Operator extensions, Friedrichs model and some fitted zero-range solvable models

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Introduction

Only a slim part of linear partial differential equations have solutions represented by explicit formulas or by approximate explicit expressions, with a controllable errors. Strong demand for reliable data on spectral and transport characteristics of the corresponding quantum and classical dynamical systems is nowadays satisfied based on results of expensive and resource consuming scanning over the multi-dimensional space of essential geometrical and physical parameters of the systems. The area of the scanning may be essentially reduced based on use of the fitted solvable models. A large class of the "zero-range models" can be constructed based on John von Neumann operator extension technique. The corresponding theory, developed in previous century, is technically demanding and hard to fit into a compact text for the first reading. Fortunately there exist a representative *Friedrichs model* of a linear dynamical system where almost all typical difficulties of the corresponding theory can be reviewed as well as advanced techniques used to overcome them.

In this paper we provide an extended insight into the Friedrichs model, as an universal tool of the analytic perturbation theory and give a state-of-art review of others fitted zero-range models. The text can be considered as a motivation and a practical introduction into the area of applied spectral analysis of linear dynamical systems.

It is commonly recognized that pure mathematics grows rather "in depth" than "in breadth". The problem of fading connections between neighboring branches of pure mathematics was noticed long ago. Great analyst Mark Krein, in his talk on the Moscow Congress of Mathematicians,1966, declared that, to restore the connections, you should lure pure mathematicians from their "private caves", where they live quietly, in deep harmony with their own special problems. He sketched in his talk a broad program of common prospects of the spectral analysis of nonselfadjoint operators and ones of the theory of analytic functions. His prediction fed the area of modern spectral analysis for almost 50 years: we witnessed joint growing fruits of the spectral theory of functions and functional models of nonselfadjoint operators during the second half of 20-th century.

Extending the Mark Krein program in the applied direction, we hope to attract with this text an attention of colleagues from neighboring area of applied mathematics, engineering and

mathematical modeling, to modern problems of mathematical physics and perturbation theory, by providing advanced tools developed in operator theory for applied spectral analysis.

In attempt to produce a short readable text, we choose here a method of delivering of the material based on analysis of representative examples rather than in didactic form of a logical sequence of relevant results. For details we send the reader to the extended list of references attached.

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Outline

1 Operator extension procedure

Operator theory appeared as a tool of quantum mechanics just in time and in almost optimal form to provide an excellent service to physicists. In particular the fundamental Weyl-Titchmarsh function and the spectral expansion for the Schrödinger operator were discovered even before E. Schrödinger proposed to use the second order linear differential operator as mathematical tools of the new theory, see basic fact and the essential history in [1]. By the beginning of thirties of previous century John von Neumann analyzed in detail the difference between the symmetric and the selfadjoint operators, see [2]. But this analysis was never taken in serious by physicists and engineers: they always prefered to discover important mathematical facts and ideas independently from mathematicians. A classical example of the independence was the whole story connected with interpretation of the zero-range potential, discovered by E. Fermi, see [3], and the corresponding subsection below. Fermi conjectured that the zero-range model can be fitted to results of scattering experiment of neutrons by the nucleons under assumption of presence of a bound state in the scattering system with a small negative energy level. But in subsequent papers and books, see [4, 5, 6] concerning the zero-range potentials

the fitting problem was not discussed. On the other hand the solvable models based on J. von Neumann operator extension theory, reveal a surprising universality and allow a reasonably good fitting of zero-range model on a certain interval of energy in the corresponding scattering problems.

Properly fitted zero-range models may serve as first step $A_0 \to A_\varepsilon^0$ in the corresponding analytic perturbation procedures, allowing "to eliminate the dangerous resonances", as proposed by H. Poincare in [13]. Due to the elimination the analytic perturbation procedure $A_0 \longrightarrow A_\varepsilon$ is transformed to the two-steps algorithm :

$$A_0 \longrightarrow A_\varepsilon^0 \longrightarrow A_\varepsilon,$$

corresponding to the chain-decomposition of the scattering matrix $S(A_\varepsilon, A_0)$:

$$S(A_\varepsilon, A_0) = S(A_\varepsilon, A_\varepsilon^0) S(A_\varepsilon^0, A_0),$$

with the first step corresponding to the "jump-start" $A_\varepsilon^0 \longleftarrow A_0$ and the second step corresponding to the standard geometrically convergent analytic perturbation procedure $A_\varepsilon \longleftarrow A_\varepsilon^0$. The idea of the two-steps procedure was suggested by H. Poincare, see [13], without any suggestions how the intermediate operator A_ε^0) could be constructed. I. Prigogine attempted to find the intermediate operator in form of a function of an unperturbed A_0, but eventually came to the conclusion that it is impossible. Nevertheless the idea of H. Poincare survived, and an analog of the intermediate operator was constructed for quantum networks as a finite-dimensional perturbation of A_ε, see [7, 8]. In [8] the *Intermediate Hamiltonian* is constructed based on symplectic version of the John von Neumann operator extension procedure, see [9] for standard von Neumann operator extension procedure based on Caley transform.

