norms are defined. The symbol $[\cdot, \cdot]$ is used for the inner product in $L^2(\Gamma)$. We denote by $v_i$ the derivative of a function $v$.
with respect to $x_i$ for $i = 1, 2, 3$. Throughout this paper, we use $C$ and $c$ as generic numerical positive constants independent of $h$, which may take a different value at each occurrence.

3. A SET OF VARIANT HERMITE ELEMENTS

In this section, after reviewing the conventional Lagrange and Hermite elements [3], we introduce a set of variant tetrahedral elements for three-dimensional problems. As for a set of variant triangular elements for two-dimensional problems we refer to [6].

3.1.LAGRANGE AND HERMITE ELEMENTS

We denote by $\mathcal{P}_3$ and $\mathcal{H}_3$ the Lagrange element and the Hermite element of degree three, respectively. Let $\mathcal{P}_3$ denote the space of polynomials of degree $\leq 3$. Let us set $\hat{a}_{ij} = \frac{1}{2} (\hat{a}_i + \hat{a}_j) \ (1 \leq i, j \leq 4, i \neq j)$ and $\hat{a}_{ijk} = \frac{1}{4} (\hat{a}_i + \hat{a}_j + \hat{a}_k) \ (1 \leq i < j < k \leq 4)$. We let the components of a point or a vector $a_i \in \mathbb{R}^3$ as $\{a_{ij}^{(l)}\}_{1 \leq m \leq 3}$. Let $\{\lambda_i\}_{i=1}^4$ be the system of barycentric coordinates of $\hat{K}$. We define four index sets by

\begin{align*}
(3a) & \quad I_v^0 = \{i; \; 1 \leq i \leq 4\}, \\
(3b) & \quad I_E = \{(i, j); \; 1 \leq i \leq 4, 1 \leq j \leq 4, i \neq j\}, \\
(3c) & \quad I_F = \{(i, j, k); \; 1 \leq i < j < k \leq 4\}, \\
(3d) & \quad I_v^1 = \{(i, m); \; 1 \leq i \leq 4, 1 \leq m \leq 3\}.
\end{align*}

The numbers of DOF in $\mathcal{P}_3$ and $\mathcal{H}_3$ on $\hat{K}$ are 20, and their $\mathcal{P}_3$-unisolvent sets are

\[ \hat{\Sigma}^L = \{ p(\hat{a}_i); \; i \in I_v^0 \} \cup \{ p(\hat{a}_{ij}); \; (i, j) \in I_E \} \]

\[ \quad \cup \{ p(\hat{a}_{ijk}); \; (i, j, k) \in I_F \}, \]

\[ \hat{\Sigma}^H = \{ p(\hat{a}_i); \; i \in I_v^0 \} \cup \{ p(\hat{a}_{ijk}); \; (i, j, k) \in I_F \} \]

\[ \quad \cup \{ p_m(\hat{a}_i); \; (i, m) \in I_v^1 \}. \]

Let $\hat{\Phi}^L$ and $\hat{\Phi}^H$ be sets of functions defined by

\[ \hat{\Phi}^L = \left\{ \phi_i^{0,L}; \; i \in I_v^0 \right\} \cup \left\{ \phi_{ijk}^{0,L}; \; (i, j) \in I_E \right\} \]

\[ \quad \cup \left\{ \phi_{ijk}^{0,L}; \; (i, j, k) \in I_F \right\}, \]

\[ \hat{\Phi}^H = \left\{ \phi_i^{0,H}; \; i \in I_v^0 \right\} \cup \left\{ \phi_{ijk}^{0,H}; \; (i, j, k) \in I_F \right\} \]

\[ \quad \cup \left\{ \phi_{im}^{1,H}; \; (i, m) \in I_v^1 \right\}, \]

where

\[ \phi_i^{0,L} = \frac{1}{2} \lambda_i (3\lambda_i - 1)(3\lambda_i - 2), \]

\[ \phi_{ijk}^{0,L} = \frac{9}{2} \lambda_i \lambda_j (3\lambda_i - 1), \quad \phi_{ijk}^{0,H} = 2\lambda_i \lambda_j \lambda_k, \]

\[ \phi_i^{0,H} = -2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{1 \leq j < k \leq 4} \lambda_j \lambda_k. \]

The following lemma has been proven on these elements.

**Lemma 1.** (Ciarlet-Raviart [4]) $\hat{\Sigma}^L$ and $\hat{\Sigma}^H$ are $\mathcal{P}_3$-unisolvent, and $\hat{\Phi}^L$ and $\hat{\Phi}^H$ form the basis functions, respectively.

