

**Volume Integral Equation Method in Problems of  
Mathematical Physics**

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## Preface

In our course we will consider the volume integral equations in the following form

$$a(x)u(x) + \int_Q \frac{K(x-y)}{|x-y|^m} b(y)u(y)dy = f(x), m \leq 3.$$

Many important classes of the wave scattering problems can be described by equations of this form; for example, this is the case for problems of electromagnetic and acoustic scattering on 3D transparent bodies. The corresponding integral operator is compact ( $m=1$ ) in acoustic problems and singular ( $m=3$ ) in electromagnetic problems.

Why do we want to consider integral equations though the initial problems usually are formulated as boundary value problems for partial differential equations? For that there are two main reasons.

I. One of the ways for the mathematical investigation (proof of the existence and uniqueness theorems, etc.) of the initial problem of mathematical physics is the following. We reduce the initial boundary value problem to an integral equation. Then we establish the equivalence of the differential formulation of the problem and the corresponding integral equation. It means that any solution of the integral equation (maybe with some restriction on the parameters of the problem) satisfies the partial differential equations and boundary condition and back. Based on the integral inequalities which are usually obtained from the differential formulation, we prove the uniqueness theorem of our problem. Then using the theory of solvability of integral equations (Fredholm integral equation or singular integral equation theories) in appropriate (from the physical point of view) functional space we prove the existence and uniqueness theorem and others facts for the initial problem of mathematical physics. In our course we will follow these steps.

II. We will construct the methods and algorithms for the numerical solution of the initial problems by using integral equations. At the first glance the partial differential equations are more appropriate for the numerical solution because after discretization we receive the system of linear algebraic equations (SLAE) with sparse matrix in comparison with the full matrix which we obtain in the integral equation case. But for the wave scattering problems the solution must satisfy the radiation condition at infinity. Therefore for the good accuracy we need to find numerically the unknown wave field in the domain which is sufficiently greater than scattering object  $Q$  and due to 3D of the initial problem the dimension of the SLAE will be huge. Using discrete fast Fourier transform techniques and taking into account that the kernels of integral equations depend only on the difference of arguments we may construct a fast algorithm for the multiplication of the SLAE matrix and the vector. Then applying iterative methods we can build the effective methods and algorithms for the numerical solution of the initial problems based on the integral equations.

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# PART I

## Mathematical properties of integral equations

### CHAPTER 1

#### Acoustic scattering problems

##### 1.1 Formulation of the problem

For acoustic and quantum mechanics cases, an appropriate mathematical problem is treated as follows. Find the scalar field  $u$  which satisfies the Helmholtz equation

$$\Delta u + k^2 u = f, \quad (1.1)$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

and the radiation condition at infinity

$$\lim_{r \rightarrow \infty} \left[ r \left( \frac{\partial u}{\partial r} - ik_0 u \right) \right] = 0, \quad (1.2)$$

$$r = |x| = \sqrt{(x_1^2 + x_2^2 + x_3^2)},$$

where  $k$  is the wave number, which is the function of coordinates in a bounded 3D domain  $Q$  in Euclidean space  $E_3$ ;  $k = k_0 = \text{const}$  outside  $Q$ ,  $\text{Im } k_0 \geq 0$ ,  $\text{Re } k_0 \geq 0$ ; the source of the field  $f$  is a given function of coordinates; and  $x_1, x_2, x_3$  are the Cartesian coordinates.

##### 1.2 Integral equation

From the theory of the Helmholtz equation we have that the integral representation

$$u_0(x) = - \int f(y) G(R) dy, \quad G(R) = \frac{\exp(ik_0 R)}{4\pi R}, \quad (1.3)$$

$$R = |x - y|, x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), dy = dy_1 dy_2 dy_3$$

satisfies the equation

$$\Delta u_0 + k_0^2 u_0 = f \quad (1.4)$$

in the domain where  $f$  is a Holder-continuous function (see Definition in Section 3.2). It follows obviously from (1.3) that  $u_0(x)$  also satisfies the radiation condition at infinity (1.2). The function  $G$  is called Green's function for the Helmholtz equation. Rewrite Eq. (1.1) in the following form

$$\Delta u + k_0^2 u = f - (k^2 - k_0^2)u. \quad (1.5)$$

Unknown field  $u$  (see (1.3)-(1.4)) can be presented as

$$u(x) = -\int f(y)G(R)dy + \int_Q (k^2(y) - k_0^2)u(y)G(R)dy, \quad x \in E_3. \quad (1.6)$$

Then the initial differential problem can be reduced to the following integral equation for the unknown field  $u$  in the domain  $Q$

$$u(x) - \int_Q (k^2(y) - k_0^2)u(y)G(R)dy = u_0(x), \quad x \in Q. \quad (1.7)$$

Here  $u_0$  is the wave field generated by the source function  $f$  in the free space with the constant wave number  $k_0$ . If we find a solution of the integral equation (1.7) in the domain  $Q$  then we may calculate  $u(x)$  outside  $Q$  by using representation (1.6).

### 1.3 Uniqueness statement

Assume that the functions  $f(x)$  and  $k(x)$  are Holder-continuous everywhere. Then any solution of the integral equation (1.7) satisfies the Helmholtz equation (1.1) and the radiation condition and back.

Let us prove the uniqueness statement for the initial problem (1.1) - (1.2). From (1.1) we have

$$\Delta u^* + (k^2)^* u^* = f^*. \quad (1.8)$$

Here symbol  $*$  means the complex conjugation. Let  $\Omega_r$  be the ball of radius  $r$  which contains the domain  $Q$ . Then from (1.1) and (1.8) we obtain the equality

$$\int_{\Omega_r} (u^* f - u f^*) dv = \int_{\Omega_r} (u^* \Delta u - u \Delta u^*) dv + \int_{\Omega_r} (k^2 - (k^2)^*) |u|^2 dv. \quad (1.9)$$

We now apply Green's formula to the first integral on the right-hand side of (1.9) and consider its limit as  $r \rightarrow \infty$  taking radiation condition (1.2) into account. Considering the imaginary part of the resultant we finally derive

$$-\text{Im} \int_{E_3} u f^* dv = \text{Im} \int_{E_3} k^2 |u|^2 dv + \text{Re} k_0 \lim_{r \rightarrow \infty} \int_{S_r} |u|^2 dS, \quad (1.10)$$

where  $S_r$  is the sphere of radius  $r$ .

Relation (1.10) describes the energy conservation law for the wave field. Note that from the physical meaning of the scattering problems  $\text{Im} k^2 \geq 0$  and  $\text{Re} k_0 > 0$ .

Let us put that source function  $f$  equal zero everywhere. From (1.10) we have  $u = 0$  in the domain  $Q$  if  $\text{Im} k^2(x) > 0$ ,  $x \in Q$ . Then by using integral representation (1.6) we obtain that  $u = 0$  in the whole space. Based on the integral inequalities (which are obtained from (1.1) - (1.2), although tangled without any explicit physical meaning) we may prove that homogeneous equation (1.1) with radiation condition has only the trivial solution [4], i.e.  $u = 0$  in the whole space  $E_3$  if

$$\text{Im } k^2(x) \geq 0, \quad x \in Q. \quad (1.11)$$

Thus, we have proved the following statement: under abovementioned restrictions the solution of the initial problem (1.1) - (1.2) is unique if it exists.

#### 1.4 Existence statements

To answer the existence question we have to use some results from functional analysis. First, we have to specify appropriate functional space. Generally speaking, one can choose different spaces, and the choice governs the results of analysis. It is reasonable to apply the following criterion: functional space must be sufficiently wide, providing the consideration of all physically admissible solutions; however, the space should not be too wide, because in this case, for example, the uniqueness may be violated due to the presence of solutions that have no physical meaning. The integrals of squared field characteristics stand in the conservation law for the scattering problems (1.10). Therefore, one may assume that the space of square-integrable functions  $L_2(Q)$  is the most appropriate from the physical viewpoint as applied to the analysis of the integral equations (1.7).

Thus (see Theorem 3.1) we have the following statement.

**Theorem 1.1** *There exists the unique solution of the initial problem (1.1) – (1.2) if  $k(x)$  and  $f(x)$  are Holder-continuous everywhere and condition (1.11) is satisfied.*

Below we will consider the integral equation (1.7) imposing minimal restrictions on the wave function  $k(x)$ ; namely we will assume that the function  $k(x)$  is only a bounded function of coordinates in domain  $Q$ .

Rewrite (1.7) in the symbolic form

$$u - \hat{S}[(k^2 - k_0^2)u] = f, \quad (1.12)$$

where  $\hat{S}$  is a linear and continuous operator in  $L_2(Q)$ .

First we show that for any  $f \in L_2(Q)$  the following inequality holds

$$\text{Im} \int_Q f^* (\hat{S}f) dQ \geq 0. \quad (1.13)$$

Let  $f$  be a differentiable function of coordinates. Then, taking into account (1.3), (1.7) and (1.12), we see that  $u = -\hat{S}f$  satisfies everywhere the Helmholtz equation (1.4) and condition at infinity (1.2). Then from (1.10), taking into account that the function  $f$  is equal zero outside  $Q$  and  $k(x) = k_0$  everywhere, we obtain the inequality (1.13). Further, any element  $f \in L_2(Q)$  is a limit of a sequence of differentiable functions. Therefore, if we pass to the limit and take into account that the operator  $\hat{S}$  is continuous, we obtain (1.13) for any  $f \in L_2(Q)$ .

**Theorem 1.2 [11].** *Let  $k(x)$  be a bounded function of coordinates in  $Q$  such that  $\text{Im } k^2(x) > \text{Im } k_0^2$ ,  $x \in Q$ . Then the solution of the integral equation (1.7) exists and is unique in  $L_2(Q)$ .*

*Proof.* Let us multiply both side of Eq. (1.12) by  $(k^2 - k_0^2)^*$ :

$$(k^2 - k_0^2)^* u - (k^2 - k_0^2)^* \hat{S}[(k^2 - k_0^2)u] = (k^2 - k_0^2)^* f. \quad (1.14)$$

Function  $\delta = (k^2 - k_0^2)$  has an inverse function due to  $\text{Im } k^2(x) > \text{Im } k_0^2$ . Then we have that Eqs. (1.12) and (1.14) are equivalent, i.e. any solution of (1.12) is a solution of (1.14) and back. Let  $\hat{A}$  denote the operator of Eq. (1.14). For any  $u \in L_2(Q)$  we have

$$\begin{aligned} (\hat{A}u, u) &= \int_Q \delta^* |u|^2 dQ - \int_Q \delta^* \hat{S}(\delta u) u^* dQ = \\ &= \int_Q \text{Re} \delta |u|^2 dQ - i \int_Q \text{Im} \delta |u|^2 dQ - \int_Q (\delta u)^* [\hat{S}(\delta u)] dQ. \end{aligned}$$

We conclude from (1.13) that

$$\left| (\hat{A}u, u) \right| \geq \left| \text{Im}(\hat{A}u, u) \right| \geq \min_{x \in Q} \text{Im} \delta(x) (u, u). \quad (1.15)$$

The theorem now follows from Theorem 4.3 and equivalence of Eq. (1.12) and Eq. (1.14). The conditions of Theorem 1.2 are more strict than condition of Theorem 1.1 (they are virtually identical if  $\text{Im } k_0 = 0$ , which takes place in most real problems). However, no smoothness of the function  $k(x)$  is required in Theorem 1.2; therefore in this case the solution of Eq. (1.7) satisfies Eq. (1.1) in a generalized sense.



## CHAPTER 2

### Electromagnetic scattering problems

#### 2.1 Formulation of the problem

Now we will consider the next class of wave scattering – electromagnetic scattering. The medium in a finite 3D domain  $Q$  is characterized by a dielectric permittivity tensor function  $\hat{\varepsilon}$  and constant ( $\varepsilon = \varepsilon_0 = \text{const}$ ) outside  $Q$ ; the permeability is constant everywhere,  $\mu = \mu_0 = \text{const}$ . The problem is to find the electromagnetic field excited in the medium by an external field with time dependence given by the factor  $\exp(-i\omega t)$ . The corresponding mathematical problem is stated as follows: find unknown vector functions  $\vec{E}$  and  $\vec{H}$  satisfying Maxwell equations

$$\text{rot } \vec{H} = -i\omega \hat{\varepsilon} \vec{E} + \vec{J}^0, \quad \text{rot } \vec{E} = i\omega \mu_0 \vec{H} \quad (2.1)$$

and the radiation condition at infinity (1.2), where  $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$ . In (2.1)  $\vec{J}^0$  is the external current generating the external field  $\vec{E}^0, \vec{H}^0$ ; and  $\text{Im } \varepsilon_0 \geq 0, \text{Im } \mu_0 \geq 0, \text{and } \text{Im } k_0 \geq 0$ .

#### 2.2 Integro-differential equation

Write equations (2.1) in the equivalent form

$$\text{rot } \vec{H} = -i\omega \varepsilon_0 \vec{E} + \vec{J}, \quad \text{rot } \vec{E} = i\omega \mu_0 \vec{H}. \quad (2.2)$$

Here

$$\vec{J} = \vec{J}^0 + \vec{J}^p, \quad (2.3)$$

where

$$\vec{J}^p = -i\omega(\hat{\varepsilon} - \varepsilon_0 \hat{I}) \vec{E} \quad (2.4)$$

is the electric current of polarization which does not equal zero only in domain  $Q$ .