Definition 1.1 *For given complex value* λ, $\Im\lambda \neq 0$ *of the spectral parameter the* **deficiency subspaces** *are*

$$N_\lambda := H \ominus \overline{[\mathcal{A}_0 - \lambda I]\ D_0},$$
$$N_{\bar\lambda} := H \ominus \overline{\left[\mathcal{A}_0 - \bar\lambda I\right]\ D_0}.$$

The dimension of N_λ, $N_{\bar\lambda}$ is constant on the whole upper and lower half-planes $\Im\lambda > 0$, $\Im\lambda < 0$ respectively.

Definition 1.2 *Introduce the deficiency index* $(dim N_\lambda,\ dim N_{\bar\lambda}) := (n_+, n_-)$ *of the operator* $\mathcal{A}_0$.

J. von Neumann proved that

Theorem 1.1 *The hermitian operator* $\mathcal{A}_0$ *has a self-adjoint extension if and only if* $n_+ = n_- =: n \leq \infty$.

The idea of construction of the extension is based on the following theorems von Neumann, see for instance [9]:

Theorem 1.2 *The domain of the adjoint operator is represented as a direct sum of the domain $D_{\bar{A}_0}$ of the closure and the deficiency subspaces, in particular:*

$$D_{A_0^+} = D_{\bar{A}_0} + N_i + N_{-i}.$$

The deficiency subspaces of the densely-defined operator are the eigen-spaces of the adjoint operator:

$$\mathcal{A}_0^+ e_i = -ig_i,\ g_i \in N_i,\ \mathcal{A}_0^+ g_{-i} = ig_{-i},\ g_{-i} \in N_{-i}. \tag{1}$$

Theorem 1.3 *If $\mathcal{A}_0$ is an Hermitian operator with equal deficiency indices (n_+, n_-), and V is an isometry $V : N_i \to N_{-i}$. Then the isometry V defines a self-adjoint extension $\mathcal{A}_V$ of $\mathcal{A}_0$, acting on the domain*

$$D_{A_V} = D_{\bar{A}_0} + \{g_i + Vg_i,\ g_i \in N_i\}$$

as a restriction of $\mathcal{A}_0^+$ onto D_{A_V}:

$$A_V : u_0 + g_i + Vg_i \to \bar{\mathcal{A}}_0 u_0 - ig_i + iVg_i.$$

J. von Neumann reduced the construction of the extension of the symmetric operator $\mathcal{A}_0$ to an equivalent problem of construction of an extension of the corresponding isometrical operator - the Caley transform of $\mathcal{A}_0$, see [9]. It is much more convenient, for differential operators, to construct the extensions based on so-called **boundary form**.

Example. Symplectic extension procedure for the differential operator Consider the second order differential operator

$$L_0 u = -\frac{d^2 u}{dx^2},$$

defined on all square integrable functions, $u \in L_2(0, \infty)$, with square- integrable derivatives of the first and second order and vanishing near the origin. This operator is symmetric and it's adjoint L_0^+ is defined by the same differential expression on all square integrable functions with square integrable derivatives of the first and second order and no additional boundary condition at the origin. This operator is not symmetric: its boundary form

$$\mathcal{J}(u, v) = \langle L_0^+ u, v\rangle - \langle u, L_0^+ v\rangle = u'(0)\bar{v}(0) - u(0)\bar{v}'(0),\ u, v \in D_{L_0^+}$$

is generally non equal to zero for $u, v \in D_{L_0^+}$. But it vanishes on a "Lagrangian plane" $\mathcal{P}_\gamma \subset D_{L_0^+}$ defined by the boundary condition

$$u'(0) = \gamma u(0),\ \gamma = \bar{\gamma}.$$

The restriction L_γ of the L_0^+ onto the Lagrangian plane $\mathcal{P}_\gamma$ is a self-adjoint operator in $L_2(0, \infty)$: it is symmetric, and the inverse of it $(L_\gamma - \lambda I)^{-1}$, at each complex spectral point λ, exists and is defined on the whole space $L_2(0, \infty)$.

The operator extension procedure used above for the differential operator, can be applied to general symmetric operators and serves a convenient alternative for construction of solvable models of orthogonal sums of differential operators and non-densely defined operators- finite matrices. We call the abstract analog of the above extension procedure the *symplectic extension procedure*. Let A be a self-adjoint operator in a finite-dimensional Hilbert space E, dim $E = d$, and $N_i := N$ is a subspace of E, dim $N = n < d/2$, which does not overlap with $\frac{A+iI}{A-iI}N_i := N_{-i}$:

$$N_i \cap N_{-i} = \{0\}.$$

Define the operator A_0 as a restriction of A onto $D_0 := \frac{I}{A-iI}E \ominus N$. This operator is symmetric, and the subspaces $N_{\pm i}$ play roles of it's deficiency subspaces. The operator can A_0 can be extended to the self-adjoint operator $A_\Gamma supset A_0$ via symplectic extension procedure involving the corresponding boundary form.