Let $\Pi_K^L : C^0(\hat{K}) \rightarrow \mathcal{P}_3(\hat{K})$ be the Lagrange interpolation operator and $\Pi_K^H : C^1(\hat{K}) \rightarrow \mathcal{P}_3(\hat{K})$ be the Hermite one. The interpolation error estimates for general $\mathcal{P}_3$-interpolations $\Pi_K$ onto $\mathcal{P}_3(\hat{K})$ have been given in [3], [4].

**Lemma 2.** (Ciarlet-Raviart [4]) Let $(\hat{K}, \Sigma_K, \hat{\Phi}_K)$ be a finite element of 3-simplex where $\Sigma_K$ is $\mathcal{P}_3$-unisolvent. Let $\Pi_K^{(l)} : C^l(\hat{K}) \rightarrow \mathcal{P}_3(\hat{K})$ denote its interpolation, where $l = 0, 1$ is the maximum order of derivatives occurring in the definition of $\Sigma_K$. Let $q$ be an integer with $2 + l \leq q \leq 4$. Under the assumptions (T1) and (T2), it holds that, for $v \in H^q(\hat{K})$,

\[ \| v - \Pi_K^{(l)} v \|_{m,K} \leq C h^{q-m} \| v \|_{q,K} \quad (m = 0, \ldots, q). \]

Therefore, the interpolation estimates for the Lagrange and Hermite elements are

\[ \| v - \Pi_K^L v \|_{m,K} \leq C h^{q-m} \| v \|_{q,K} \quad (q = 2, 3, 4), \]

\[ \| v - \Pi_K^H v \|_{m,K} \leq C h^{q-m} \| v \|_{q,K} \quad (q = 3, 4) \]

for $v \in H^q(\hat{K})$ and $m \in [0, q]$.

3.2. VARIANT HERMITE ELEMENTS

We consider finite element computations of partial differential equations subject to Dirichlet boundary conditions, employing $\mathcal{P}_3$-elements. In the case of the Hermite element there are four DOF at each vertex, one function value and three derivatives. At vertices on the boundary three of them, one function value and two tangential derivatives, are essential boundary conditions and the last one is a natural boundary condition. (In polyhedral domains the fourth DOF may also become a tangential derivative at corner nodes. Then, all DOF are essential boundary conditions.) In the case of the Lagrange element, all DOF are function values, and they are essential boundary conditions on all boundary nodes, which makes the programming much easier. On the other hand the total number of DOF of the Hermite element is far smaller than that of the Lagrange element as shown in Tables 1 and 3.

Here we introduce variant Hermite elements, which connect the Lagrange element and the Hermite element. Thus we can use the Lagrange element in the simplices intersecting $\Gamma$ and the Hermite element in an interior part of $\Omega$. Let $a_i$ be an internal vertex of $\partial \Omega$ ($a_i \notin \Gamma$), and $a_j$ be any vertex connected to $a_i$. We collect all DOF, $\{p(a_{ij})\}_{j}$, of the Lagrange type at $a_{ij} \equiv (2a_i + a_j)/3$, and replace
them by Hermitian type DOF, \(\{p_m(a_i)\}_{m=1}^3\) (see Fig. 2).

We number local indices of the vertices so that

\[ I = \{(i,j); 1 \leq i \leq 4 - s, 1 \leq j \leq 4, i \neq j\}, \]

\[ I^r = \{(i,m); 5 \leq s \leq i \leq 4, 1 \leq m \leq 3\}. \]

We note that \(I_E(0) = I_E\) and \(I^r(4) = I^r\), where \(I_E\) and \(I^r\) are index sets defined in (3b) and (3d), respectively. We number local indices of the vertices so that \(a_i \in I\) \((1 \leq i \leq 4 - s)\) and \(a_i \in \Omega\) \((5 \leq s \leq i \leq 4)\). Let us define a set of variant elements \(L(1 - \frac{s}{4})H(\frac{s}{4}) (0 \leq s \leq 4)\) on \(K\), where \(s = 0\) and \(s = 4\) correspond to the Lagrangian element and the Hermite one, respectively. The pairs of DOF and basis functions are defined by

\[
\left(\Sigma(\frac{s}{4}), \Phi(\frac{s}{4})\right) \equiv \left\{(p(\hat{a}_i), \phi_i^0); i \in I^r\right\}
\cup \left\{(p(\hat{a}_{ij}), \phi_{ij}^0); (i, j) \in I_E(s)\right\}
\cup \left\{(p(\hat{a}_{ijk}), \phi_{ijk}^0); (i, j, k) \in V_F\right\}
\cup \left\{(p_m(\hat{a}_i), \phi_{im}^0); (i, m) \in I^r(s)\right\}
\]