We may formally consider equations (2.2) as Maxwell equations in homogeneous medium; i.e. assuming that the electromagnetic field is produced by current  $\vec{J}$ . Express the solution of (2.2) that satisfies the radiation condition at infinity in terms of vector potential  $\vec{A}$  using the known formulas

$$\vec{A}(x) = \int \vec{J}(y) G(R) dy, \quad (2.5)$$

$$\vec{E} = i\omega \mu_0 \vec{A} - \frac{1}{i\omega \varepsilon_0} \text{grad div } \vec{A}, \quad \vec{H} = \text{rot } \vec{A}. \quad (2.6)$$

In (2.5)  $G$  is the Green function of the Helmholtz equation

$$G(R) = \frac{\exp(ik_0 R)}{4\pi R}. \quad (2.7)$$

From (2.3) - (2.7) we obtain that unknown electromagnetic field can be presented as

$$\vec{E}(x) = \vec{E}^0(x) + k_0^2 \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy + \text{grad div} \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy, \quad x \in E_3 \quad (2.8)$$

$$\vec{H}(x) = \vec{H}^0(x) - i\omega\varepsilon_0 \text{rot} \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy, \quad x \in E_3, \quad (2.9)$$

where  $R = |x - y|$ ;  $x = (x_1, x_2, x_3)$ ;  $y = (y_1, y_2, y_3)$ ;  $\hat{\varepsilon}_r = \hat{\varepsilon} / \varepsilon_0$ .

In (2.8) - (2.9)  $\vec{E}^0(x), \vec{H}^0(x)$  is the electromagnetic field generated by known current  $\vec{J}^0$  in the homogeneous space with parameters  $\varepsilon_0$  and  $\mu_0$ . If we know the electric field we may calculate magnetic field by using (2.9).

Because  $\hat{\varepsilon}_r = \hat{I}$  outside  $Q$  we can reduce the initial problem to the volume integro-differential equation with respect to field  $\vec{E}$  in the domain  $Q$

$$\vec{E}(x) - k_0^2 \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy - \text{grad div} \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy = \vec{E}^0(x), \quad x \in Q. \quad (2.10)$$

If we have obtained solution to the equation (2.10) in the domain  $Q$  then we may calculate electromagnetic field outside  $Q$  by using representation (2.8), (2.9).

Note that we cannot apply *grad div* under the integral sign in (2.10) because in this case one must differentiate function  $G$  twice with respect to coordinates which yields the term  $\sim 1/R^3$  in the kernel of the integral equation and the corresponding improper integrals diverge. However, the *rot* can be applied under the integral (2.9) because in this case we will have a weak singularity  $\sim 1/R^2$ .

### 2.3 Singular volume integral equation

Represent function  $G(R)$  as

$$G(R) = G_0(R) + G_1(R), \quad G_0(R) = \frac{\exp(ik_0R) - 1}{4\pi R}, \quad G_1 = \frac{1}{4\pi R}. \quad (2.11)$$

Then from (2.10), we have

$$\begin{aligned} & \vec{E}(x) - k_0^2 \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G(R) dy - \text{grad div} \int_Q (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y) G_1(R) dy - \\ & \int_Q \left( (\hat{\varepsilon}_r - \hat{I}) \vec{E}(y), \text{grad} \right) \text{grad} G_0(R) dy = \vec{E}^0(x), \quad x \in Q, \end{aligned} \quad (2.12)$$

Here the symbol  $(*,*)$  denotes the inner product of the vectors.

From (3.13), (3.17) (see Section 3.2), we can reduce Eq. (2.12) to the singular volume integral equation

$$\begin{aligned} & \vec{E}(x) + \frac{1}{3} (\hat{\varepsilon}_r(x) - \hat{I}) \vec{E}(x) - p.v. \int_Q \left( (\hat{\varepsilon}_r(y) - \hat{I}) \vec{E}(y), \text{grad} \right) \text{grad} G(R) dy - \\ & k_0^2 \int_Q (\hat{\varepsilon}_r(y) - \hat{I}) \vec{E}(y) G(R) dy = \vec{E}^0(x), \quad x \in Q. \end{aligned} \quad (2.13)$$

## 2.4 Equivalence statement

Singular integral equation (2.13) is obtained from equation (2.10). Therefore, in order to simplify the analysis, we will sometimes use equation (2.10). Now we shall consider the equivalence of Maxwell equations (2.1) and the integral equation. We will assume that the electromagnetic field satisfies the Maxwell equations in the usual sense. Such solutions of the initial problem will be called *classical solutions*.

It is clear that every solution to the Maxwell equations satisfying the condition at infinity is a solution of (2.13). In order to justify the converse statement we will assume that (I) the components of tensor  $\hat{\varepsilon}$  are Holder-continuous functions everywhere as well as the considered solutions of (2.13); and (II) external field satisfies the Maxwell equations for the homogeneous space, i.e.

$$\text{rot } \vec{H}^0 = -i\omega\varepsilon_0 \vec{E}^0 + \vec{J}^0, \quad \text{rot } \vec{E}^0 = i\omega\mu_0 \vec{H}^0. \quad (2.14)$$

Introduce the notation

$$\vec{V}(x) = \int_Q [\hat{\varepsilon}_r(y) - \hat{I}] \vec{E}(y) G(R) dy. \quad (2.15)$$

Substituting  $\vec{E}(x)$  and  $\vec{H}(x)$  from (2.8) and (2.9) into the first equation (2.1) and taking into account (2.14), we have

$$\begin{aligned} \text{rot } \vec{H} + i\omega\hat{\varepsilon} \vec{E} - \vec{J}^0 &= \text{rot } \vec{H}^0 - i\omega\varepsilon_0 \text{rot rot } \vec{V} + i\omega\hat{\varepsilon} \vec{E}^0 + i\omega\hat{\varepsilon} k_0^2 \vec{V} + i\omega\hat{\varepsilon} \text{grad div } \vec{V} - \vec{J}^0 = \\ &= i\omega(\hat{\varepsilon} - \varepsilon_0 \hat{I})[\vec{E}^0 + k_0^2 \vec{V} + \text{grad div } \vec{V}] + i\omega\varepsilon_0 [-\text{rot rot } \vec{V} + \text{grad div } \vec{V} + k_0^2 \vec{V}]. \end{aligned} \quad (2.16)$$

Next, vector-function  $\vec{V}$  specified by (2.15) satisfies the vector Helmholtz equation

$$\Delta \vec{V} + k_0^2 \vec{V} = -[\hat{\varepsilon}_r - \hat{I}] \vec{E}.$$

Then taking into account the vector identity

$$\text{rot rot } \vec{B} = \text{grad div } \vec{B} - \Delta \vec{B},$$

we obtain

$$[-\text{rot rot } \vec{V} + \text{grad div } \vec{V} + k_0^2 \vec{V}] = -\frac{1}{\varepsilon_0} [\hat{\varepsilon} - \varepsilon_0 \hat{I}] \vec{E}.$$

Then from (2.16), we have

$$\text{rot } \vec{H} + i\omega\hat{\varepsilon} \vec{E} - \vec{J}^0 = i\omega(\hat{\varepsilon} - \varepsilon_0 \hat{I})[-\vec{E} + \vec{E}^0 + k_0^2 \vec{V} + \text{grad div } \vec{V}]. \quad (2.17)$$

From (2.8) and (2.15), it follows that the right hand side of (2.17) is identically zero.

Applying the similar reasoning, we can show that the second equation of (2.1) transforms to an identity as a result of substituting  $\vec{E}(x)$  and  $\vec{H}(x)$  given by (2.8) and (2.9). Then any solution of integral equation (2.13) satisfies the Maxwell equations provided that the above conditions are fulfilled.

## 2.5 Uniqueness statement

Let us prove the uniqueness statement for the initial problem. From second equation of (2.1) we have

$$\operatorname{rot} \bar{E}^* = -i\omega\mu_0^* \bar{H}^*. \quad (2.18)$$

Let  $\Omega_r$  be the ball of radius  $r$  which contains the domain  $Q$ . Multiplying first equation of (2.1) by  $\bar{E}^*$  and equation (2.18) by  $\bar{H}$ , we obtain the equality

$$-\int_{\Omega_r} \bar{E}^* \bar{J}^0 dv = \int_{\Omega_r} \left( \bar{H} \operatorname{rot} \bar{E}^* - \bar{E}^* \operatorname{rot} \bar{H} \right) dv - i\omega \int_{\Omega_r} \bar{E}^* \hat{\varepsilon} \bar{E} dv + i\omega\mu_0^* \int_{\Omega_r} |\bar{H}|^2 dv. \quad (2.19)$$

We now apply vectorial Green's formula to the first integral in the right-hand side of (2.19) and consider its limit as  $r \rightarrow \infty$ . Considering the real part of the resultant we finally derive

$$\begin{aligned} -\operatorname{Re} \int_{E_3} \bar{E}^* \bar{J}^0 dv = & \omega \operatorname{Im} \int_Q \bar{E}^* \hat{\varepsilon} \bar{E} dv + \omega \operatorname{Im} \varepsilon_0 \int_{E_3 \setminus Q} |\bar{E}|^2 dv + \\ & \omega \operatorname{Im} \mu_0 \int_{E_3} |\bar{H}|^2 dv + \lim_{r \rightarrow \infty} \int_{S_r} \left( \operatorname{Re} [\bar{E}, \bar{H}^*], \bar{n} \right) dS, \end{aligned} \quad (2.20)$$

where  $S_r$  is the sphere of radius  $r$  and  $\bar{n}$  is the external normal to this sphere;  $[*,*]$  denote the cross product of vectors.

Relation (2.20) describes the energy conservation law for the electromagnetic field. By virtue of radiation conditions the last term on the right hand side of (2.20) is nonnegative and coincides with the energy flux of the electromagnetic field to infinity. The first term in the left hand side of (2.20) and first three terms in the right hand side of (2.20) have the physical meaning of power loss or generation of energy in matter. We have  $\operatorname{Im} \varepsilon_0 \geq 0$  and  $\operatorname{Im} \mu_0 \geq 0$ ; therefore, only power loss may occur in domain  $E_3 \setminus Q$ .

Let us put the external current  $\bar{J}^0$ , generating the external field, equals zero everywhere. From (2.20) we have  $\bar{E} = 0$  in the domain  $Q$  if the Hermitian tensor  $(\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x))/(2i)$  is positive definite at every point of  $Q$ . This condition has the following physical meaning: the matter in  $Q$  has the losses. In the isotropic case, the condition means that  $\operatorname{Im} \varepsilon(x) > 0$  for all  $x \in Q$ . Then by using integral representations (2.8), (2.9) with  $\bar{E}^0, \bar{H}^0 = 0$ , we obtain that the electromagnetic field equals zero in the whole space.

Based on the theory of elliptic equation [1, 3], we can prove that homogeneous equations (2.1) with radiation condition has only the trivial solution in the whole space  $E_3$ , if Hermitian tensor  $(\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x))/(2i)$  is nonnegative definite at every point of  $Q$ , condition (2.25) is satisfied and  $\hat{\varepsilon}(x)$  is three times continuously differentiable function of coordinates. The physical meaning of the first condition: the matter in  $Q$  cannot generate electromagnetic energy. In the isotropic case, the condition means that  $\operatorname{Im} \varepsilon(x) \geq 0$  for all  $x \in Q$ .

Thus, we have proved the following statement: under abovementioned restrictions the solution of the initial electromagnetic problem is unique if it exists.

## 2.6 Existence statements

To answer the existence question we have to use some results from functional analysis. First, we have to specify appropriate functional space. The integrals of squared field characteristics stand in the conservation law for electromagnetic scattering problems (2.20). Therefore, as in the acoustic case, one may assume that the space of square-integrable vector-functions  $\bar{L}_2(Q)$  with the inner product

$$(\bar{U}, \bar{V}) = \int_Q \bar{U}(x) \bar{V}^*(x) dx \quad (2.21)$$

is the most appropriate from the physical viewpoint as applied to the analysis of the integral equations (2.13).

### 2.6.1 Classical solutions

Below we will assume that the components of tensor  $\hat{\varepsilon}$  are Holder-continuous functions everywhere. Consider singular integral equation in the Hilbert space  $\bar{L}_2(E_3)$

$$\begin{aligned} \bar{E}(x) + \frac{1}{3}(\hat{\varepsilon}_r(x) - \hat{I})\bar{E}(x) - p.v. \int_{E_3} ((\hat{\varepsilon}_r(y) - \hat{I})\bar{E}(y), \text{grad}) \text{grad} G(R) dy - \\ k_0^2 \int_{E_3} (\hat{\varepsilon}_r(y) - \hat{I})\bar{E}(y) G(R) dy = \bar{E}^0(x), \quad x \in Q. \end{aligned} \quad (2.22)$$

Tensor-function  $(\hat{\varepsilon}_r(x) - \hat{I}) = 0$  if  $x \in E_3 \setminus Q$ . Therefore singular volume integral equations (2.13) and (2.22) are equivalent.

From (3.24) and (3.30), we find (see Section 3.3) that the elements of symbol matrix  $\Phi = \{\Phi_{nm}\}$  of singular operator (2.22) in the Cartesian coordinate system have the following form

$$\Phi_{nm}(x, \beta) = \delta_{nm} + \frac{1}{\varepsilon_0} \sum_{l=1}^3 \varepsilon_{lm}(x) \beta_l \beta_m - \beta_n \beta_m, \quad n, m = 1, 2, 3. \quad (2.23)$$

Here  $\varepsilon_{lm}(x)$  are the components of tensor-function  $\hat{\varepsilon}(x)$  in the Cartesian coordinate system. From (2.23) we find

$$\det[\Phi] = \frac{1}{\varepsilon_0} \sum_{n,m=1}^3 \varepsilon_{nm}(x) \beta_n \beta_m. \quad (2.24)$$

Now, applying Theorem 3.4 for singular equations (2.22) and the equivalence of equations (2.12) and (2.22), we obtain the following statement.