Select a basis $\{g_s^+\}_{s=1}^n =: g_s \in N_i$, and consider the dual basis $\left\{\frac{A+iI}{A-iI}g_s = g_s^-\right\}_{s=1}^n \in N_i$. Introduce, following [10], another basis in the defect $N = N_i + N_{-i}$

$$W_s^+ = \frac{1}{2}\left[g_s + \frac{A+iI}{A-iI}g_s\right],\ W_s^- = \frac{1}{2i}\left[g_s - \frac{A+iI}{A-iI}g_s\right].$$

Due to $A_0^+ g_s + ig_s = 0,\ [A_0^+ - iI]\frac{A+iI}{A-iI}g_s = 0$ we have,

$$A_0^+ W_s^+ = W_s^-,\ A_0^+ W_s^- = -W_s^+.$$

Following [10] we use the representation of elements from the domain of the adjoint operator by an expansion on the new basis:

$$u = u_0 + \sum_s \eta_+^s W_s^+ + \eta_-^s W_s^-,$$

with $u_0 \in D(A_0)$ and symplectic coordinates $\eta_\pm^s$.

We also introduce the *boundary vectors* of elements from $D(A_0^+)$

$$\vec{\eta}_\pm =: \sum_s \eta_\pm^s g_s \in N_i,$$

$$u = u_0 + \frac{A}{A-iI}\vec{\eta}_+^u - \frac{I}{A-iI}\vec{\eta}_-^u := u_0 + n^u,\ u_0 \in D(A_0)\, n^u \in N.$$

Define the formal adjoint operator A_0^+ on the defect $\mathbf{N} = N_i + N_{-i}$ as:

$$A_0^+ g_+ = -ig_+, \text{ for } g_+ \in N_i,\ A_0^+ g_- = ig_-, \text{ for } g_+ \in N_{-i},$$

$$A_0^+(g_+ + g_-) = -ig_+ + ig_-.$$

Then we have:

$$A_0^+ W_s^+ = W_s^-,\ A_0^+ W_s^- = -W_s^+.$$

Following [10], we will use the representation of elements from the domain of the adjoint operator by the expansion on the new basis:

$$u = u_0 + \sum_s \eta_+^s W_s^+ + \eta_-^s W_s^-,$$

with $u_0 \in D(A_0)$ and symplectic coordinates $\eta_\pm^s$. We also introduce the *boundary vectors* of elements from $D(A_0^+)$

$$\vec{\xi}_\pm := \sum_s \eta_\pm^s g_s \in N_i,$$

$$u = u_0 + \frac{A}{A - iI}\vec{\eta}_+^{\,u} - \frac{I}{A - iI}\vec{\eta}_-^{\,u} := u_0 + n^u,\ u_0 \in D(A_0)\, n^u \in N.$$

The adjoint operator is defined if A_0^+ is densely defined. But if N_i, N_{-i} do not overlap, the formal adjoint on the defect $N_i + N_{-i}$ can be defined based on the J. von Neumann formula.

Lemma 1.1 *[10] The boundary form of the formal adjoint operator is calculated in terms of symplectic variables $\eta_\pm^{\,u}$, $\eta_\pm^{\,v}$ as*

$$J_{_A}(u, v) = \langle A^+ u,\, v\rangle - \langle u,\, A^+ v\rangle = \langle \eta_+^{\,u},\, \eta_-^{\,v}\rangle - \langle \eta_-^{\,u},\, \eta_+^{\,v}\rangle$$

and it depends only on the parts of the vectors u, v in the defect.

The self-adjoint extension can be defined on the defect via standard J. von Neumann procedure, for given isometry V and then extended to maximal domain via adding the closure of A_0.

Theorem 1.4 Krein formula *Consider a closed symmetric operator A_0 in the Hilbert space $\mathcal{H}$, obtained via restriction of the self-adjoint operator A onto the dense domain $D(A_0)$, with finite-dimensional deficiency subspaces $N_{\mp i}$, $P_{N_i} := P_N$, dim N_i = dim N_{-i}. Then the resolvent of the selfadjoint extension A_Γ defined by the boundary conditions with hermitian operator $\Gamma : N_i \to N_i$:*

$$\vec{\eta}_+ = \Gamma\vec{\eta}_- \tag{2}$$

is represented, at regular points of A_Γ, by the formula:

$$(A_\Gamma - \lambda I)^{-1} = \frac{I}{A - \lambda I} - \frac{A + iI}{A - \lambda I} P_+ \Gamma \frac{I}{I + P_N \frac{I + \lambda A}{A - \lambda I} P_N \Gamma} P_N \frac{A - iI}{A - \lambda I}$$