where

\[
\phi_i^0 = \begin{cases} 
\phi_i^{L,0} - \frac{1}{s} \sum_{j=1}^{s-1} \phi_{jj}^{L,0} & (1 \leq i \leq 4 - s), \\
\phi_i^H, & (5 - s \leq i \leq 4), 
\end{cases}
\]

\[
\phi_{ij}^0 = \begin{cases} 
\phi_{ij}^{L,0} & (1 \leq i, j \leq 4 - s, i \neq j), \\
\phi_{ij}^H & (1 \leq i \leq 4 - s, 5 \leq j \leq 4), 
\end{cases}
\]

\[
\phi_{ijk}^0 = \phi_{ijk}^H,
\]

\[
\phi_{im}^0 = \phi_{im}^L - \frac{1}{s} \sum_{j=1}^{s-1} (\hat{a}_{ij}^{(j)} - \hat{a}_m^{(i)}) \phi_{jj}^L.
\]

The tetrahedral elements \(L(1 - \frac{s}{4})H(\frac{s}{4})\) are shown in Fig. 3. We have the following property whose proof is given in Appendix A.1.

**Lemma 3.** For \(s \in [0, 4]\), \(\Sigma(\frac{s}{4})\) is \(P_3\)-unisolvent and \(\Phi(\frac{s}{4})\) forms the basis functions.

### 3.3. Interpolation error analysis

We consider the \(L(1 - \frac{s}{4})H(\frac{s}{4})\) finite element, \(s \in [1, 3]\). Let \(N_0^s\) and \(N_1^s\) be the numbers of nodes in \(K\), where function values and derivative values are imposed,

\[
(\Sigma_s, \Phi_s) \equiv \{(p(\hat{a}_i^0), \phi_i^0); 1 \leq i \leq N_0^s\}
\cup \{(p_m(\hat{a}_i^1), \phi_{im}^1); 1 \leq i \leq N_1^s, 1 \leq m \leq 3\}.
\]

Let \(\hat{\Pi} : C^1(K) \rightarrow P_3(K)\) be the interpolation operator defined by, for \(\hat{v} \in C^1(K)\),

\[
\hat{\Pi} \hat{v} = \sum_{i=1}^{N_0^s} \hat{v}(\hat{a}_i^0) \phi_i^0 + \sum_{i=1}^{N_1^s} \sum_{m=1}^{3} \hat{v}_m(\hat{a}_i^1) \phi_{im}^1.
\]

The set of DOF and bases in an element \(K\) is denoted by

\[
(\Sigma_K, \Phi_K) \equiv \{(p(a_i^0), \phi_i^{0,K}); 1 \leq i \leq N_0^s\}
\cup \{(p_m(a_i^1), \phi_{im}^{1,K}); 1 \leq i \leq N_1^s, 1 \leq m \leq 3\},
\]

Figure 2: The nodes around a vertex.
(a) left: Lagrangian type (b) right: Hermitian type

Figure 3: The family of the variant tetrahedral elements.
where the nodes and basis functions are defined by
\begin{align}
(15) & \quad a_i^{0,K} = F_K(a_i^0), \quad a_i^{1,K} = F_K(a_i^1), \\
(16) & \quad \phi_i^{0,K} = \phi_i^0 \circ F_K^{-1}, \\
(17) & \quad \phi_{lm}^{1,K} = \sum_{i=1}^3 (F_K)_{m,i}(a_i^1) \hat{\phi}_{il}^1 \circ F_K^{-1}.
\end{align}

The local interpolation in $K$, $\Pi_K : C^1(K) \to \mathcal{P}_3(K)$ is defined by, for $v \in C^1(K)$,
\begin{align}
(18) & \quad \Pi_K v = \sum_{i=1}^{N_0^K} v(a_i^0) \phi_i^0 + \sum_{i=1}^{N_3^K} v_m(a_i^1) \phi_{im}^1.
\end{align}

Then it holds that, for $v \in C^1(K)$,
\begin{align}
(19) & \quad (\Pi_K v) \circ F_K = \hat{\Pi}(v \circ F_K),
\end{align}

which is easily proven by substituting (15)–(17) into (18).

From Lemma 3, we have interpolation error estimates corresponding to (4):
\begin{corollary}
Let $1 \leq s \leq 3$ be an integer and $K$ be an $L((1 - \frac{1}{s})H(\frac{s}{s}))$ element. Then, under the assumptions (T1) and (T2) it holds that, for $v \in H^s(K)$ and $q = 3, 4$,
\begin{align}
(20) & \quad \|v - \Pi_K v\|_{m,K} \leq Ch_q^{s-m} [v]_{q,K} \quad (m = 0, \ldots, q).
\end{align}

Let $N_D = N_D^0 + 3N_D^{(1)}$ be the total numbers of DOF, where $N_D^0$ and $N_D^{(1)}$ are the number of function value DOF and the number of sets of derivative value DOF, respectively.