**Theorem 2.1 [1, 9]** *The operator of the singular integral equation (2.13) is a Noether operator in  $\bar{L}_2(Q)$  if and only if the following condition is satisfied*

$$\sum_{n,m=1}^3 \varepsilon_{nm}(x) \beta_n \beta_m \neq 0, \quad x \in Q, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \quad (2.25)$$

It can be proved using Theorem 3.5 that if the inequality

$$\text{Im} \sum_{n,m=1}^3 \varepsilon_{nm}(x) \beta_n \beta_m \geq 0, \quad x \in Q, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.. \quad (2.26)$$

holds then the singular integral equation (2.13) has the Fredholm property. For isotropic case (2.26) has the form

$$\text{Im } \varepsilon(x) \geq 0. \quad (2.27)$$

Conditions (2.26), (2.27) are satisfied for any passive media (that do not generate energy).

Now from uniqueness statement and Theorem 2.1 we obtain the existence and uniqueness theorem.

**Theorem 2.2 [1, 9]** *Let Hermitian tensor  $(\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x))/(2i)$  is nonnegative definite at every point of  $Q$  and conditions (2.25) and (2.26) are satisfied. Then there exists the unique solution of integral equation (2.13) in  $\bar{L}_2(Q)$  if one of the following conditions is fulfilled:*

(A) *tensor  $(\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x))/(2i)$  is positive definite at every point of  $Q$ ;*

(B)  *$\hat{\varepsilon}(x)$  is a three times continuously differentiable function.*

*This solution also satisfies Maxwell equations (2.1) and radiation conditions at infinity.*

### 2.6.2 Generalized solutions [1, 11]

Below we will consider the integral equation (2.13) imposing minimal restrictions on the dielectric permittivity tensor function  $\hat{\varepsilon}(x)$ ; namely we will assume that all the components of tensor function  $\hat{\varepsilon}(x)$  are only bounded functions of coordinates in domain  $Q$ . In this case, according to theorem 3.2, the operator of integral equation (2.13) is bounded.

Rewrite integral equation (2.13) in the symbolic form

$$\vec{E} - \frac{1}{\varepsilon_0} \hat{S}((\hat{\varepsilon} - \varepsilon_0 \hat{I}) \vec{E}) = \vec{E}^0. \quad (2.28)$$

First we show that for any  $\vec{J} \in \bar{L}_2(Q)$  the following inequality holds

$$\text{Im} \frac{1}{\varepsilon_0} \int_Q \vec{J}^* (\hat{S} \vec{J}) dQ \geq 0. \quad (2.29)$$

Taking into account (2.28), (2.10) and (2.5), (2.6) we see that  $\vec{E} = -\frac{1}{i\omega\varepsilon_0} \hat{S} \vec{J}$ ,  $\vec{H} = \frac{1}{i\omega\mu_0} \text{rot } \vec{E}$

satisfy everywhere the Maxwell equations (2.2) and condition at infinity (1.2). Then from (2.20), taking into account that the vector function  $\vec{J}$  is equal zero outside  $Q$  and  $\hat{\varepsilon}(x) = \varepsilon_0 \hat{I}$  everywhere, we obtain the inequality (2.29).

**Theorem 2.3.** *Let  $\hat{\varepsilon}(x)$  be a bounded tensor function of coordinates in  $Q$  and Hermitian tensor function  $(\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x) - 2i \text{Im } \varepsilon_0 \hat{I})/(2i)$  be positive definite at every point of  $Q$ . Then integral equation (2.13) has the unique solution in  $\bar{L}_2(Q)$ .*

*Proof.* Let us multiply both side of Eq. (2.28) by  $(\hat{\varepsilon} - \varepsilon_0 \hat{I})^*$ :

$$(\hat{\varepsilon} - \varepsilon_0 \hat{I})^* \vec{E} - (\hat{\varepsilon} - \varepsilon_0 \hat{I})^* \hat{S}[(\hat{\varepsilon} - \varepsilon_0 \hat{I}) \vec{E}] = (\hat{\varepsilon} - \varepsilon_0 \hat{I})^* \vec{E}^0. \quad (2.30)$$

Tensor function  $\hat{\delta} = (\hat{\varepsilon} - \varepsilon_0 \hat{I})$  has an inverse function due to the condition of the theorem. Then we have that Eqs. (2.28) and (2.30) are equivalent. Let  $\hat{A}$  denote the operator of Eq. (2.30). From (2.28), (2.30) and (2.21) we have for any  $\bar{u} \in \bar{L}_2(Q)$

$$\begin{aligned} (\hat{A}\bar{u}, \bar{u}) &= \int_Q \hat{\delta}^* |\bar{u}|^2 dQ - \int_Q \hat{\delta}^* \hat{S}(\hat{\delta}\bar{u}) \bar{u}^* dQ = \\ &= \int_Q \left( \frac{\hat{\delta} + \hat{\delta}^*}{2} \right) |\bar{u}|^2 dQ - i \int_Q \left( \frac{\hat{\delta} - \hat{\delta}^*}{2i} \right) |\bar{u}|^2 dQ - \int_Q (\hat{\delta}\bar{u})^* [\hat{S}(\hat{\delta}\bar{u})] dQ. \end{aligned}$$

We conclude from (2.29) that

$$\left| (\hat{A}\bar{u}, \bar{u}) \right| \geq \left| \text{Im}(\hat{A}\bar{u}, \bar{u}) \right| \geq \min_{x \in Q} \lambda(x)(u, u) = p_0(u, u), \quad p_0 > 0. \quad (2.31)$$

where  $\lambda(x)$  is the minimum eigenvalue of Hermitian tensor

$$\hat{\delta}_2(x) = \frac{(\hat{\delta}(x) - \hat{\delta}^*(x))}{2i} = \frac{(\hat{\varepsilon}(x) - \hat{\varepsilon}(x) - 2i \text{Im } \varepsilon_0)}{2i}.$$

Note that if  $\hat{\varepsilon}(x) = \varepsilon(x) \hat{I}$ , i.e., the medium is isotropic, then  $p_0 = \min_{x \in Q} \text{Im}(\varepsilon(x) - \varepsilon_0)$ .

The theorem now follows from Theorem 4.3 and equivalence of Eq. (2.28) and Eq. (2.30). The conditions of Theorem 2.3 is more strict than condition of Theorem 2.2 (they are virtually identical if  $\text{Im} \varepsilon_0 = 0$ , which takes place in most of real problems). However, no smoothness of the tensor function  $\hat{\varepsilon}(x)$  is required in Theorem 2.3 therefore in this case the solution of Eq. (2.13) satisfies Maxwell equation in generalized sense.

## 2.7 Spectrum of integral operator [15, 16]

The *spectrum* of the operator  $\hat{A}$  on the complex plane  $Z$  is the set of points  $\lambda$  such that the operator  $(\hat{A} - \lambda \hat{I})$  does not have an inverse defined everywhere in the Hilbert space  $H$ . The points  $\lambda$  such that the operator  $(\hat{A} - \lambda \hat{I})$  is not Noether belong to the continuous part of the spectrum of  $\hat{A}$ . The points  $\lambda$  such that  $(\hat{A} - \lambda \hat{I})$  is a Fredholm operator and there exists a nontrivial solution  $u$ ,  $\hat{A}u - \lambda u = 0$  belong to the discrete part of the spectrum of  $\hat{A}$ .

### 2.7.1 Continuous part of spectrum

Rewrite integral equation (2.13) in the symbolic form

$$\hat{A}u \equiv u - \hat{S}((\hat{\varepsilon}_r - \hat{I})u) = f. \quad (2.32)$$

Obviously

$$\hat{A} - \lambda \hat{I} = (1 - \lambda) \left[ \hat{I} - \hat{S} \left( \frac{\hat{\varepsilon}_r - \lambda I}{1 - \lambda} - I \right) \right]. \quad (2.33)$$

By comparing (2.32) and (2.33) from Theorem 2.1 we find that the continuous part of the spectrum of the operator in equation (2.13) contains the set  $\sigma_1$  of points on the complex plane given by the formula

$$\lambda = \frac{1}{\varepsilon_0} \sum_{n,m=1}^3 \varepsilon_{nm}(x) \beta_n \beta_m, \quad x \in Q, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \quad (2.34)$$

It follows that the point  $\lambda = 1$  belongs to  $\sigma_1$  since  $\varepsilon_{nm} = \delta_{nm} \varepsilon_0$  on the boundary of the domain  $Q$ . Note that, by virtue of the Holder continuity of the permittivity tensor function  $\hat{\varepsilon}_r(x)$ , the set  $\sigma_1$  is a connected subset of the complex plane.

For isotropic case ( $\hat{\varepsilon}_r(x) = \varepsilon_r(x) \hat{I}$ ) we have the following formula

$$\lambda = \varepsilon_r(x), \quad x \in Q. \quad (2.35)$$

### 2.7.2 Spectrum for low-frequency case

Now we will assume that  $\hat{\varepsilon}_r(x) = \varepsilon_r(x) \hat{I}$  (we consider isotropic case). In this case integral equation (2.13) has the form

$$\begin{aligned} \vec{E}(x) + \frac{1}{3}(\varepsilon_r(x) - 1) \vec{E}(x) - p.v. \int_Q ((\varepsilon_r(y) - 1) \vec{E}(y), \text{grad}) \text{grad} G(R) dy - \\ k_0^2 \int_Q (\varepsilon_r(y) - 1) \vec{E}(y) G(R) dy = \vec{E}^0(x), \quad x \in Q. \end{aligned} \quad (2.36)$$

From Theorem 3.5 and (2.35) we have the following statement.

**Lemma 2.1.** *The operator of equation (2.36) is Fredholm in the space  $\vec{L}_2(Q)$  if the set  $\sigma_1$  of values  $\varepsilon_r(x)$ ,  $x \in Q$  does not contain a closed curve surrounding the point  $\lambda = 0$  on the complex plane.*

Denote

$$\varepsilon_r^+(\lambda, x) = (\varepsilon_r(x) - \lambda) / (1 - \lambda), \quad \lambda \notin \sigma_1. \quad (2.37)$$

It follows from (2.32), (2.33) and (2.37) that the operator  $(\hat{A} - \lambda \hat{I})$  is Fredholm if the function  $\varepsilon_r^+(\lambda, x)$  satisfies the assumptions of Lemma 2.1. We denote the boundary of  $\sigma_1$  by  $\gamma_1$  and the set of all points of the complex plane  $Z$  lying on and inside the boundary  $\gamma_1$  by  $\sigma^+$ . If, in particular, the set  $\sigma_1$  is a non closed curve, then  $\sigma^+ = \sigma_1 = \gamma_1$ . Therefore, taking into account the preceding considerations and performing simple computations, we obtain the following assertion.

**Lemma 2.2** *The operator  $(\hat{A} - \lambda \hat{I})$  is Fredholm in the space  $\vec{L}_2(Q)$  if  $\lambda \in Z \setminus \sigma^+$ .*

In the general case it is impossible to describe the localization domain of the discrete spectrum of the operator accurately. However, this can be done in a special case which is important in



practice. Consider low-frequency electromagnetic wave scattering problems such that the diameter of  $Q$  is much less than the wavelength,  $D \ll \lambda$ , where  $\lambda = 2\pi/k_0$ .

Equation (2.36) can be applied when the wave number  $k_0 = 0$ , i.e., for the static case. Obviously, all preceding assertions remain valid in the static case. It follows from (2.36) that

$$\left(\hat{A}(k_0) - \hat{A}(0)\right)\vec{V} = -k_0^2 \int_Q (\varepsilon_r - 1)\vec{E}(y)G(R)dy - \int_Q ((\varepsilon_r - 1)\vec{V}(y), \text{grad}) \text{grad } G_0(R)dy, \quad (2.38)$$

where  $\hat{A}(k_0)$  and  $\hat{A}(0)$  are the operators in the integral equations for the stationary and static cases, respectively, and  $G_0(R)$  is determined by (2.11). The second integral operator in (2.38) does not contain a singular integral since the kernel of this operator has no singularity at  $x=y$  and is a smooth function of the coordinates. Therefore, from (2.38) we obtain

$$\lim_{r \rightarrow \infty} \left\| \hat{A}(k_0) - \hat{A}(0) \right\| = 0. \quad (2.39)$$

From (2.39) we have the following assertion.

**Lemma 2.3** *The spectrum of the low-frequency integral operator  $\hat{A}(k_0)$  tends to the spectrum of the static operator  $\hat{A}(0)$  as  $k_0 \rightarrow 0$ .*

In the static case, the integral equation (2.10) can be represented in the form

$$\vec{E}(x) - \text{grad div} \int_Q (\varepsilon_r(y) - 1)\vec{E}(y)(1/4\pi R)dy = \vec{E}^0(x). \quad (2.40)$$

The solution of the homogeneous equation (2.40) satisfies the differential equations

$$\text{rot } \vec{E} = 0, \quad \text{div}(\varepsilon_r \vec{E}) = 0 \quad (2.41)$$

The first equation (2.41) follows from the identity  $\text{rot grad} = 0$ , and the second equation follows from the identities  $\text{grad div} = \text{rot rot} + \Delta$  and  $\text{div rot rot} = 0$  and the differential equation  $\Delta \vec{A} = -\vec{J}$  which is valid for the volume potential  $\vec{A}(x) = \int \vec{J}(y)(1/4\pi R)dy$ .