Proof. Solution of the homogeneous equation $(A^+ - \lambda I)\, u = f$ is reduced to finding $u_0, \vec{\xi}_\pm$ from the equation

$$(A - \lambda I)u_0 - \frac{I + \lambda A}{A - iI}\vec{\eta}_+^{\,u} - \frac{A - \lambda I}{A - iI}\vec{\eta}_-^{\,u} = f.$$

Applying to this expression the operator $\frac{A-iI}{A-\lambda I}$, due to $(A-iI)u_0 \perp N_i$, we obtain

$$\vec{\eta}_- = -\frac{I}{I + P_N \frac{I+\lambda A}{A-\lambda I} P_+} P_N \frac{A-iI}{A-\lambda I} f.$$

Then, from the above equation (1) and from the boundary condition (2), we derive:

$$u_0 = \frac{1}{A-iI}\left[\frac{I+\lambda A}{A-\lambda I}\vec{\eta}_+ + \vec{\eta}_-\right] + \frac{I}{A-\lambda I} f,$$

and

$$u = u_0 + \frac{A}{A-iI}\vec{\eta}_+ - \frac{I}{A-iI}\vec{\eta}_- = \tag{3}$$

$$\frac{I}{A-\lambda I} f - \frac{A+iI}{A-\lambda I} P_N \Gamma \frac{I}{I + P_N \frac{I+\lambda A}{A-\lambda I} P_N \Gamma} P_N \frac{A-iI}{A-\lambda I} f$$

The end of the proof

See below, in next section, a joint extension of a pair of operators constructed via imposing a boundary condition onto the corresponding symplectic variables, see Theorem 2.1.

2 Jump start for the Friedrichs model.

The standard technique of the analytic perturbation theory is developed for additive perturbations $\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon\mathcal{B}$ of operators with discrete spectrum, see for instance [11]. It is well known, see [12], that the analytic perturbation procedure, for operators with continuous spectrum, is convergent only for small values of the perturbation parameter ε - "below the threshold of creation of resonances", but can't be extended beyond this limit. H. Poincaré connected the divergence of the analytic perturbation procedure in celestial mechanics with resonances, which appear in [13] as small denominators. H. Poincare conjectured that the improving of convergence of the analytic perturbation procedure can be achieved via "elimination of dangerous resonances", but he did not suggest any standard procedure for that. Following H. Poincaré, I. Prigogine attempted to elaborate a technical procedure to improve the convergence of the perturbation procedure on continuous spectrum by a special choice of the "intermediate operator" on the first step of the procedure. He tried to find an intermediate operator $\mathcal{C}_\varepsilon$ in the commutant of the non-perturbed operator, see [14, 15, 17, 16, 18], such that the analytic perturbation procedure is convergent for the pair $(\mathcal{C}_\varepsilon, \mathcal{A}_\varepsilon)$. No such operator was found, and the idea was abandoned. It arose again when considering the scattering problems on Quantum Neworks, see [19, 7, 8]. It appeared that the intermediate operator for the scattering problem on the quantum network can be constructed via Glazman's "splitting" by imposing a semi-transparent boundary condition on some sections of the semi-infinite wires. This boundary condition splits the original Hamiltonian into two parts: the trivial part in the *open channels* and the non-trivial part on the union of the *closed channels* in the wires and in the energy normed space of the Cauchy data

on the *compact part of the network*. In contrast to the assumption of Prigogine, this operator does not commute with the non-perturbed operator. More important that the corresponding solvable model, see [8], emulates the perturbed operator locally, in the spectral subspace associated with the conductivity band. But it is already connected with the perturbed operator by the standard convergent analytic perturbation procedure serving a first step of the analytic perturbation procedure. Important detail of the construction suggested in [8] is the ladder-like continuous spectrum of the quantum network, with a countable sequence of finite jumps of the multiplicity at the thresholds. The semi-transparent boundary condition defines the splitting of the perturbed operator into orthogonal sum of two parts corresponding to the open and closed channels. The solvable model of a junction [8] is obtained via binding the trivial part (in open channels) with the resonance component of the nontrivial part (in closed channels).

The Friedrichs model - a compact perturbation of the momentum $P = i\,\frac{d}{dx}$, see for instance [12, 20, 21, 22, 23, 24, 25, 26]

$$\mathcal{P}_{_\varepsilon} = \mathcal{P} + \varepsilon A$$

is a highly representative example in the perturbation theory. The continuous spectrum of it has constant multiplicity and can't be split by semi-transparent boundary conditions into the parts corresponding to the open and closed channels. The standard procedure of analytic perturbation suggested in [12] for the Friedichs model is convergent only for small values of the perturbation parameter ε. Extension of the perturbation procedure beyond the "threshold of creation of resonances" requires the non-analytic techniques of the Mathematical Scattering Theory, see for instance [27, 28, 29].