We denote the nodes in $T_h$ by $\{a_j^{0,h}\}_{j=1}^{N_D^0}$ (of function values) and by $\{a_j^{1,h}\}_{j=1}^{3N_D^{(1)}}$ (of derivatives), where $a_j^{1,h}$ is satisfied for $m = 1, 2$ and $j = 1, \ldots, N_D^{(1)}$.

Let $j$ be the local number of the node such that $a_j^{0,h} = \hat{a}_j^0$ or $a_j^{1,h} = \hat{a}_j^1$. The corresponding basis functions $\{\phi_j^{0,h}\}_{j=1}^{N_D^0}$ and $\{\phi_j^{1,h}\}_{j=1}^{3N_D^{(1)}}$ are defined by
\begin{align}
(21) & \quad \phi_j^{0,h} = \begin{cases} \phi_j^{0,K}, & (a_j^{0,h} \in \mathcal{E}K), \\
0 & \text{(otherwise)} \end{cases} \\
& \quad \phi_j^{1,h} = \begin{cases} \phi_j^{1,K}, & (a_j^{1,h} \in \mathcal{E}K), \\
0 & \text{(otherwise)} \end{cases}
\end{align}

for $j = 1, \ldots, N_D^0$, and by
\begin{align}
(22) & \quad \phi_{j+1,m}^{1,h} = \begin{cases} \phi_{j}^{1,K}, & (a_{j,m}^{1,h} \in \mathcal{E}K), \\
0 & \text{(otherwise)} \end{cases}
\end{align}

for $j = 1, \ldots, N_D^{(1)}$ and $l = 1, 2, 3$. It can be easily shown that, for $j = 1, \ldots, N_D^0$ and $m = 1, 2, 3$,
\begin{align}
\phi_j^{0,h}(a_j^{0,h}) = d_{ij}, & \quad (i = 1, \ldots, N_D^0), \\
\phi_j^{0,h}(a_1^{1,h}) = 0, & \quad (i = 1, \ldots, N_D^{(1)}),
\end{align}

and that, for $j = 1, \ldots, N_D^{(1)}$ and $m = 1, 2, 3$,
\begin{align}
\phi_{j+1,m}^{1,h}(a_{j+1,m}^{1,h}) = 0, & \quad (i = 1, \ldots, N_D^{(1)}), \\
(\phi_{j+1,m}^{1,h})_{m,j}^{1,h} = d_{ij} d_{lm}, & \quad (i = 1, \ldots, N_D^{(1)}).
\end{align}

We define the HP3V element space $X_h$ by
\begin{align}
(23) & \quad X_h = \left\{ \sum_{j=1}^{N_0^h} c_j^0 \phi_j^{0,h} + \sum_{j=1}^{3N_1^{(1)}} c_j^{1,h} \phi_j^{1,h} : c_j^0, c_j^{1,h} \in \mathbb{R} \right\},
\end{align}

which is shown to be $H^1$-conforming.

**Lemma 4.** It holds that
\begin{align}
(24) & \quad \phi_{l,m}^{1,K} = \sum_{i=1}^3 e_i(l) \phi_{il}^{1,K}.
\end{align}

On the surface $L$ it holds that, for $i \neq i_*$,
\begin{align}
(25) & \quad \phi_{l,i_*,L}^{0,K} = \phi_{l,i_*,L}^{0,K} = \phi_{l,i_*k}^{0,K} = \phi_{l,i_*k}^{0,K} = \phi_{l,i_*k}^{0,K} = 0.
\end{align}

Here, the total number of bases vanishing on $L$ is 10. Since any function in the space $\mathcal{P}_3(L)$ is uniquely determined by the rest 10 DOF, $v_h|_L$ is determined by them. This shows the continuity of the function $v_h$ on $L$.

The HP3V interpolation $\Pi_h : C^1(\Omega) \cap C^0(\Omega) \to X_h$ is defined by
\begin{align}
(26) & \quad \Pi_h v = \sum_{i=1}^{N_0^h} v(a_i^{0,h}) \phi_i^{0,h} + \sum_{i=1}^{N_1^{(1)}} \sum_{m=1}^3 v_m(a_i^{1,h}) \phi_{i,m}^{1,h} - \sum_{(m-1)N_D^{(1)}}^{3N_D^{(1)}}.
\end{align}

**Remark 1.** We note that $(\Pi_h v)|_{K} = \Pi_K(v|_{K})$ for $v \in H^3(\Omega)$ and $K \in T_h$, and $\Pi_h v = 0$ on $\Gamma$ for $v \in H^3(\Omega) \cap H^1_0(\Omega)$.