From the first equation in (2.41), we have  $\vec{E} = \text{grad } \varphi$ . Then equations (2.41) can be reduced to a second-order differential equation for the function  $\varphi$

$$\text{div}(\varepsilon_r \text{grad } \varphi) = 0. \quad (2.42)$$

Let  $\psi$  be an everywhere-defined differentiable function. We have an obvious identity

$$\text{div}(\psi \varepsilon_r \text{grad } \varphi) = \psi \text{div}(\varepsilon_r \text{grad } \varphi) + (\text{grad } \psi, \varepsilon_r \text{grad } \varphi). \quad (2.43)$$

Let  $\psi = \bar{\varphi}$ . Then, by integrating relation (2.43) over the space and by taking into account (2.42) and the divergence theorem, we obtain the integral relation

$$\int \varepsilon_r |\text{grad } \varphi|^2 d\nu = \lim_{R \rightarrow \infty} \int_{S_R} \bar{\varphi} \frac{\partial \varphi}{\partial n} dS, \quad (2.44)$$

where  $S_R$  is the sphere of radius  $R$  centered at the origin and  $n$  is the normal to the sphere. Since  $\varphi$  is a harmonic function outside  $Q$ , it follows that  $\bar{\varphi} \partial \varphi / \partial n$  decreases as  $R^{-3}$  at infinity. Therefore, the limit on the right-hand side in (2.44) is zero, and each solution of the homogeneous equation (2.40) satisfies the integral relation

$$\int \varepsilon_r |\text{grad } \varphi|^2 d\nu = 0. \quad (2.45)$$

It follows from (2.33) and (2.37) that  $\lambda$  is a point of the discrete spectrum of the operator (2.40) if there exists a nonzero solution  $\varphi$  of Eq. (2.42) with permittivity  $\varepsilon_r^+(\lambda, x)$ . Moreover, it follows from (2.45) and (2.37) that the corresponding value of  $\lambda$  is given by the formula

$$\lambda = \frac{\int \varepsilon_r |\text{grad } \varphi|^2 d\nu}{\int |\text{grad } \varphi|^2 d\nu}. \quad (2.46)$$

It is impossible to find the corresponding functions  $\varphi$ . However, using formula (2.46), one can find the localization domain of points of the discrete spectrum on the complex plane: the points of the discrete spectrum of the integral operator (2.40) can lie only inside the convex envelope of the set  $\sigma_1$  given by formula (2.35). Set  $\sigma^+$  lies inside convex envelope of the set  $\sigma_1$ . Therefore we arrive to the statement.

**Theorem 2.4** *Spectrum of the integral operator (2.40) can lie only inside the convex envelope of the set  $\sigma_1$  given by formula (2.35).*

Theorem 2.4 and Lemma 2.3 provide approximate information about a convex envelope of the spectrum of the integral operator for the low-frequency case.

### 2.7.3 Example

Let domain  $Q$  in the equation (2.36) be a ball and suppose that function of dielectric permittivity has the following form in the spherical system of coordinates

$$\frac{\varepsilon(r)}{\varepsilon_0} = \begin{cases} \varepsilon_2, & d_2 \geq r \geq 0 \\ \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \frac{r - d_2}{d_1 - d_2}, & d_1 \geq r \geq d_2 \\ \varepsilon_1 + (1 - \varepsilon_1) \frac{r - d_1}{R - d_1}, & R \geq r \geq d_1 \end{cases}. \quad (2.47)$$

In (2.47)  $R$  is a radius of a ball, and  $R > d_1 > d_2 > 0$ .

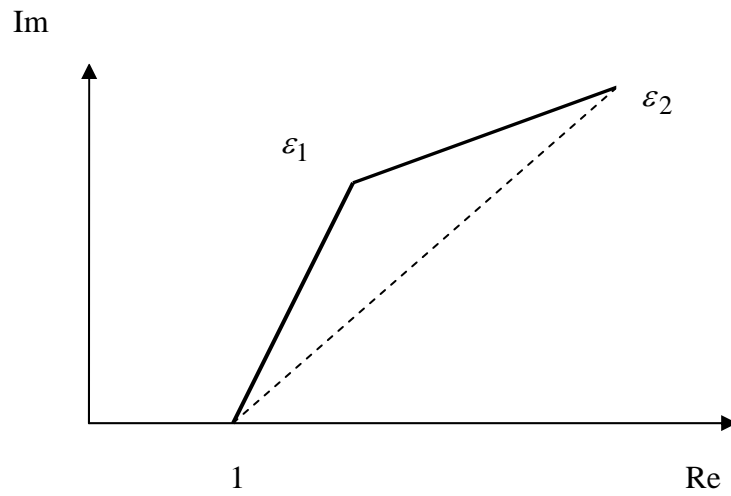


Figure 2.1: Spectrum of integral operator.

On Fig. 2.1 fat solid line schematically outline the continuous part of spectrum for the case (2.47). The spectrum of the integral operator for the low-frequency case lies inside the triangle.

# CHAPTER 3

## Certain results of mathematical analysis

### 3.1 Some functional analysis

**Definition.** Set  $X$  is called a linear space if for each two elements  $u$  and  $v$  from  $X$  their sum  $(u+v)$  is defined and is also an element of  $X$ . Also for each element  $u \in X$  and number  $\lambda$  product  $\lambda u$  is defined which is also an element of  $X$ . Moreover, these operations satisfy the following properties:

1.  $(u+v)+f = u+(v+f)$ ,  $u, v, f \in X$ ;
2.  $u+v = v+u$ ;
3. In  $X$  there is a zero element  $\theta$ , such that for any  $u \in X$ , we have  $0u = \theta$ ;  
 $u + \theta = u$ ;
4.  $(\lambda + \mu)u = \lambda u + \mu u$ ;
5.  $\lambda(u+v) = \lambda u + \lambda v$ ;
6.  $(\lambda\mu)u = \lambda(\mu u)$ ;
7.  $1u = u$ .

**Definition.** A norm on a linear space  $X$  is a real-valued function whose value at  $u \in X$  is denoted by  $\|u\|$ , and which has the following properties:

$$\|u\| \geq 0; \quad \|u\| = 0, \text{ if and only if } u = \theta; \quad \|\alpha u\| = |\alpha| \|u\|; \quad \|u+v\| \leq \|u\| + \|v\|.$$

Here  $u$  and  $v$  are arbitrary elements of  $X$  and  $\alpha$  is any scalar.

If a linear space has a norm then we have a *normed linear space*.

If for any fundamental sequence  $v_m, m=1,2,\dots$  from a normed linear space  $X$  there exists a limit  $v \in X$ , then we have a *complete space* (a *Banach space*).

**Definition.** Let  $B$  be a Banach space. Mapping  $A: B \rightarrow B$  is called a linear operator if for any  $u, v \in B$  and  $\lambda, \mu$  we have

$$A(\lambda u + \mu v) = \lambda Au + \mu Av.$$

**Definition.** Operator  $A$  is said to be bounded in the Banach space  $B$  if there is a real number  $C$  such that for any  $u \in B$   $\|Au\| \leq C \|u\|$ .

Minimal value of such numbers  $C$  is a norm  $\|A\|$  of a linear operator.

If a linear operator is bounded then it is continuous. It means that if  $v = \lim_{m \rightarrow \infty} v_m$  then

$$\lim_{m \rightarrow \infty} Av_m = Av.$$

If in a Banach space  $B$  for any elements  $u, v \in B$  we can define the inner product  $(u, v)$  with the following properties:

$$(u+v, w) = (u, w) + (v, w), \quad (\alpha u, v) = \alpha (u, v), \quad (u, v) = (v, u)^*,$$

$$(u, u) \geq 0, \quad (u, u) = 0 \text{ if and only if } u = \theta,$$

then we have a *Hilbert space*. In this case  $\|u\| = \sqrt{(u, u)}$ .

For the Hilbert space  $L_2(Q)$  of square-integrable functions we have the following inner product

$$(u, v) = \int_Q u(x)v(x)^* dx. \quad (3.1)$$

The integral operator acting in the Hilbert space  $L_2(Q)$  ( $Q$  is a bounded 3D domain) has the following form

$$(Au)(x) = \int_Q \frac{K(x, y)}{R^m} u(y) dy, \quad x \in Q. \quad (3.2)$$

Here  $K(x, y)$  is a bounded function of the coordinates  $x$  and  $y$ ,  $R = |x - y|$ , and  $3 > m \geq 0$ . If  $3 > m > 0$  then we have a weakly singular integral operator. All such operators are bounded in the Hilbert space  $L_2(Q)$ .

Let us formulate several definitions.

**Definition.** Let  $A$  be a linear operator acting in the Hilbert space  $H$ . Then operator  $A^*$ , which is also defined in  $H$ , is called conjugate to  $A$  if the equality

$$(A f, g) = (f, A^* g)$$

holds for all  $f, g \in H$ .

The solutions of the homogeneous equation  $Au = 0$  will be called zeros of operator  $A$ . Dimension of the subspace of zeros will be denoted by  $n(A)$ . Then  $n(A^*)$  is the dimension of the subspace of zeros of the conjugate operator  $A^*$ . The difference

$$\text{Ind } A = n(A) - n(A^*)$$

is called the index of operator  $A$ .

**Definition.** A linear operator acting in the Hilbert space  $H$  is called normally solvable if the domain of images of  $A$  is an orthogonal completion to the subspace of zeros of the operator  $A^*$ .

Thus, if  $A$  is a normally solvable operator, then the equation  $Au = f$  is solvable if and only if its free term  $f$  is orthogonal to all zeros of the conjugate operator  $A^*$ .

**Definition.** Linear operator  $A$  is called the Noether operator if it is normally solvable and its index is finite.

**Definition.** Linear operator  $A$  is called the Fredholm operator if it is a Noether operator and its index is equal to zero.

According to this definition, we can formulate a sufficient solvability condition: if  $A$  is a Fredholm operator, then the equation  $Au = f$  is uniquely solvable for any  $f \in H$  if the homogeneous equation  $Au = 0$  has only the trivial solution.

Note that sometimes the Noether operator is called the Fredholm operator and the Fredholm operator is called the Fredholm operator with the zero index.

**Definition.** A linear operator  $K$  acting in the Hilbert space  $H$  is called compact operator if for every bounded subset  $M$  of  $H$ , the image  $K(M)$  is compact.

Operators of the form  $(I + K)$  are Fredholm operators.

Linear integral operator (3.2) is a compact operator in the Hilbert space  $L_2(Q)$  if  $m < 3$ .

Let us consider the following integral equation.

$$u(x) + \int_Q \frac{K(x, y)}{R^m} u(y) dy = f(x), \quad x \in Q, \quad m < 3. \quad (3.3)$$

Then we have the following statement.

**Theorem 3.1** There exists the unique solution of the integral equation (3.3) in  $L_2(Q)$  for any  $f \in L_2(Q)$  if homogeneous equation ( $f = 0$ ) has only the trivial solution.

### 3.2 Derivatives of a weakly singular integral

Consider the following expression in the Cartesian coordinate system

$$V(x) = \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} \int_Q \frac{U(y)}{|x-y|} dy, \quad n, m = 1, 2, 3, \quad (3.4)$$

where  $U(x)$  equals zero outside domain  $Q$ .

We can write

$$\begin{aligned} V(x) &= \frac{\partial}{\partial x_n} \int_Q \frac{\partial}{\partial x_m} \left[ \frac{1}{|x-y|} \right] U(y) dy = \\ &= - \frac{\partial}{\partial x_n} \int_Q |x-y|^{-2} \frac{(x_m - y_m)}{|x-y|} U(y) dy. \end{aligned} \quad (3.5)$$

Now we have to be very careful, since should we differentiate one more time, a term with  $|x-y|^{-3}$  will appear under the integral sign. Indeed, denote

$$\alpha_m = \frac{x_m - y_m}{|x-y|} \quad (3.6)$$

and compute explicitly

$$- \frac{\partial}{\partial x_n} \left( \frac{\alpha_m}{|x-y|^2} \right) = \frac{1}{|x-y|^3} (3\alpha_n \alpha_m - \delta_{nm}), \quad (3.7)$$

where  $\delta_{nm}$  is the Kroneker's symbol and  $\alpha_n, n = 1, 2, 3$ , are the Cartesian coordinates of the points of the unit sphere  $S$ . In the spherical system of coordinates

$$\alpha_1 = \sin \theta \cos \varphi, \alpha_2 = \sin \theta \sin \varphi, \alpha_3 = \cos \theta. \quad (3.8)$$

Then, we arrive at a *strongly singular* integral. A mathematical trick which we are going to use now consists of splitting the domain of integration into two parts: one with singularity and one without. Namely, for every fixed  $x$  we represent (3.5) as a sum of two integrals:

$$V(x) = \frac{\partial}{\partial x_n} \int_{Q \setminus |x-y| < \varepsilon} \left[ \frac{-\alpha_m}{|x-y|^2} \right] U(y) dy + \frac{\partial}{\partial x_n} \int_{|x-y| < \varepsilon} \left[ \frac{-\alpha_m}{|x-y|^2} \right] U(y) dy.$$

Next, we take a limit  $\varepsilon \rightarrow 0$ , i.e.

$$V(x) = \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial x_n} \int_{Q \setminus |x-y| < \varepsilon} \left[ \frac{-\alpha_m}{|x-y|^2} \right] U(y) dy + \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial x_n} \int_{|x-y| < \varepsilon} \left[ \frac{-\alpha_m}{|x-y|^2} \right] U(y) dy. \quad (3.9)$$

This will help us to obtain an explicit expression for the second integral. Notice that for any  $\varepsilon \neq 0$  the first integral does not contain any singularity and we can substitute (3.7) into this expression. First consider the second integral. We add and subtract  $U(x)$  under the integral sign and arrive at

$$\lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \left\{ \frac{\partial}{\partial x_n} \frac{-\alpha_m}{|x-y|^2} \right\} [U(y) - U(x) + U(x)] dy = \quad (3.10)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \left\{ \frac{\partial}{\partial x_n} \left[ \frac{-\alpha_m}{|x-y|^2} \right] \right\} [U(y) - U(x)] dy + \lim_{\varepsilon \rightarrow 0} U(x) \int_{|x-y| < \varepsilon} \left\{ \frac{\partial}{\partial x_n} \left[ \frac{-\alpha_m}{|x-y|^2} \right] \right\} dy.$$

**Definition.** Function  $u$  is said to be *Holder-continuous* in a domain  $D$  if the inequality

$$|u(y) - u(x)| \leq C|x-y|^\delta, \quad C = \text{const}, \delta > 0$$

is valid for any  $x, y \in D$ .