In this section we consider the scattering problem for the Friedrichs model extending the two-steps jump-start technique to the operator with Lebesgue spectrum:

$$\mathcal{P}_0 \longrightarrow \mathcal{P}_0^\varepsilon \longrightarrow \mathcal{P}_\varepsilon.$$

On the first step the scattering matrix $S(\mathcal{P}_0^\varepsilon, \mathcal{P}_0)$ consists of a single Blaschke factor which is *non-analytic* with respect to ε. The scattering matrix $S(\mathcal{P}_\varepsilon, \mathcal{P}_0^\varepsilon)$ on the second step is analytic with respect to ε, see the discussion in subsection 2.3.

In agreement with the anticipation of Poincaré [13], the "dangerous resonances" appear as an essential detail in our construction for the Friedrichs model. But the construction of the "jump start" requires *exact data* of the resonance. In Friedrichs model we may obtain these data via solving an algebraic equation. But in general case finding of the exact data is a hard problem. On the other hand substitution of the exact data by some *approximate data* hinders the analyticity of the intermediate scattering matrix $S(\mathcal{P}_\varepsilon, \mathcal{P}_0^\varepsilon)$

2.1 Scattering problem for the Friedrichs model.

We discuss in this paper the Friedrichs model obtained as a perturbation $\mathcal{P}_{_\beta}$ of the orthogonal sum $\mathcal{P} \oplus A$ of the momentum $\mathcal{P} = i\,\frac{d}{dx}$ in $L_2(R, E)$, $\dim E = m < \infty$ and a finite hermitian matrix A acting in the space K, $A : K \to K$, $\dim K = k < \infty$. We construct $\mathcal{P}_{_\beta}$ via the

symplectic operator extension procedure, see previous section, by restriction of the momentum $\mathcal{P} \to \mathcal{P}_{_0}$ onto the domain $D_0 = W_2^{1,0}(R, E)$ of all smooth functions taking values in E and vanishing at the origin. The operator $\mathcal{P}_{_0}$ is symmetric, with deficiency indices (m, m), $m =$ dim E. The corresponding adjoint operator $\mathcal{P}_{_0}^{^+}$ is defined on $W_2^1(R_-) \oplus W_2^1(R_+)$ of functions with non - correlated boundary data at the right side and left-side origin $0^+, 0^-$. The boundary form of the adjoint operator defined in [10, 33, 30] via integration by parts

$$J(u, v) = \langle \mathcal{P}_0^+ u,\, v\rangle - \langle u,\, \mathcal{P}_{_0}^{^+} v\rangle = i\left[u\bar{v}(0^-)\right] - u\bar{v}(0^+)$$

can be represented in terms of the corresponding symplectic variables $\xi_\pm$:

$$\xi_+^u = \frac{u(0^+) + u(0^-)}{2},\ \xi_-^u = i\left[u(0^+) - u(0^-)\right]$$

as

$$J_p(u, v) = \langle \xi_+^u,\ \bar{\xi}_-^v\rangle_{_E} - \langle \xi_-^u,\, \xi_+^v\rangle_{_E}. \tag{4}$$

The restricted hermitian matrix is not densely defined, hence the adjoint operator does not exist. Yet, if the selected pair of the deficiency subspaces N_i, N_{-i} does not not overlap $N_i \cap N_{-i} = 0$, one can define, based on (1), the formal adjoint operator A_0^+ on the defect $N_i + N_{-i}$. The boundary form of A_0^+ can't be calculated via integration by parts, but can be calculated based on (1). A version of the operator extension theory for non-densely defined operators was suggested in [31]. The symplectic rescription of it, based on use of the formal adjoint operator, was suggested in [10] see also [30, 32]. The main obstacle to the extension procedure in this case - absence of the adjoint operator - is avoided by reducing the construction of the extension onto the defect $N_{_i} + N_{_{-i}}$ - the sum of deficiency subspaces $N_{_i} = N,\ N_{_{-i}} = \frac{A+iI}{A-iI}N$. In this paper we assume that dim $N = n \leq k/2$, and $N_{_i} \cap N_{_{-i}} = 0$, which is automatically fulfilled if $n = 1$. We derive an expression for the Scattering matrix in case when dim $N = n \leq k/2$. In fact the derived expression for the scattering matrix remains valid also in the case $k = 1$, see [34, 35].

The restricted operator A_0 is defined on the non-dense domain $D_{_{A_0}} = \frac{1}{A-iI}K \ominus N$. We define the formal adjoint A_0^+ on the defect in agreement with the above von Neumann theorem:

$$[A_0^+ + iI]N_{_{-i}} = [A_0^+ - iI]N_{_i} = 0.$$

Consider an orthogonal sum $P_0 \oplus A_0$ of the restricted operators, and construct a Lagrangian plane $\mathcal{L}_{_\beta}$ parametrized by the Hermitian matrix B connecting the symplectic coordinates $\xi_{_\pm}$ of the "outer" component with the symplectic coordinates $\eta_{_\pm}$ of the "inner" component

$$B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix},\ \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} = B \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}. \tag{5}$$

Here $\beta_{10}^{^+} = \beta_{_{01}} \in C_{_m} \times C_{_n},\ \beta_{_{00}} \in C_{_m} \times C_{_m},\ \beta_{_{11}} \in C_{_n} \times C_{_n},$.