4. APPLICATION TO THE POISSON PROBLEM

We apply the HP3V finite element to Poisson problems subject to Dirichlet boundary conditions. We establish a priori error estimates of the finite element solution in Theorem 1.

The problem we consider is to find a function $u : \Omega \to \mathbb{R}$ such that
\begin{align}
(27) & \quad \Delta u = f \quad \text{in } \Omega, \\
& \quad u = g \quad \text{on } \Gamma,
\end{align}

where $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$, and $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ are given functions.
4.1. Finite element scheme and a priori error estimates

Let us define the space $V_h$ and its affine set $V_h(g)$ as follows:

\begin{align}
(27) & \quad V_h(g) \equiv \{ v_h \in X_h; v_h(a_j) = g(a_j), \forall a_j : \text{node on } \Gamma \}, \\
(28) & \quad V_h \equiv V_h(0).
\end{align}

We solve the Poisson problem (26) by an HP3V finite element scheme. The discrete problem we deal with is to find $u_h \in V_h(g)$ which satisfies

\begin{equation}
(29) \quad a(u_h, v_h) = (f_h, v_h), \quad \forall v_h \in V_h,
\end{equation}

where

\begin{align}
(30) & \quad a(u_h, v_h) \equiv \int_{\Omega} \nabla u_h \cdot \nabla v_h dx, \\
(31) & \quad (f_h, v_h) \equiv \int_{\Omega} (\Pi_h f) v_h dx.
\end{align}

**Theorem 1.** Let $\Omega$ be a convex polyhedron. Let $u_h$ be the HP3V finite element solution of (29). Under the assumptions (T1)–(T3), $u \in H^4(\Omega)$, and $f \in H^{4+m}(\Omega)$ ($m = 0, 1$), it follows that

\begin{equation}
\| u - u_h \|_{1, \Omega} \leq C h^3 \left( \| u \|_{4, \Omega} + h^m \| f \|_{3+m, \Omega} \right).
\end{equation}

Moreover, if $f \in H^4(\Omega)$ and $g \in H^4(\Gamma)$, then the following holds:

\begin{equation}
(33) \quad \| u - u_h \|_{0, \Omega} \leq C h^4 \left( \| u \|_{4, \Omega} + \| f \|_{4, \Omega} + \| g \|_{4, \Gamma} \right).
\end{equation}

Since the $H^1$-error estimate is proven by the standard theory (for instance, see [3]), we omit the proof. The $L^2$-error estimate (33) is proven by the Aubin-Nitsche process [1], which is shown here for completeness.

Let us consider the following Poisson problem. Find $w : \Omega \to \mathbb{R}$ such that

\begin{equation}
(34) \quad \begin{cases} \\
-D\Delta w = u - u_h & \text{in } \Omega, \\
w = 0 & \text{on } \Gamma.
\end{cases}
\end{equation}

Since $\Omega$ is a convex polyhedron, we have $w \in H^2(\Omega)$ and

\begin{equation}
\| w \|_{2, \Omega} \leq C \| u - u_h \|_{0, \Omega}.
\end{equation}

We use the Clément $P_3$-interpolation $\Pi_{H0}^3 : H^1_0(\Omega) \to V_h$.

**Lemma 5.** (Clément [5]) Let $q$ and $m$ be integers, $q \in [0, 4]$ and $m \in [0, q]$. Under the assumptions (T1)–(T3), it holds that, for $v \in H^q(\Omega) \cap H^m_0(\Omega)$,

\begin{equation}
(35) \quad \left( \sum_{K \in T_h} \| v - \Pi_{H0}^3 v \|_{m, K}^2 \right)^{\frac{1}{2}} \leq C h^{q-m} \| v \|_{q, \Omega}.
\end{equation}