Suppose that  $U(x)$  is a Holder-continuous function in  $Q$ .

The first integral of the last expression (3.10) contains a difference  $[U(y) - U(x)]$ .

Then, we would have the following

$$\begin{aligned} & \left| \lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \left\{ \frac{\partial}{\partial x_n} \left[ \frac{-\alpha_m}{|x-y|^2} \right] \right\} [U(y) - U(x)] dy \right| \leq \\ & C \lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \left| \frac{\partial}{\partial x_n} \left[ \frac{-\alpha_m}{|x-y|^2} \right] \right| |x-y|^\delta dy = C \lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \frac{|3\alpha_n \alpha_m - \delta_{nm}|}{|x-y|^{3-\delta}} dy = 0 \end{aligned} \quad (3.11)$$

The zero at the end follows from the fact that the resulting integral is weakly singular for  $\delta > 0$  (since  $3 - \delta < 3$ ) and that the domain of integration tends to zero in the limit. Hence, we are left with the last integral of (3.10) which we transform in the following way (we use the Gauss theorem here, since the domain of integration is a ball and has a spherical boundary  $S_\varepsilon$  with normal  $\nu_n = -\alpha_n$ , where  $\alpha_n$  are determined by (3.8)):

$$\begin{aligned}
U(x) \lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \frac{\partial}{\partial x_n} \left[ \frac{-\alpha_m}{|x-y|^2} \right] dy &= U(x) \lim_{\varepsilon \rightarrow 0} \int_{|x-y| < \varepsilon} \frac{\partial}{\partial y_n} \left[ \frac{\alpha_m}{|x-y|^2} \right] dy = \\
U(x) \lim_{\varepsilon \rightarrow 0} \int_{y \in S_\varepsilon} \frac{\nu_n \alpha_m}{|x-y|^2} dS &= U(x) \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^{2\pi} \int_0^\pi \frac{-\alpha_n \alpha_m}{\varepsilon^2} \sin \theta d\theta d\varphi = -U(x) \frac{4\pi}{3} \delta_{nm}.
\end{aligned} \tag{3.12}$$

Thus, from (3.9)-(3.12), it follows that the total expression for the two spatial derivatives of a weakly singular integral (3.5) is

$$V(x) = \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} \int_Q \frac{U(y)}{|x-y|} dy = \lim_{\varepsilon \rightarrow 0} \int_{Q \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] U(y) dy - \frac{4\pi}{3} U(x) \delta_{nm}. \tag{3.13}$$

Now we will show that the limit of the integral in expression (3.13) exists. Let  $\Omega(x)$  be a ball with the center at the point  $x \in Q$ ,  $Q \subset \Omega(x)$ . Because function  $U(x)$  does not equal zero only in the domain  $Q$  we can write for all  $x \in Q$

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{Q \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] U(y) dy &= \\
\lim_{\varepsilon \rightarrow 0} \int_{\Omega(x) \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] U(y) dy &= \\
\lim_{\varepsilon \rightarrow 0} \int_{\Omega(x) \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] [U(y) - U(x)] dy + \\
U(x) \lim_{\varepsilon \rightarrow 0} \int_{\Omega(x) \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] dy &.
\end{aligned} \tag{3.14}$$

At first we will calculate the last integral (3.14) by using spherical system of coordinates with the origin at the point  $x$ . We have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega(x) \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] dy = ,$$



$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^R \frac{1}{r} \left[ \int_0^{\pi} \int_0^{2\pi} [3\alpha_n \alpha_m - \delta_{nm}] \sin \theta d\theta d\varphi \right] dr, \quad (3.15)$$

where  $R$  is a radius of the ball  $\Omega(x)$ .

Obviously, from (3.8) it follows that integral (3.15) equals zero.

Consider the first integral in the last expression (3.14). Since  $U(x)$  is a Holder-continuous function we have

$$[U(y) - U(x)] = \varphi(x, y)|x - y|^{\delta},$$

where  $\varphi(x, y)$  is a bounded function of coordinates. Then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega(x) \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] [U(y) - U(x)] dy = \\ \int_{\Omega(x)} \frac{1}{|x-y|^{3-\delta}} [3\alpha_n \alpha_m - \delta_{nm}] \varphi(x, y) dy. \end{aligned} \quad (3.16)$$

The integral in (3.16) exists in the usual sense. So, the limit of the integral in expression (3.13) exists. This integral is also known as the principal value (*p.v.*) of a strongly singular integral

$$p.v. \int_Q \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] U(y) dy = \lim_{\varepsilon \rightarrow 0} \int_{Q \setminus |x-y| < \varepsilon} \frac{1}{|x-y|^3} [3\alpha_n \alpha_m - \delta_{nm}] U(y) dy \quad (3.17)$$

### 3.3 Elements of the theory of singular volume integral equations [2]

Let  $Q$  be a domain in  $E_3$ . This domain may be finite, infinite and, in particular, coincide with  $E_3$ . We will consider the following singular integrals

$$V(x) = p.v. \int_Q u(y) \frac{f(y, \alpha)}{R^3} dy = \lim_{\varepsilon \rightarrow 0} \int_{Q \setminus (R < \varepsilon)} u(y) \frac{f(y, \alpha)}{R^3} dy. \quad (3.18)$$

Here  $R = |x - y|$  is the distance between points  $x$  and  $y$  and  $\alpha = (x - y)/R$  is a point on a unit sphere  $S$  centered at the origin.

Point  $x$  is called a pole of the singular integral,  $f(x, \alpha)$  the characteristic, and  $u(y)$  the density. Everywhere below, we will assume that the characteristic satisfies the condition

$$\int_S f(x, \alpha) dS = 0. \quad (3.19)$$

It is easy to verify that integral (3.18) does not exist if condition (3.19) is violated.

Below, we will consider singular integrals as operators in a Hilbert space  $L_2(E_3)$ .

**Theorem 3.2.** Let characteristic  $f(x, \alpha)$  satisfy condition (3.19) and the inequality

$$\int_S |f(x, \alpha)|^2 dS \leq C = \text{const} \quad (3.20)$$

be valid for every point  $x$ . Then, if  $u \in L_2(E_3)$ , singular integral (3.18) exists at almost all points  $x \in E_3$ ; in addition,  $v \in L_2(E_3)$  and singular operator is bounded in  $L_2(E_3)$ .

**Definition.** A linear operator  $\hat{A}$  considered in a Hilbert space  $L_2(E_3)$  and defined as

$$(\hat{A}u)(u) = a(x)u(x) + p.v. \int_{E_3} \frac{f(y, \alpha)}{R^3} u(y) d(y) + (\hat{K}u)(x), \quad (3.21)$$

where  $\hat{K}$  is a compact operator in  $L_2(E_3)$ , will be called the singular integral operator in  $L_2(E_3)$ .

We will assume below that characteristic is a Holder-continuous function with respect to  $x$  and  $f(x, \alpha) = 0$  if  $x \in E_3 \setminus Q$  where  $Q$  is a finite domain in  $E_3$ .

The analysis of solvability of linear equations with singular operator (3.21) is much more complicated than in the case of integral equations with a Fredholm kernel. Therefore, the corresponding theory was elaborated substantially later and in less detail. Singular integral equations are analyzed using the notion of the symbol of a singular operator  $\hat{A}$ . The symbol is defined as a function  $\Phi_A(x, \beta)$  of points  $x \in E_3$  and  $\beta \in S$ , where  $S$  is the unit sphere.

The symbol must satisfy the following three conditions:

- (I) the symbol of every compact operator is zero;
- (II) the symbol of the sum of two singular operators equals the sum of their symbols;
- (III) the symbol of the product of two singular operators equals the product of their symbols.

The symbol is defined using several different formulas. We will present one of them, which is the most convenient for the analysis of specific singular integral equations.

For most applications characteristic can be represented as  $f(x, \alpha) = f_1(x) f_2(\alpha)$ . The symbol of the singular integral operator (3.21) is given by the formula

$$\Phi_A(x, \beta) = a(x) + f_1(x) \frac{F \left[ p.v. \int_{E_3} \frac{f_2(\alpha)}{R^3} u(y) dy \right]}{F[u(x)]}; \quad \beta = \frac{k}{|k|}, \quad (3.22)$$

where  $F$  denotes the Fourier transform

$$F[v(x)]_{(k)} = \int_{E_3} \exp(ikx) v(x) dx \quad (3.23)$$

Here  $kx = k_1x_1 + k_2x_2 + k_3x_3$  and  $dx = dx_1 dx_2 dx_3$ .

In many cases, the singular integral is a result of derivation of a weakly singular integral

$$V(x) = \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} \int_Q u(y) \frac{f(y)}{R} dy \quad (3.24)$$

Then, according to (3.13) and (3.17) we obtain

$$V(x) = p.v. \int_Q u(y) \frac{f(y)(3\alpha_n \alpha_m - \delta_{nm})}{R^3} dy - \frac{4\pi}{3} f(x) u(x) \delta_{nm} \quad (3.25)$$

The function

$$V_0(x) = \int_{E_3} \frac{1}{4\pi R} U(y) dy \quad (3.26)$$

satisfies the Laplas equation

$$\Delta V_0 = -U. \quad (3.27)$$

Then, we have the following relationships

$$\begin{aligned} F \left[ \Delta \int_{E_3} \frac{1}{4\pi R} U(y) dy \right] &= -|k|^2 F \left[ \int_{E_3} \frac{1}{4\pi R} U(y) dy \right] = F[-U], \\ F \left[ \int_{E_3} \frac{1}{4\pi R} U(y) dy \right] &= \frac{F(U)}{|k|^2}, \\ F \left[ \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} \int_{E_3} \frac{1}{4\pi R} U(y) dy \right] &= -\frac{k_n k_m}{|k|^2} F[U] = -\beta_n \beta_m F[U]. \end{aligned} \quad (3.28)$$

Finally, from (3.24)-(3.28), we find that the symbol of the singular operator

$$(\hat{A}_0 u)(x) = p.v. \int_Q u(y) \frac{f(y)(3\alpha_n \alpha_m - \delta_{nm})}{R^3} dy \quad (3.29)$$

has the following form

$$\text{symbol}(\hat{A}_0) = \Phi_0(x, \beta) = -4\pi f(x) \beta_n \beta_m + \frac{4\pi}{3} f(x) \delta_{nm}. \quad (3.30)$$

The following statement is valid.

**Theorem 3.3.** *Let  $\hat{A}$  be a singular integral operator of the form (3.21). Then  $\hat{A}$  is a Fredholm operator in  $L_2(E_3)$  if and only if the symbol does not degenerate, i.e.,*

$$\inf_{x \in E_3, \beta \in S} |\Phi(x, \beta)| > 0. \quad (3.31)$$

Consider a system of singular integral equations

$$\sum_{n=1}^3 \hat{A}_{nm} u_m = f_n; \quad n = 1, 2, 3, \quad (3.32)$$

where  $u_n, f_n \in L_2(E_3)$  and  $\hat{A}_{nm}$  are singular integral operators (3.21). System (3.32) can be represented as a vector operator equation

$$\hat{A}\vec{u} = \vec{f}.$$

Here  $\vec{u}$  and  $\vec{f}$  are vector-functions and

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad (3.33)$$

is a matrix singular operator.

We denote the symbol of operator  $\hat{A}_{nm}$  by  $\Phi_{nm}(x, \beta)$ . The matrix

$$\Phi(x, \beta) = \begin{pmatrix} \Phi_{11}(x, \beta) & \Phi_{12}(x, \beta) & \Phi_{13}(x, \beta) \\ \Phi_{21}(x, \beta) & \Phi_{22}(x, \beta) & \Phi_{23}(x, \beta) \\ \Phi_{31}(x, \beta) & \Phi_{32}(x, \beta) & \Phi_{33}(x, \beta) \end{pmatrix} \quad (3.34)$$

will be called the matrix symbol or simply the symbol of matrix operator  $\hat{A}$ .

**Theorem 3.4.** *Let  $\hat{A}$  be a matrix singular integral operator of the form (3.33). Then  $\hat{A}$  is a Noether operator in  $\bar{L}_2(E_3)$  if and only if matrix symbol does not degenerate, i.e.,*

$$\inf_{x \in E_3, \beta \in S} |\det \Phi(x, \beta)| > 0 \quad (3.35)$$

Unlike the case of one equation, the index of system (3.32) may not be equal to zero. The following theorem establishes sufficient conditions for the Fredholm property.