Theorem 2.1 *The joint boundary form $J_p(u, v) + J_A(\mathbf{u}, \mathbf{v})$ vanishes on the Lagrangian plane $\mathcal{L}_{_\beta}$ described by the equation (5). This Lagrangian plane defines the corresponding self-adjoint extension P_β of $\mathcal{P}_0 \oplus A_0$.*

Spectral properties of finite-dimensional perturbations of multiplication and differentiation operators were intensely studied in numerous papers, vere intensely studied since mid-eighties [4, 5, 6, 12, 20, 21, 22, 23, 24, 25, 26] based on explicit expressions for Green functions, eigenvalues and eigenfunctions. Our aim is the construction of fitted solvable models and the use them as first step in the analytic perturbation procedures. Correspondingly we omit the proofs of completeness and orthogonality of the scattered waves, but concentrate on asymptotic properties of the scattering matrix at the typical values of energy $E \approx E_F$, or, equivalently, on the fitting of the model on the essential spectral interval.

For the Friedrichs model of our type the scattered waves $\mathbf{\Psi}$ have two components: in the "outer" space $L_2(R,\, E)$ and in the inner space K, $\mathbf{\Psi} = \{\Psi_{_0},\, \Psi_1\}$. The components $\Psi_{_0},\, \Psi_K$ satisfy the adjoint homogeneous equations

$$i\frac{d\Psi_0}{dx} = p\Psi_0,\ (A^+ - pI)\Psi_{_K}(\nu) = 0. \tag{6}$$

which means, in particular, that the symplectic coordinates $\eta_\pm$ are connected by the corresponding Krein function: $(n \times n)$ matrix - function $\mathcal{M}(p) = P_N \frac{I+pA}{A-pI} P_N$, see [**?**]:

$$\eta_{_-} = -\mathcal{M}(p)\eta_{_+}.$$

The Krein function is an abstract analog of the Weyl-Titchmarsh function [1], which is an important characteristic of the differential operators. Hereafter we call it Krein-Weyl-Titchmarsh (KWT) function. It belongs to Nevanlinna class (is analytic and has a positive imaginary part in the upper half-plane $\Im p > 0$). Then, presenting the outer component of the "incoming" eigenfunction by the Ansatz

$$\Psi_{_0}(x,\, \nu) = \left\{ \begin{array}{ccc} e^{-ipx}\nu & \text{for} & x < 0, \\ e^{-ipx}S\ \nu & \text{for} & x > 0, \end{array} \right. , \tag{7}$$

and the inner component defined as a solution of the corresponding homogeneous adjoint equation under the above boundary condition (5).

Theorem 2.2 *The scattering matrix defined as the transmission coefficient S in the exterior component of the scattered waves (7) is represented as:*

$$S_{_\beta}(p) = \frac{iI + \frac{1}{2}\left[\beta_{00} - \beta_{_{01}}\,\mathcal{M}\,(I + \beta_{11}\,\mathcal{M})^{^{-1}}\,\beta_{_{10}}\right]}{iI - \frac{1}{2}\left[\beta_{00} - \beta_{_{01}}\,\mathcal{M}\,(I + \beta_{11}\,\mathcal{M})^{^{-1}}\,\beta_{_{10}}\right]}. \tag{8}$$

In particular, if $\beta_{11} = 0 = \beta_{00} + \beta_{01}P_NAP_N\beta_{10}$, then

$$S_{_\beta}(p) = \frac{I + \frac{1}{2i}\beta_{_{01}}P_N\frac{I+A^2}{A-pI}P_N\beta_{_{10}}}{I - \frac{1}{2i}\beta_{_{01}}P_N\frac{I+A^2}{A-pI}P_N\beta_{_{10}}}, \quad \textit{and}\ S_\beta(\infty) = I, \tag{9}$$

with the scattered waves of the perturbed operator defined by (7,6) as:

$$\mathbf{\Psi}^\beta = \left(\begin{array}{c} \Psi_{_0}^\beta \\ \\ \Psi_{_K}^\beta \end{array} \right) = \left(\begin{array}{c} \Psi_{_0}^\beta \\ \\ \frac{A-iI}{A-pI}\beta_{10}\left[I - \frac{i}{2}\beta_{01}P_N\frac{I+A^2}{A-pI}P_N\,\beta_{10}\right]^{-1}\ \nu \end{array} \right). \tag{10}$$