4.2. Proof of (33) in Theorem 1

Since $w \in H^2(\Omega)$ and $\Pi_{H0}^3 w \in H^1_0(\Omega)$, we deduce that

\begin{equation}
(37) \quad \| u - u_h \|_{2, \Omega}^2 = (u - u_h, -\Delta w)_\Omega \\
= (u - u_h, \nabla w)_\Omega - \left[ u - u_h, \frac{\partial w}{\partial n} \right]_\Gamma \\
= (\nabla (u - u_h), \nabla (w - \Pi_{H0}^3 w))_\Omega \\
+ (\nabla (u - u_h), \nabla \Pi_{H0}^3 w)_\Omega - \left[ u - \Pi_{H0} u, \frac{\partial w}{\partial n} \right]_\Gamma \\
= (\nabla (u - u_h), \nabla (w - \Pi_{H0}^3 w))_\Omega \\
+ \left[ f - \Pi_{H0} f, \Pi_{H0}^3 w \right]_\Omega - \left[ u - \Pi_{H0} u, \frac{\partial w}{\partial n} \right]_\Gamma \\
\leq \| \nabla (u - u_h) \|_{0, \Omega} \| \nabla (w - \Pi_{H0}^3 w) \|_{0, \Omega} \\
+ \| f - \Pi_{H0} f \|_{0, \Omega} \| \Pi_{H0}^3 w \|_{0, \Omega} + \| u - \Pi_{H0} u \|_{0, \Omega} \left\| \frac{\partial w}{\partial n} \right\|_{0, \Gamma} \\
\equiv I_1 + I_2 + I_3.
\end{equation}

Here, in the third equality, we have used the fact that $u_h = \Pi_{H0} u$ on $\Gamma$. At first, we observe

\begin{align}
(38) & \quad \| \nabla (w - \Pi_{H0}^3 w) \|_{0, \Omega} \leq c h \| w \|_{2, \Omega}, \\
(39) & \quad \| \Pi_{H0}^3 w \|_{0, \Omega} \leq c \| w \|_{0, \Omega}, \\
(40) & \quad \left\| \frac{\partial w}{\partial n} \right\|_{0, \Gamma} \leq c \| w \|_{2, \Omega},
\end{align}

where (38) and (39) are derived from Lemma 5 and (40) is the consequence of the trace theorem. The first term $I_1$ is estimated by using $H^1$-error estimate (32):

\begin{equation}
(41) \quad I_1 \leq c h^3 \left( \| u \|_{4, \Omega} + h \| f \|_{4, \Omega} \right) h \| w \|_{2, \Omega} \\
\leq c h^4 \left( \| u \|_{4, \Omega} + h \| f \|_{4, \Omega} \right) \| u - u_h \|_{0, \Omega}.
\end{equation}

As for $I_2$, we use Corollary 1 to get

\begin{equation}
(42) \quad I_2 \leq c h^4 \| f \|_{4, \Omega} \| w \|_{0, \Omega} \\
\leq c h^4 \| f \|_{4, \Omega} \| u - u_h \|_{0, \Omega}.
\end{equation}

In $I_3$, each $\Pi_K$ is equivalent to the Lagrange $P_3$-interpolation on $L \equiv K \cap \Gamma$: $H^2(L) \to P_3(L)$. Hence, we obtain

\begin{equation}
(43) \quad I_3 \leq c h^4 \| g \|_{4, \Gamma} \| w \|_{2, \Omega} \\
\leq c h^4 \| g \|_{4, \Gamma} \| u - u_h \|_{0, \Omega}.
\end{equation}

Combining (37) with (41)–(43), we obtain the inequality (33).

5. Numerical examples

We show two numerical results to observe the effectiveness of our scheme by solving (26). Given a domain $\Omega$ and an exact solution $u$, we calculate $f$ and $g$. Solving (26) by the HP3V scheme (29), we obtain the finite element solution...
We compute the relative errors in $H_0^1$- and $L^2$-norms defined by

\[
(a) \frac{\|\nabla (\Pi_h u - u_h)\|_0,\Omega}{\|\nabla (\Pi_h u)\|_0,\Omega}, \quad (b) \frac{\|\Pi_h u - u_h\|_0,\Omega}{\|\Pi_h u\|_0,\Omega}.
\]

Figure 4: A simplicial decomposition $\mathcal{T}_h$ when $N = 12$ in Example 1.

Table 1: The numbers of DOF in HP3, HP3V and LP3 in Example 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>HP3</th>
<th>HP3V</th>
<th>LP3</th>
<th>HP3V/LP3</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4,894</td>
<td>6,346</td>
<td>8,056</td>
<td>0.788</td>
</tr>
<tr>
<td>12</td>
<td>36,526</td>
<td>42,627</td>
<td>60,976</td>
<td>0.700</td>
</tr>
<tr>
<td>24</td>
<td>275,524</td>
<td>300,612</td>
<td>462,601</td>
<td>0.650</td>
</tr>
<tr>
<td>48</td>
<td>2,134,282</td>
<td>2,235,544</td>
<td>3,592,810</td>
<td>0.622</td>
</tr>
</tbody>
</table>