**Theorem 3.5.** *Let  $\hat{A}$  be a matrix singular integral operator of the form (3.33) satisfying the condition of Theorem 3.4. Then  $\hat{A}$  is a Fredholm operator in  $\bar{L}_2(E_3)$  if there is a smooth curve in the complex  $\lambda$ -plane that joins  $\lambda = 0$  and  $\lambda = \infty$  and has no common points with the set of eigenvalues of the symbol matrix.*

# PART II

## Numerical methods for integral equations

### Introduction

To solve the integral equation numerically, one reduces it to a system of linear algebraic equations (SLAE). The solution of that system must approximate the solution of the original problem with a prescribed accuracy. Let  $h$  be a typical length on which the solution  $u$  varies only slightly. The specific values of  $h$  are determined by the desired accuracy of the solution. As a rule, *a priori* estimates of  $h$  necessary for obtaining the desired accuracy of the solution are known in specific problems. Then the dimension  $N$  of SLAE can be estimated as  $N \approx (mesQ/h^3)$ . It turns out that  $N$  is very large,  $N \gg 1000$ . Then it is virtually impossible to use direct methods since this would require performing  $T \sim N^3$  arithmetic operations and storing  $N^2$  entries of the matrix of the SLAE in computer memory.

It is clear that we must apply an iteration method. Number  $T$  of arithmetic operations that guarantees the required accuracy of solution and memory volume required for the implementation of the algorithm are the main efficiency criteria for any numerical algorithm. Multiplication of matrix SLAE by vector is the most laborious operation of the iteration method. Therefore, the number of multiplications for the implementation of a particular algorithm will be called the number of iterations. The value of  $T$  is estimated by the formula

$$T \approx L(T_A + T_0)$$

Here,  $L$  is the number of iterations,  $T_A$  is the number of arithmetic operations required for multiplication of a matrix by a vector, and  $T_0$  is the number of arithmetic operations required for other computations. As a rule  $T_0 \ll T_A$ .

Our main purpose is the minimization of the values  $T_A$  and  $L$ .

# CHAPTER 4

## Iteration methods

### 4.1 Simple iteration method

In the Banach space  $B$ , we consider the linear operator equation

$$\hat{A}u = f, \quad (4.1)$$

where  $\hat{A}$  is a bounded operator.

Rewrite Eq. (4.1) in the equivalent form

$$u - \hat{B}_\mu u = f / \mu. \quad (4.2)$$

Here  $\hat{B}_\mu$  is the linear operator given by the formula  $\hat{B}_\mu = (\mu \hat{I} - \hat{A}) / \mu$  and  $\mu \neq 0$  is an arbitrary complex number.

The successive approximations

$$u_{n+1} = \hat{B}_\mu u_n + f / \mu, \quad n = 0, 1, \dots \quad (4.3)$$

converge to the solution of Eq. (4.2) and hence of Eq. (4.1) for any  $u_0, f \in B$  provided that

$$\rho_0(\mu) = \sup_{\eta \in \sigma(\hat{B}_\mu)} |\eta| < 1, \quad \eta \in \sigma(\hat{B}_\mu). \quad (4.4)$$

One can readily show that there is a one-to-one correspondence between points of the spectrum  $\sigma(\hat{A})$  of the operator  $\hat{A}$  and points of the spectrum  $\sigma(\hat{B}_\mu)$  of the operator  $\hat{B}_\mu$ ; this correspondence is given by the formula

$$\eta = (\mu - \lambda) / \mu, \quad \lambda \in \sigma(\hat{A}), \quad \eta \in \sigma(\hat{B}_\mu). \quad (4.5)$$

The iterations (4.3) can be represented in the simpler form

$$u_{n+1} = u_n - \frac{1}{\mu} (\hat{A}u_n - f), \quad n = 0, 1, \dots \quad (4.6)$$

One can prove the following statement [8].

**Theorem 4.1.** *A necessary and sufficient condition for the existence of complex number  $\mu$  such that the iterations (4.6) converge to the solution of Eq. (4.1) for arbitrary  $u_0, f \in B$  is that the origin of the complex plane lies outside a convex envelope of the spectrum of  $\hat{A}$ .*

The convex envelope is illustrated in Fig. 4.1.

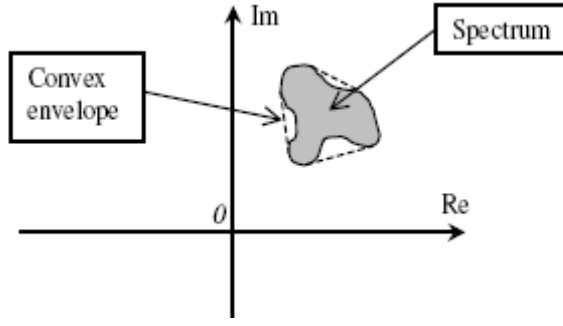


Figure 4.1: Spectrum and its convex envelope.

The iteration converges to the solution at the rate of a geometric progression; i.e.

$$\|u_n - u\| \leq C [\rho_0(\mu)]^n, \quad C = \text{const}, \quad (4.7)$$

where, by (4.4) and (4.5),  $\rho_0(\mu)$  is given by the formula

$$\rho_0(\mu) = \frac{\sup |\mu - \lambda|}{|\mu|}, \quad \lambda \in \sigma(\hat{A}). \quad (4.8)$$

Obviously, the best convergence of the iterations is attached at the value of  $\mu$  for which the function  $\rho_0(\mu)$  takes the minimum value. By  $S_\mu$  we denote the disk on the complex plane with center  $\mu$  and the least radius  $R$  which contains all points of the spectrum of  $\hat{A}$ . Obviously,  $R = \sup |\mu - \lambda|$ ,  $\lambda \in \sigma(\hat{A})$ . From the origin, we draw the tangents to the disk  $S_\mu$  and denote the angle between them by  $\alpha$ . Then it follows from (4.8) that  $\rho_0(\mu) = \sin(\alpha/2)$ . Thus we have proved the following assertion.

**Theorem 4.2** *Let the origin of the complex plane lie outside the convex envelope of the spectrum of  $\hat{A}$ . Let  $S_0$  be the disk which contains all points of the spectrum of  $\hat{A}$  and is "seen" from the origin at minimal angle  $\alpha_0$ . Then the best convergence of the iterations (4.6) to the solution of Eq. (4.1) is attained at the complex value  $\mu_0$  which is the center of the disk  $S_0$ . The iterations converge to the solution at the rate of a geometric progression with the denominator  $\rho_0 = \sin(\alpha_0/2)$ .*

The convex envelope of the spectrum of an integral operator on a complex plane depending on the form of the dielectric permittivity function has been defined in Chapter 2 in the case of low-frequency electromagnetic scattering problems. It follows from Theorems 4.1 and 4.2 and relation (2.35) that the simple iteration method can be used for Eq. (2.36) for arbitrary real media; moreover, we can readily evaluate the optimal iterative parameter  $\mu_0$ . Numerical experiments have shown that this method is a very effective for the numerical solution of low-frequency scattering problems.

#### 4.2 Minimal residual iteration method

Consider equation (4.1) with a linear bounded (in general case non-selfadjoint) operator  $\hat{A}$  in the Hilbert space  $H$ . Define the sequence  $\{u_n\}$ ,  $u_n \in H$  by the formula

$$u_{n+1} = u_n - \tau_n (\hat{A}u_n - f); \quad n = 0, 1, \dots \quad (4.9)$$

with the complex iteration parameters  $\tau_n$ .

Denote

$$h_n = \hat{A}u_n - f. \quad (4.10)$$

Then multiplying both side of (4.9) by  $\hat{A}$  we obtain the relationship

$$h_{n+1} = h_n - \tau_n \hat{A}h_n. \quad (4.11)$$

Therefore, we determine iteration parameters  $\tau_n$  so as to provide the minimum of the functions

$$\gamma_n(\tau) = \frac{\|h_{n+1}\|}{\|h_n\|} = \frac{\|h_n - \tau \hat{A}h_n\|}{\|h_n\|}. \quad (4.12)$$

Represent  $h_n$  as a sum of two elements; one of them is a projection of  $h_n$  on element  $\hat{A}h_n$  and the other is orthogonal to  $\hat{A}h_n$ . This decomposition is uniquely determined from the obvious formulas

$$\begin{aligned} h_n &= \tau_n \hat{A}h_n + d_n, \\ \tau_n &= \frac{(h_n, \hat{A}h_n)}{(\hat{A}h_n, \hat{A}h_n)}. \end{aligned} \quad (4.13)$$

Here  $(f, g)$  denotes the inner product of  $f, g \in H$ .

Substituting (4.13) into (4.12) and taking into account that  $(\hat{A}h_n, d_n) = 0$ , we have

$$\gamma_n^2(\tau) = \frac{|\tau_n - \tau|^2 \|\hat{A}h_n\|^2 + \|d_n\|^2}{\|h_n\|^2}. \quad (4.14)$$

Now it is clear that (4.14) reach the minimum at the complex  $\tau = \tau_n$ . The corresponding value of  $\gamma_n(\tau_n)$  is given by the expression

$$\gamma_n(\tau_n) = \sqrt{1 - \frac{|(h_n, \hat{A}h_n)|^2}{(h_n, h_n)(\hat{A}h_n, \hat{A}h_n)}}. \quad (4.15)$$

Note that  $\gamma_n(\tau_n) \leq 1$ , equality takes place only when  $(h_n, \hat{A}h_n) = 0$ , i.e., elements  $h_n$  and  $\hat{A}h_n$  are orthogonal.

Let us prove the following statement [1, 9].

**Theorem 4.3.** *Let  $\hat{A}$  be a linear bounded operator acting in the Hilbert space  $H$ . Assume that for any  $v \in H$*

$$|(\hat{A}v, v)| \geq p_0(v, v), \quad p_0 > 0. \quad (4.16)$$



Then (I) there exists the unique solution to equation (4.1) in  $H$ ; (II) the norm of the inverse operator  $\hat{A}^{-1}$  is estimated as  $\|\hat{A}^{-1}\| \leq 1/p_0$ ; (III) iterations (4.9), (4.10) and (4.13) converge to the solution of (4.1) for any initial approximation  $u_0 \in H$ ; and (IV) the rate of convergence is estimated by

$$\|u_n - u\| \leq \frac{1}{\rho_0} \left[ \sqrt{1 - \frac{\rho_0^2}{M_0^2}} \right]^n \|\hat{A}u_0 - f\|, \quad (4.17)$$

where  $M_0$  is the upper estimate for the norm of operator  $\hat{A}$ , i.e.,  $\|\hat{A}\| \leq M_0$ .

*Proof.* We have

$$\frac{\|h_n\|}{\|h_0\|} = \frac{\|h_n\|}{\|h_{n-1}\|} \frac{\|h_{n-1}\|}{\|h_{n-2}\|} \dots \frac{\|h_1\|}{\|h_0\|}. \quad (4.18)$$

From (4.12), (4.15) and (4.18), it follows that

$$\|h_n\| \leq \left[ \prod_{m=0}^{n-1} \gamma_m(\tau_m) \right] \|h_0\|. \quad (4.19)$$

(4.15) yields an inequality

$$\gamma_m(\tau_m) \leq \sqrt{1 - \frac{\rho_0^2}{M_0^2}} \equiv q < 1, \quad (4.20)$$

which is valid for any  $m$ . Note that by virtue of (4.15)  $p_0/M_0 \leq 1$ . From (4.19) and (4.20), it follows that residuals  $h_n$  tend to zero as  $n \rightarrow \infty$ .

Since  $\left| \left( \hat{A}v, v \right) \right| \leq \|\hat{A}v\| \|v\|$  for  $\forall v \in H$ , (4.16) yields

$$\|\hat{A}v\| \geq p_0 \|v\|. \quad (4.21)$$

From (4.9), (4.10), (4.13) and (4.21), we obtain

$$\|u_{n+1} - u_n\| \leq \frac{M_0}{p_0^2} \|h_n\|.$$

Now for arbitrary  $m > n$ ,

$$\|u_m - u_n\| = \|(u_m - u_{m-1}) + \dots + (u_{n+1} - u_n)\| \leq \frac{q^n}{1-q} \frac{M_0}{p_0^2} \|h_0\|.$$

This means that  $\{u_n\}$  is a fundamental sequence in  $H$ ; therefore due to the completeness of the Hilbert space,  $u_n$  tends to a limit  $u \in H$ . Since  $\hat{A}$  is a continuous operator because  $\hat{A}$  is a linear

and bounded and  $\lim_{n \rightarrow \infty} h_n = 0$ , the equality  $\hat{A}u_n - f = h_n$  implies that  $u = \lim_{n \rightarrow \infty} u_n$  is a solution to equation (4.1).

Thus, (4.1) has a solution in  $H$  for any  $f \in H$ , and according to (4.21), this solution is unique.

Using inequality (4.21) we conclude that  $\hat{A}^{-1}$  is a bounded operator and  $\|\hat{A}^{-1}\| \leq 1/p_0$ .

We have

$$\|h_n\| = \|\hat{A}u_n - f\| = \|\hat{A}u_n - \hat{A}u\| \geq p_0\|u_n - u\|.$$

These relationships are considered in combination with (4.19) and (4.20) ensure the validity of (4.17). The theorem is proved.

Note that Theorem 4.3 specifies not only the applicability of the method for solving linear operator equations but also a method that facilitates the proof of the existence and uniqueness of solution for particular problems (see, for example Theorems 1.2 and 2.3).