Proof. We can re-write the equation (5) as

$$\left(\begin{array}{c} i(S\nu-\nu) \\ \eta_{+}(\nu) \end{array} \right) = \left(\begin{array}{cc} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{array} \right) \left(\begin{array}{c} \frac{S+1}{2}\nu \\ -\mathcal{M}\ (p)\ \eta_{+}(\nu) \end{array} \right),$$

hence $\eta_{+}(\nu) = [I + \beta_{11}\,\mathcal{M}]^{-1}\beta_{10}\frac{I+S}{2}\,\nu$ and

$$2i\,(S-1)\,\nu = \left[\beta_{00} - \beta_{01}\ \mathcal{M}\,(1+\beta_{11}\,\mathcal{M})^{-1}\,\beta_{01}\right]\,[S+I]\,\nu,$$

which implies

$$\eta_{+}(\nu) = \frac{1}{I+\beta_{11}\mathcal{M}}\beta_{10}\frac{I+S}{2}\nu. \tag{11}$$

and the announced expression for the outer component of the scattered wave. It remains to check that the inner component Ψ_1 of the scattered wave is represented by (6). Indeed (6) can be represented as

$$\frac{A+iI}{A-pI}\eta_{+}(\nu) = \frac{A}{A-iI}\eta_{+} + \frac{I}{A-iI}\ \frac{I+pA}{A-pI}\eta_{+} =$$

$$\frac{A}{A-iI}\eta_{+} + \frac{I}{A-iI}P_N\frac{I+pA}{A-pI}\eta_{+} + \frac{I}{A-iI}(I-P_N)\frac{I+pA}{A-pI}\eta. \tag{12}$$

According to the Neumann decomposition (3), which is valid even for non-densely defined operators (but becomes not unique !), we may set: $P_N\frac{I+pA}{A-pI}\eta_{+} = -\eta_{-}$ and $\frac{I}{A-iI}(I-P_N)\frac{I+pA}{A-pI}\eta_{=} : u_0$. Then, according to above Theorem 1.4 $u = \frac{A+iI}{A-pI}\eta_{+}(\nu)$ is a solution of the adjoint equation $(A^{+}-pI)u = 0$. It satisfies the boundary conditions, if we choose the scattering matrix as suggested.

The end of the proof

2.2 Blaschke structure of the scattering matrix.

Let us consider a simplest scattering matrix constructed for a one-dimensional operator $A = \alpha_0\ \ \mathbf{e}_0\rangle\,\langle\mathbf{e}_0$. The general formula for the scattering matrix is applicable in this case too, with $P = I, \vec{\beta}_{01} := \beta_{01}P\mathbf{e}_0, \beta_{11} = 0$ and $\beta_{00} = -\beta_{01}PAP\beta_{10}$:

$$S(p) = \frac{2i + \frac{1+\alpha_0^2}{\alpha_0-p}\vec{\beta}_{01}\rangle\,\langle\vec{\beta}_{01}}{2i - \frac{1+\alpha_0^2}{\alpha_0-p}\vec{\beta}_{01}\rangle\,\langle\vec{\beta}_{01}} = \frac{2i + \frac{1+\alpha_0^2}{\alpha_0-p}\ \ B^2\ \ \vec{\nu}_0\rangle\,\langle\vec{\nu}_0}{2i - \frac{1+\alpha_0^2}{\alpha_0-p}\ \ B^2\ \ \vec{\nu}_0\rangle\,\langle\vec{\nu}_0}, \tag{13}$$

with $B = \parallel \vec{\beta}_{01} \parallel$, $\vec{\beta}_{01} =: B\ \vec{\nu}_0$. Then the scattering matrix of the one-dimensional perturbation of the momentum can be represented as a multi-dimensional Blaschke-factor, see [65], with a single zero in the lower half -plane $\Im p_0 < 0$:

$$S(p) = \frac{p - [\alpha_0 - \frac{i}{2}(1+\alpha_0^2)B^2P_0]}{p - [\alpha_0 + \frac{i}{2}(1+\alpha_0^2)B^2P_0]} = \frac{p - [\alpha_0 - \frac{i}{2}(1+\alpha_0^2)B^2]}{p - [\alpha_0 + \frac{i}{2}(1+\alpha_0^2)B^2]}P_0 + P_0^{\perp} =: \frac{p-p_0}{p-\bar{p}_0}P_0 + P_0^{\perp} \tag{14}$$

with $p_0 = \alpha_0 - \frac{i}{2}(1+\alpha_0^2)B^2$, $P_0 = \vec{\nu}_0\rangle\langle\vec{\nu}_0$ and $P_0^{\perp} = I - P_0$.
Vice versa, consider a standard Blaschke factor

$$S(p) = \frac{p-p_0}{p-\bar{p}_0}P_{_0} + P_{_0}^{^{\perp}}, \tag{15}$$

with a one-dimensional projection $P_0 = \vec{\nu}_0\rangle\langle\vec{\nu}_0$. This factor can be obtained as a scattering matrix for the pair consisting of the conventional momentum operator in the space $L_2(R,E)$ and the perturbed momentum operator obtained via breeding $\mathcal{P}$ with a one-dimensional operator $A_0 = \alpha_0\ \mathbf{e}_0\rangle\langle\mathbf{e}_0$ with the eigen-value $\alpha_0 = \Re p_0$, the boundary matrix