Table 2: Relative errors and slopes in Example 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$h$</th>
<th>(a)</th>
<th>(b)</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6.26E-01</td>
<td>3.81E-02</td>
<td>7.04E-03</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.00E-01</td>
<td>5.99E-03</td>
<td>6.91E-04</td>
<td>3.19</td>
</tr>
<tr>
<td>24</td>
<td>1.52E-01</td>
<td>8.71E-04</td>
<td>5.55E-05</td>
<td>3.70</td>
</tr>
<tr>
<td>48</td>
<td>8.12E-02</td>
<td>1.18E-04</td>
<td>3.82E-06</td>
<td>4.27</td>
</tr>
</tbody>
</table>

Example 1. The domain $\Omega$ and $u$ are

$\Omega = \{x \in \mathbb{R}^3; |x_i| < 1 \ (i = 1, 2, 3)\},$

$u(x_1, x_2, x_3) = \sin \pi x_1 \cos \pi x_2 \exp x_3.$

We make use of simplicial decompositions $\mathcal{T}_h$ with

$N = 6, 12, 24, 48,$

where $N$ indicates the number of division of each edge. A mesh is shown in Fig. 4. The numbers of DOF in HP3, HP3V and LP3 are shown in Table 1. Fig. 5 shows graphs of the relative errors vs. the mesh size $h$ in logarithmic scale, and Table 2 shows the slopes $m$ of the segments of the graphs. Those results reflect the theoretical convergence orders, $m = 3, 4,$ given in (32) and (33).

Table 3: The numbers of DOF in HP3, HP3V and LP3 in Example 2.

<table>
<thead>
<tr>
<th>$N$</th>
<th>HP3</th>
<th>HP3V</th>
<th>LP3</th>
<th>HP3V/LP3</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>67,900</td>
<td>77,510</td>
<td>113,590</td>
<td>0.682</td>
</tr>
<tr>
<td>18</td>
<td>534,808</td>
<td>574,940</td>
<td>898,849</td>
<td>0.640</td>
</tr>
<tr>
<td>36</td>
<td>4,170,352</td>
<td>4,332,012</td>
<td>7,023,655</td>
<td>0.617</td>
</tr>
</tbody>
</table>

Example 2. The domain $\Omega$ is a prism shown in Fig. 6, and $u$ is given as

$u(x_1, x_2, x_3) = \sin \pi x_1 \cos \pi x_2 \exp x_3.$

We make use of simplicial decompositions $\mathcal{T}_h$ with

$N = 9, 18, 36,$

where $N$ is the number of division of the shortest edge between the points $(0, 2, 3)$ and $(1, 4, 3).$ A decomposition
We have introduced a set of variant Hermite tetrahedral elements of degree three, which connect the Hermite element and the Lagrange element. We have discussed the approximation property of the HP3V element constructed from the elements. Applying the element to Poisson problems, we have proven a priori convergence error estimates (32) and (33) in the $H^1$- and $L^2$-norms with rates of $O(h^3)$ and $O(h^4)$, respectively. The HP3V element can be applied not only to Poisson problems but also to various problems. In order to maintain the high convergence order for domains of general shape the isoparametric element procedure is required. We will discuss isoparametric HP3V elements for curved domains in forthcoming papers.

### Appendix

#### A.1. Proof of Lemma 3

Proof is required only for $s \in [1,3]$. The number of DOF is 20 in each variant element and is equal to $\dim P_3$. It is sufficient to check for $\hat{p} \in P_3(K)$,

$$
\hat{p} = \sum_{i \in I_0^p} \hat{p}(\hat{a}_i) \phi_i^0 + \sum_{(i,j) \in F(s)} \hat{p}(\hat{a}_{ij}) \phi_{ij}^0
$$

$$
+ \sum_{(i,j,k) \in F_1(s)} \hat{p}(\hat{a}_{ijk}) \phi_{ijk}^1 + \sum_{(i,m) \in F_2(s)} \hat{p}(\hat{a}_i) \phi_m^1.
$$