It follows from Theorem 4.3 and inequalities (1.15) and (2.31) that the minimal residual method can be used for integral equations (1.7) and (2.13). Numerical experiments have shown that this method is an effective tool for the numerical solution of scattering problems.

### 4.3 Multistep minimal residual iteration method

Consider a generalization of the minimal residual iteration method. The iterative parameter  $\tau_n$  in (4.9) is defined from the orthogonal condition of the  $(n+1)$ -st residual  $h_{n+1}$  to the element  $\hat{A}h_n$ .

Consider the sequence  $\{u_n\}$ ,  $u_n \in H$ , which is defined by the formulas

$$u_{n+1} = u_n - \sum_{l=1}^m \tau_{nl} \hat{A}^{l-1} h_n, \quad n = 0, 1, \dots, \quad (4.22)$$

$$m = \text{const}, \quad m \geq 1.$$

Here  $\{\tau_{nl}\}$  is the set of the iterative parameters used to pass from  $n$ -th to  $(n+1)$ -st layer.

Because the residuals are related by  $h_{n+1} = h_n - \sum_{l=1}^m \tau_{nl} \hat{A}^l h_n$ , we define the iterative parameters

from the orthogonal condition of  $h_{n+1}$  to the subspace, which is generated by the elements  $\{\hat{A}h_n, \dots, \hat{A}^m h_n\}$  (the Krylov subspace). The iterative procedure is completely defined by the formulas [1, 6, 10]

$$h_n^{(0)} \equiv h_n = \hat{A}u_n - f, \quad u_n^{(0)} \equiv u_n,$$

$$v_l = \begin{cases} h_n^{(0)}, & l = 1, \\ h_n^{(l-1)} - \sum_{k=1}^{l-1} \beta_{lk} v_k, & l > 1, \end{cases}$$

$$\beta_{lk} = \frac{(\hat{A}h_n^{(l-1)}, \hat{A}v_k)}{\|\hat{A}v_k\|^2}, \quad \tau_i = \frac{(h_n^{(l-1)}, \hat{A}h_n^{(l-1)})}{\|\hat{A}v_l\|^2},$$

$$u_n^{(l)} = u_n^{(l-1)} - \tau_l v_l, \quad h_n^{(l)} = h_n^{(l-1)} - \tau_l \hat{A} v_l, \quad u_{n+1} \equiv u_n^{(m)},$$

$$n = 0, 1, \dots; \quad l = 1, \dots, m. \quad (4.23)$$

Iterative procedure (4.23), which may be named as multistep minimal residual method (for short it is called as GMRES method) converge to the solution of (4.1) if condition (4.16) is satisfied.

It is clear that the use of one layer in (4.23) with  $m > 1$  leads to a larger decrease of the residual than that in  $m$  sequential iterations (4.9), (4.10) and (4.13). The more general assertion holds, namely, the more  $m$  in the iterative procedure the faster the convergence. Let  $k$  be the number of layers in (4.23). Then the quantity  $L=km$  defines the amount of multiplications of a matrix by a vector in the algorithm and hence  $L$  is the number of iterations in this case (see Introduction to Part II). However, the increase in the convergence rate is due to the increase in the amount of memory required. Therefore, when solving a particular problem it is necessary to take into account available computational resources (e.g., speed and memory), and maybe other restrictions. Note that the use of even small  $m=2, 3$  leads to sharp increase in the convergence rate as compared with (4.9), (4.10) and (4.13).

In most real applications the quantity  $L$  lies from 20 to 1000.

# CHAPTER 5

## Discretization methods

### 5.1 Galerkin method

Consider Eq. (4.1) with the linear bounded operator  $\hat{A}$  acting in the Hilbert space  $H$ . Let  $H_N \subset H$  be the finite-dimensional Hilbert space with the basis  $\{v_n^{(N)}\}$ ,  $n = 1, \dots, N$ . We seek the approximate solution to Eq. (4.1) by the Galerkin method as

$$u_N = \sum_{n=1}^N \alpha_n^{(N)} v_n^{(N)}, \quad (5.1)$$

where unknown coefficients  $\alpha_n^{(N)}$  are defined from the system of  $N$  linear algebraic equations, namely,

$$\sum_{n=1}^N \alpha_n^{(N)} A_{nm}^{(N)} = (f, v_m^{(N)}), \quad (5.2)$$

$$A_{nm}^{(N)} = (\hat{A} v_n^{(N)}, v_m^{(N)}). \quad (5.3)$$

Equations (5.2), (5.3) are obtained from the requirement: residual of the approximate solution  $h_N = \hat{A} u_N - f$  must be orthogonal to the subspace  $H_N$ .

Formulate the following condition.

**Condition A.** *The sequence of the subspace  $\{H_N\}$  is ultimately dense in  $H$  if for any  $u \in H$  there exist the elements  $\tilde{u}_N \in H_N$ ,  $N = 1, 2, \dots$ , such that*

$$\|u - \tilde{u}_N\| = \inf_{\omega \in H_N} \|u - \omega\| \leq \varepsilon(u, N),$$

where  $\varepsilon(u, N)$  is the bound of the discrepancy,  $\varepsilon(u, N) \rightarrow 0$  as  $N \rightarrow \infty$ .

The following theorem holds [1, 17].

**Theorem 5.1.** *Let for any  $u \in H$  inequality (4.16) and Condition A hold. Then the solution to the system of linear equations (5.2), (5.3) can be found by the iterative method of minimal residuals. Moreover,  $u_N$  converges to the solution of Eq. (4.1), i.e.  $\lim_{N \rightarrow \infty} \|u - u_N\| = 0$ .*

*Proof.* Let  $b^{(N)} = \{b_n^{(N)}\}$  be an element of the space of  $N$ -dimensional vectors  $E_N$  and let  $A^{(N)} = \{A_{nm}^{(N)}\}$  be a linear operator in  $E_N$ . By (5.3) and (4.16) we have

$$\left| (A^{(N)} b^{(N)}, b^{(N)})_{E_N} \right| = \left| \sum_{n,m=1}^N A_{nm}^{(N)} b_n^{(N)} (b_m^{(N)})^* \right| =$$

$$\left| \left( \hat{A} \left[ \sum_{n=1}^N b_n^{(N)} v_n^{(N)} \right], \left[ \sum_{m=1}^N b_m^{(N)} v_m^{(N)} \right] \right) \right|_H \geq \rho_0 q_0 (b^{(N)}, b^{(N)})_{E_N}$$

where  $q_0$  is the least eigenvalue of the Hermitian positive definite matrix  $\beta_{nm} = (v_n^{(N)}, v_m^{(N)})$ ,  $n, m = 1, \dots, N$ .

Hence, by Theorem 4.3, the solution to (5.2), (5.3) exists, is unique, and can be obtained by the iterative method of minimal residuals.

We multiply (5.2) by  $(\alpha_m^{(N)})^*$  and sum the resultant with respect to  $m$ , taking (5.1), (5.3) into account. We have

$$(\hat{A}u_N, u_N) = (f, u_N) \quad (5.4)$$

whence by (4.16) we have

$$\|u_N\| \leq p_0^{-1} \|f\|. \quad (5.5)$$

If Condition A is satisfied, then the solution  $u$  to Eq. (4.1), which exists and is unique in  $H$  by Theorem 4.3, can be represented as

$$\begin{aligned} u &= \tilde{u}_N + x_N, \quad \tilde{u}_N \in H_N, \\ \lim_{N \rightarrow \infty} \|x_N\| &= 0. \end{aligned} \quad (5.6)$$

From (4.1), (4.16) and (5.4) we get

$$\begin{aligned} p_0 \|u - u_N\|^2 &\leq |(\hat{A}(u - u_N), (u - u_N))| = |(f - \hat{A}u_N, u - u_N)| = |(f - \hat{A}u_N, u)| = \\ &= |(f - \hat{A}u_N, \tilde{u}_N) + (f - \hat{A}u_N, x_N)|. \end{aligned}$$

Since  $\tilde{u}_N \in H_N$ , then  $(f - \hat{A}u_N, \tilde{u}_N) = 0$ . Therefore, taking into account (5.5), we find

$$\|u - u_N\|^2 \leq \frac{1}{p_0} \left[ \|f\| + \frac{\hat{A}}{p_0} \|f\| \right] \|x_N\|,$$

whence by (5.6)  $\lim_{N \rightarrow \infty} \|u - u_N\| = 0$  as  $N \rightarrow \infty$ .

The theorem is proved.

The finite-dimensional subspaces of functional spaces  $L_2(Q)$  or  $\bar{L}_2(Q)$  satisfying Condition A can be constructed in various ways for the Hilbert spaces  $H_N$ , in which the operators of Eqs. (1.7) and (2.13) act, e.g., for the spaces of the piecewise-constant functions or vector-functions. Inequalities of the form (4.16) hold for the operators of Eqs. (1.7) and (2.13). Therefore, the Galerkin method can be used to solve numerically integral equations (1.7) and (2.13), and the system of linear algebraic equations by the iterative method of minimal residuals.

For applying Galerkin method we need to calculate inner products of the form

$$(\hat{A}U, V)_{L_2}, \quad (5.7)$$

where  $U$  and  $V$  are known functions and  $\hat{A}$  is the operator of equation. It is easy to calculate (5.7) in acoustic case. For electromagnetic case it is not so clear. But, fortunately, inner product

in functional space  $\bar{L}_2(Q)$  is defined as integral (2.21). Therefore, we may use the integral equation (2.10) and apply *grad div* under the integral sign. Indeed, by using Einstein summation, we obtain

$$\left( \text{grad}_x \text{div}_x \int_Q U(y) \frac{1}{4\pi R} dy, V(x) \right)_{\bar{L}_2(Q)} = \iint_{QQ} \frac{(3\alpha_n \alpha_m - \delta_{nm})}{4\pi R^3} U_m(y) V_n^*(x) dx dy \quad (5.8)$$

where  $\alpha_1, \alpha_2, \alpha_3$  are determined by formula (3.6) or (3.8).

Last integral in (5.8) exists in the usual sense. The inner products from others integral operators of (2.10) are calculated in a similar manner.

## 5.2 Collocation method

The kernel of the integral operator depends only on the difference of Cartesian coordinates of  $x$  and  $y$ . Therefore, in the discretization, it is desirable to take account of this fact so as to obtain a matrix of the SLAE with the corresponding symmetry properties [1, 14].

Introduce a grid in the rectangular Cartesian coordinate system such that the domain  $Q$  is entirely placed in the parallelepiped  $\Pi$  with dimensions,  $N_l h$ ,  $l=1, 2, 3$ , where  $h$  is the grid step along the Cartesian coordinates. The parallelepiped  $\Pi$  is divided by the grid into cells (elementary cubes)  $\Pi(p)$ ,  $p = (p_1, p_2, p_3)$ ,  $p_l = 0, \dots, N_l - 1$ . We define domain  $\tilde{Q}$  as the union  $N_Q$  of the elementary cubes centers of which lie inside domain  $Q$ . It is obviously that  $N_Q \leq N_1 N_2 N_3 = N$ .

Then the choice of the grid is defined by the domain shape and by the requirements that the desired solution, external field, and the parameters of the medium vary slightly inside the cells (elementary cubes). The node points, at which the values of functions are given, are positioned at the cell centers  $x(p)$  with the coordinates

$$x_1(p) = p_1 h + \frac{h}{2}, \quad x_2(p) = p_2 h + \frac{h}{2}, \quad x_3(p) = p_3 h + \frac{h}{2}. \quad (5.9)$$

Then integral equation is approximated by a system of linear algebraic equations with respect to the values of unknown field taken at the nodes situated in domain  $Q$ .

This approximation for the integral equation (1.7) (acoustic case) or integral equation (2.13) (electromagnetic case) has the following form

$$u(p) - \sum_{y(q) \in Q} B(p-q) v(q) u(q) = u_0(p), \quad x(p) \in Q. \quad (5.10)$$

where

$$u(p) \equiv u(x(p)); \quad u_0(p) \equiv u_0(x(p)); \quad u(q) \equiv u(x(q)). \quad (5.11)$$

For integral equation (1.7)  $B(p-q)$  and  $v(q)$  are given by the formulas

$$B(p-q) = \int_{\Pi_q} G(R) dy, \quad (5.12)$$

$$v(q) = [k_0^2(y(q)) - k_0^2]. \quad (5.13)$$

Here

$$R = \sqrt{(y_1 - x_1(p))^2 + (y_2 - x_2(p))^2 + (y_3 - x_3(p))^2}. \quad (5.14)$$

For the singular integral equation (2.13) (electromagnetic case)  $u(p)$  and  $u_0(p)$  are the vectors,  $B(p-q)$  and  $v(q)$  are 3x3 matrix. Expressions for  $B(p-q)$  and  $v(q)$  are given by the formulas

$$v_{mk}(q) = \varepsilon_{mk}(y(q)) - \delta_{mk}, \quad (5.15)$$

$$B_{nm}(p-q) = \int_{\Pi_q} G(R) \left[ \left( \frac{3}{R^2} - \frac{2ik_0}{R} - k_0^2 \right) \alpha_n \alpha_m + \left( k_0^2 + \frac{ik_0}{R} - \frac{1}{R^2} \right) \delta_{nm} \right] dy, \quad p \neq q, \quad (5.16)$$

In (5.16)  $R$  is determined by (5.14) and  $\alpha_n$ , according (3.6), are given by formula

$$\alpha_n = \frac{x_n(p) - y_n}{|x(p) - y|}, \quad n = 1, 2, 3.$$

In order to determine  $B_{nm}(0)$ , we will consider auxiliary relationship. In [5], it is shown that when  $\vec{J} = const$  in a 3D domain  $V$ ,

$$\text{grad div}_x \int_V \frac{1}{4\pi R} \vec{J} dy = -\hat{\gamma} \vec{J}, \quad (5.17)$$

where  $\hat{\gamma}$  is a dyad specified by the shape of the boundary of domain  $V$ . If  $V$  is a parallelepiped, Cartesian axes are parallel to its edges, and point  $x$  is situated at the origin, then  $\hat{\gamma}$  is a diagonal dyad of the form

$$\gamma_{nm} = \gamma_n \delta_{nm}.$$

Here  $\gamma_n = (1/4\pi)\Omega_n$ ,  $n = 1, 2, 3$ , and  $\Omega_n$  are the doubled solid angles between the faces perpendicular to axes  $x_1, x_2, x_3$ . Obviously  $\gamma_1 + \gamma_2 + \gamma_3 = 1$ . If the domain  $V$  is a cube then, like for the ball, we have  $\gamma_1 = \gamma_2 = \gamma_3 = 1/3$ .