Since the functions $\phi_i^0, \phi_{ij}^0$ and $\phi_{ijk}^1$ are base functions of the Lagrange element, and since $\phi_i^{1,H}, \phi_{im}^{1,H}$ and $\phi_{ijk}^{0,L}$ are those of the Hermite element, the followings are obtained: For $i \in I_0^p$,

$$
\phi_i^0(\hat{a}_p) = \delta_{ip} \quad (p \in I_0^p),
$$

$$
\phi_i^0(\hat{a}_{pq}) = 0 \quad ((p,q) \in E(s)),
$$

$$
\phi_i^0(\hat{a}_{pqr}) = 0 \quad ((p,q) \in F),
$$

$$
\phi_i^0(\hat{a}_p) = 0 \quad ((p,l) \in F_1(s)).
$$

For $(i,j) \in I_E(s)$,

$$
\phi_{ij}^1(\hat{a}_p) = 0 \quad (p \in I_0^p),
$$

$$
\phi_{ij}^1(\hat{a}_{pq}) = \delta_{ip} \phi_{ij}^1 \quad ((p,q) \in E(s)),
$$

$$
\phi_{ij}^1(\hat{a}_{pqr}) = 0 \quad ((p,q) \in F),
$$

$$
\phi_{ij}^1(\hat{a}_p) = 0 \quad ((p,l) \in F_1(s)).
$$
For \((i,j,k) \in I_F\),
\[
\begin{align*}
\hat{\phi}_{i,j}^0(\hat{a}_p) &= 0 \quad (p \in I_Y^s), \\
\hat{\phi}_{i,j}^0(\hat{a}_{pq}) &= 0 \quad ((p,q) \in I_E(s)), \\
\hat{\phi}_{i,j}^0(\hat{a}_{ppq}) &= \delta_{ip}\delta_{jq}\delta_{kr} \quad ((p,q,r) \in I_F), \\
\hat{\phi}_{i,j}^0(\hat{a}_l) &= 0 \quad ((p,l) \in I_Y^l(s)).
\end{align*}
\] (49)

For \((i,m) \in I_Y^t(s)\),
\[
\begin{align*}
\hat{\phi}_{i,m}^0(\hat{a}_p) &= 0 \quad (p \in I_Y^s), \\
\hat{\phi}_{i,m}^0(\hat{a}_{pq}) &= 0 \quad ((p,q) \in I_E(s)), \\
\hat{\phi}_{i,m}^0(\hat{a}_{ppq}) &= 0 \quad ((p,q,r) \in I_F), \\
\hat{\phi}_{i,m}^0(\hat{a}_l) &= \delta_{ip}\delta_{ml} \quad ((p,l) \in I_Y^l(s)).
\end{align*}
\] (50)

Let us prove (47). From (8) we get
\[
\hat{\phi}_{i}^0(\hat{a}_p) = \hat{\phi}_{i}^0(\hat{a}_p) - \frac{1}{9} \sum_{j=5}^{4} \hat{\phi}_{j,i}^0(\hat{a}_p)
= \delta_{ip} - 0 = \delta_{ip} \quad (p \in I_Y^s),
\]
\[
\hat{\phi}_{i}^0(\hat{a}_{pq}) = \hat{\phi}_{i}^0(\hat{a}_{pq}) - \frac{1}{9} \sum_{j=5}^{4} \hat{\phi}_{j,i}^0(\hat{a}_{pq})
= 0 - \frac{1}{9} \sum_{j=5}^{4} \delta_{jp}\delta_{lq} = 0 \quad ((p,q) \in I_E(s)),
\]
\[
\hat{\phi}_{i}^0(\hat{a}_{ppq}) = \hat{\phi}_{i}^0(\hat{a}_{ppq}) - \frac{1}{9} \sum_{j=5}^{4} \hat{\phi}_{j,i}^0(\hat{a}_{ppq})
= 0 - 0 = 0 \quad ((p,q,r) \in I_F),
\]
\[
\hat{\phi}_{i}^0(\hat{a}_l) = \frac{1}{2} \lambda_i(3\lambda_i - 1)(3\lambda_i - 2) \hat{a}_l
= \frac{1}{2} \sum_{j=5}^{4} \lambda_i \lambda_j(3\lambda_j - 1) \hat{a}_l
\]
\[
= \frac{1}{2} \lambda_i(27\lambda_i^2 - 18\lambda_i + 2) \hat{a}_l
- \frac{1}{2} \sum_{j=5}^{4} (\lambda_i \lambda_j(3\lambda_j - 1) + \lambda_j \lambda_i(6\lambda_j - 1)) \hat{a}_l
+ \frac{1}{2} (9\delta_{ip} + 2) \lambda_i \hat{a}_l
- \frac{1}{2} \sum_{j=5}^{4} (\lambda_i \delta_{jp} + 3\delta_{ij} - 1) \lambda_j \hat{a}_l
= \lambda_i \hat{a}_l - \frac{1}{2} (3 - 1) \lambda_i \hat{a}_l
= \frac{1}{2} (3 - 1) \lambda_i \hat{a}_l = 0 \quad ((p,l) \in I_Y^l(s)).
\]

The others are proven similarly. (47)–(50) imply the identity (46).

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**References**