Taking the latter into account, we find

$$B_{nm}(0) = -\frac{1}{3} + \int_{\Pi_p} \left\{ k_0^2 G(R) (1 - \alpha_n^2) + \frac{k_0^2}{4\pi R} [3\Phi(R)\alpha_n^2 - \Phi(R)] \right\} dy, \quad B_{nm}(0) = 0, \quad n \neq m,$$

$$\Phi(R) = 1 + (1 - ik_0 R) \frac{\exp(ik_0 R) - 1 - ik_0 R}{(k_0 R)^2}. \quad (5.18)$$

Note that the use of a multiindex numbering for the unknowns and the right-hand side permits one to represent the symmetry properties of the matrix of the SLAE in a transparent form; furthermore, the number of distinct entries of the matrix is  $\sim N$ . Next, since the nodes are placed at cell centers, it follows that the accuracy of the approximation of the integral operator is  $\sim h^2$ .

# CHAPTER 6

## Fast algorithms

### 6.1 Discrete Fourier transform

Consider a complex-valued function of discrete argument  $f(n)$ , where  $n = 0, \pm 1, \pm 2, \dots$ . Assume that  $f(n)$  is a periodical function with period  $N$ , i.e.

$$f(n \pm N) = f(n) \quad \text{for } \forall n. \quad (6.1)$$

Discrete Fourier transform from function  $f(n)$  is defined by the formula

$$F[f] = \tilde{f}(k) = \sum_{n=0}^{N-1} \exp\left(i \frac{2\pi}{N} kn\right) f(n); \quad k = \overline{0, N-1} \quad (6.2)$$

where  $\tilde{f}(k)$  is also a periodical function with period  $N$ .

If we know function  $\tilde{f}(k)$  then we may reconstruct function  $f(n)$  by using inverse discrete Fourier transform

$$F^{-1}[\tilde{f}] = f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \exp\left(-i \frac{2\pi}{N} kn\right) \tilde{f}(k); \quad n = \overline{0, N-1}. \quad (6.3)$$

In the general case the number of arithmetic operations required for calculating discrete Fourier transform (without calculating values  $\exp\left(i \frac{2\pi}{N} kn\right)$ ) is estimated as

$$T_F \sim N^2. \quad (6.4)$$

### 6.2 Discrete Fast Fourier transform

Let  $N = N_1 N_2$ , where  $N_1$  and  $N_2$  are integer numbers. Represent  $k$  and  $n$  from formula (6.2) as  $k = k_2 + k_1 N_2$  and  $n = n_1 + n_2 N_1$ , where,  $k_1, n_1 = \overline{0, N_1-1}$ ;  $k_2, n_2 = \overline{0, N_2-1}$ . Then from (6.2) we obtain

$$\begin{aligned} \tilde{f}(k) = \tilde{f}(k_1, k_2) &= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \exp\left(i \frac{2\pi}{N_1 N_2} (k_2 + k_1 N_2) (n_1 + n_2 N_1)\right) f(n_1, n_2) = \\ &= \sum_{n_1=0}^{N_1-1} \exp\left(i \frac{2\pi}{N_1} k_1 n_1 + i \frac{2\pi}{N_1 N_2} k_2 n_1\right) \sum_{n_2=0}^{N_2-1} \exp\left(i \frac{2\pi}{N_2} k_2 n_2\right) f(n_1, n_2). \end{aligned} \quad (6.5)$$

Let  $N_1$  be a prime number. It is obvious from (6.5) that the number of arithmetic operations  $T_F(N)$  required for calculating Fourier transform (without calculating values  $\exp\left(i \frac{2\pi}{N} kn\right)$ ) is estimated as



$$T_F(N) \sim N_2 N_1^2 + N_1 T_F(N_2). \quad (6.6)$$

Rewrite (6.6) in the following form

$$\frac{T_F(N)}{N} \sim N_1 + \frac{T_F(N_2)}{N_2}. \quad (6.7)$$

Let  $N$  be represented as multiplication of prime numbers, i.e.

$$N = N_1^{l_1} N_2^{l_2} \dots N_m^{l_m}. \quad (6.8)$$

Denote

$$\text{LOG}(N) = \sum_{k=1}^m l_k N_k. \quad (6.9)$$

Then, from (6.7)-(6.9), we obtain that the number of arithmetic operations for fast Fourier transform (FFT) technique (6.5) is estimated as

$$T_{FF} \sim N \text{LOG}(N). \quad (6.10)$$

If  $N$  is a power of 2, we have

$$T_{FF} \sim N \log_2(N). \quad (6.11)$$

### 6.3 Multiplication of the matrix by vector

Let us consider a matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \dots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \dots & a_{N-3} \\ \text{-----} \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}. \quad (6.12)$$

The matrix of the form (6.12) is called the circulant matrix.

Consider the multiplication of the matrix (6.12) by an  $N$ -dimensional vector  $\vec{u}$

$$\vec{v} = A\vec{u}. \quad (6.13)$$

Introduce periodical function of the argument  $A(n)$  and put  $A(-n) = a_n, n = \overline{0, N-1}$ . Then we may rewrite (6.13) as

$$v(n) = \sum_{m=0}^{N-1} A(n-m)u(m), \quad n = \overline{0, N-1}. \quad (6.14)$$

Apply Fourier transform to both sides of (6.14). For the right-hand side of (6.14) we have

$$\begin{aligned}
F \left[ \sum_{m=0}^{N-1} A(n-m)u(m) \right] &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \exp\left(i \frac{2\pi}{N} kn\right) A(n-m)u(m) = \\
\sum_{m=0}^{N-1} \exp\left(i \frac{2\pi}{N} km\right) u(m) \sum_{n=0}^{N-1} \exp\left[i \frac{2\pi}{N} k(n-m)\right] A(n-m). & \quad (6.15)
\end{aligned}$$

Consider the second sum in the last expression of (6.15). Denoting  $q = n - m$ , we have

$$\sum_{n=0}^{N-1} \exp\left[i \frac{2\pi}{N} k(n-m)\right] A(n-m) = \sum_{q=-m}^{N-1-m} \exp\left(i \frac{2\pi}{N} kq\right) A(q).$$

Then, taking into account

$$A(q+N) = A(q), \quad \exp\left(i \frac{2\pi}{N} k(q+N)\right) = \exp\left(i \frac{2\pi}{N} kq\right)$$

we arrive at

$$\sum_{n=0}^{N-1} \exp\left[i \frac{2\pi}{N} k(n-m)\right] A(n-m) = \tilde{A}(k). \quad (6.16)$$

Then from (6.15) and (6.16) we have

$$F \left[ \sum_{m=0}^{N-1} A(n-m)u(m) \right] = \tilde{A}(k) \tilde{u}(k); \quad k = \overline{0, N-1}. \quad (6.17)$$

Finally, from (6.14), (6.17) we obtain

$$\tilde{v}(k) = \tilde{A}(k) \tilde{u}(k), \quad k = \overline{0, N-1}. \quad (6.18)$$

Thus, we can use (6.18) for the fast multiplications of the matrix (6.12) by vector due to FFT (one direct FFT and one inverse FFT).

#### 6.4 Fast algorithms for integral equations [1, 17]

The matrix of the system of linear equations (5.10) has the special form of a Toplitz matrix. If we use the Galerkin method (5.1)-(5.3) on the rectangular grid with the same basis functions inside each cell (elementary cube) then the matrix of the system of linear equations also has the structure (5.10).

It follows from (5.10) that the main computing expenses on multiplication of the matrix of the system by the vector are connected with computing sums

$$W(p) = \sum_{y(q) \in Q} B(p-q)V(q), \quad x(p) \in Q. \quad (6.19)$$

Let us complete a definition of  $V(q)$  by zero at the nodes of  $\Pi$  which do not belong  $Q$ . Then we can rewrite (6.19) as

$$W(p_1, p_2, p_3) = \sum_{q_1=0}^{N_1-1} \sum_{q_2=0}^{N_2-1} \sum_{q_3=0}^{N_3-1} B(p_1 - q_1, p_2 - q_2, p_3 - q_3) V(q_1, q_2, q_3), \quad (6.20)$$

where the function of the discrete argument  $B(p)$  is defined for

$$-(N_1 - 1) \leq p_1 \leq (N_1 - 1); \quad -(N_2 - 1) \leq p_2 \leq (N_2 - 1); \quad -(N_3 - 1) \leq p_3 \leq (N_3 - 1).$$

To decrease the amount of operations we apply the direct and inverse fast Fourier transform. Let  $\Pi_2$  denote the parallelepiped with the dimensions  $2N_1h$ ,  $2N_2h$  and  $2N_3h$ . We continue the function (or matrix function) of the discrete argument  $B(p_1, p_2, p_3)$  to all integers  $p_1, p_2, p_3$ , assuming that it is periodic with respect to each variable with the periods  $2N_1, 2N_2, 2N_3$ , respectively. We complete a definition of the function (or vector function) of the discrete argument  $V(p_1, p_2, p_3)$  setting it zero at all nodes of  $\Pi_2$  which do not belong to  $\Pi$ . Let us write the expression

$$W(p_1, p_2, p_3) = \sum_{q_1=0}^{2N_1-1} \sum_{q_2=0}^{2N_2-1} \sum_{q_3=0}^{2N_3-1} B(p_1 - q_1, p_2 - q_2, p_3 - q_3) V(q_1, q_2, q_3). \quad (6.21)$$

It is obviously that  $W(p)$  in (6.21) coincides with (6.20) for  $p \in \Pi$ . If we perform the discrete Fourier transform with respect to each variable in both sides of (6.21), we obtain (see (6.18))

$$\tilde{W}(k_1, k_2, k_3) = \tilde{B}(k_1, k_2, k_3) \tilde{V}(k_1, k_2, k_3). \quad (6.22)$$

When calculating  $W(p)$ ,  $p \in \Pi$ , from formula (6.22), the main computing expenses without taking into the account the definition of  $\tilde{B}(k)$  are connected with the fast direct and inverse Fourier transform of the functions (or vector functions in electromagnetic case) of the discrete arguments. In the direct Fourier transform the function (or vector function)  $V(p)$  is only nonzero in  $\Pi_2$  for  $p \in \Pi$ . On the other hand, when we apply the inverse Fourier transform, the function (or vector function)  $W(p)$  must be defined only at the points  $p \in \Pi$ . Taking this into account, we find that the amount of arithmetical operations required to find  $W(p)$ ,  $p \in \Pi$  is estimated by the formula

$$T_B \sim N_1 N_2 N_3 \left( \text{LOG}(N^{(1)}) + 2 \text{LOG}(N^{(2)}) + 4 \text{LOG}(N^{(3)}) \right), \quad (6.23)$$

where  $N^{(1)}, N^{(2)}, N^{(3)}$  are the quantities  $N_1, N_2, N_3$  numbered in the decreasing order. If  $N_1, N_2, N_3$  are the powers of 2, then we have

$$T_B \sim N_1 N_2 N_3 \left( \log_2(N^{(1)}) + 2 \log_2(N^{(2)}) + 4 \log_2(N^{(3)}) \right). \quad (6.24)$$

Thus (see Introduction to Part II), the considered algorithm is cost-effective for the solution of volume integral equations by using Galerkin or collocation methods and iterative techniques.

## Exercises

1. Show that the representation (1.3) satisfies radiation condition at infinity (1.2).
2. Reduce Eq. (2.12) to the singular volume integral equation (2.13).
3. Show that from (2.19) we obtain (2.20).
4. Obtain the equality (2.24).
5. Show that if the tensor  $\left(\widehat{\varepsilon}(x) - \widehat{\varepsilon}^*(x) - 2i \operatorname{Im} \varepsilon_0 \widehat{I}\right)/(2i)$  is positive definite at every point of  $Q$  then tensor function  $\widehat{\delta} = (\widehat{\varepsilon} - \varepsilon_0 \widehat{I})$  has an inverse function (see Theorem 2.3).
6. Obtain formula (2.34).
7. Prove Lemma 2.2.
8. Why do we have (2.46)?
9. Prove Theorem 4.1.
10. Find the best iteration parameter in (4.6) where the convex envelope of the spectrum is a triangle.
11. Prove relation (4.5)
12. Obtain formula (4.15).
13. Find the elements  $A_{nm}$  from (5.3) for piece-wise constant basis functions and integral operator (1.7).
14. Obtain formulas (5.16) and (5.18).
15. Show that the inverse discrete Fourier transform is given by formula (6.3).
16. Construct FFT algorithm for the general case (6.8).

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