$$B = \begin{pmatrix} \beta_{_{00}} & \beta_{_{01}} \\ \beta_{_{10}} & \beta_{_{11}} \end{pmatrix}$$

with $\beta_{11} = 0$, and β_{01} defined from the comparison of (13) with (14)

$$\Im p_0\ \vec{\nu}_0\rangle\langle\vec{\nu}_0 = \frac{1+\alpha_0^2}{2}\vec{\nu}_0\rangle\, |b_{01}|\langle\mathbf{e}_0,\mathbf{e}_0\rangle|b_{10}|\vec{\nu}_0,$$

$$\beta_{01} = \sqrt{2\frac{\Im\ p_0}{1+\alpha_0^2}}\ \vec{\nu}_0\rangle\langle\mathbf{e}_0,*\rangle,\ |\beta_{01}| = \sqrt{2\frac{\Im\ p_0}{1+\alpha_0^2}},\ \vec{\beta}_{01} = |\beta_{01}|\vec{\nu}_0. \tag{16}$$

and $\beta_{00} = -\beta_{01}\alpha_0\beta_{01}^{+}$. Then the model described by the theorems 2.1, , 2.2 has the scattering matrix S. In particular the scattered waves of the model corresponding to are found based on the Ansatz (7):

$$\mathbf{\Psi} = \begin{pmatrix} \Psi(x) \\ \frac{\alpha_0+i}{\alpha_0-p}\eta_+ e \end{pmatrix}$$

via eliminating η_- from the boundary conditions based on $\eta_- = -\frac{1+\alpha_0 p}{\alpha_0-p}\,\eta_+$. Ii particular, for η we have from (11)

$$\eta = \beta_{10}\frac{I+S}{2}\nu = \frac{1+\alpha_0^2}{\alpha_0-p}\beta_{10}\frac{I}{I-\frac{i}{2}B^2\nu_0\rangle\langle\nu_0}\ \ \nu,$$

with $B^2 = \|\ \beta_{01}P_N e_0\ \|^2$, $\nu_0 = B^{-1}\beta_{01}P_N e_0$.

Hence we derive, as a corollary of the preceding Theorem 2.2, that the scattering matrix of each one-dimensional perturbation of the momentum, with the boundary condition satisfying $\beta_{11} = 0, \beta_{00} + |\beta_{01}|^2\alpha_0 = 0$, is a Blashke-factor.

Vice versa, each 1d Blaschke-factor with a single zero $p_0 : \Im p_0 < 0$ can be obtained as a scattering matrix of the corresponding one-dimensional perturbation $A = \alpha_0\ \mathbf{e}_0\rangle\langle\mathbf{e}_0$ of the momentum and appropriate boundary condition with $\beta_{11} = 0, \beta_{00} + |\beta_{01}|^2\alpha_0 = 0$.

Assuming that the condition of the preceding theorem is fulfilled, consider the rational representation (8) of the stationary scattering matrix of the operator $\mathcal{P}_{_\beta}$. Due to $S(\infty) = 1$, it is represented, for small β, as a finite Blashke product

$$S_{_\beta}(p) = \frac{I + \frac{1}{2i}\beta_{_{01}}P_N\frac{I+A^2}{A-pI}P_N\beta_{_{10}}}{I - \frac{1}{2i}\beta_{_{01}}P_N\frac{I+A^2}{A-pI}P_N\beta_{_{10}}} = \tag{17}$$

$$\prod_s \left\{ \left[\frac{p - p_s(\beta)}{p - \bar{p}_s(\beta)} \right] P_s + P_s^{\perp} \right\} =: \prod_s B_s(\beta, p). \tag{18}$$

of elementary Blaschke factors $B_s(\beta, p)$. The following statement, similar to the chain rule for wave operators, see for instance [36], describes scattering systems with common scattering channels. In our case the role of the systems can play three Friedrichs models $\mathcal{P}_0, \mathcal{P}_\beta, \mathcal{P}_0^\beta$, where the scattering matrix $S(\mathcal{P}_{\beta_1}, \mathcal{P}) = S_1^\beta$ is an elementary Blaschke factor of the scattering matrix $S(\mathcal{P}_\beta, \mathcal{P}) = S_\beta$, see more comments in Appendix 2.

Consider the spectral problems in $L_2(R, E)$, $L_2(R, E) \oplus K_0^\beta$, $L_2(R, E) \oplus K_\beta$: with unperturbed momentum,

$$\mathcal{P}\Psi = p\Psi,\ \Psi \in W_2^1(R, E),$$

with one-dimensional perturbation $\mathcal{P}_\beta$ of the momentum, see (2.1):

$$\mathcal{P}_{\beta_0}\Psi = p\Psi,\ \Psi \in W_2^1(R, E) \oplus K_0^\beta,$$

and the scattering matrix $S_0^\beta(p)$ represented by a single Blaschke factor, and a multidimensional perturbation A_β of the momentum:

$$\mathcal{P}_\beta\Psi = \frac{1}{i}\Psi,\ \Psi \in W_2^1(R, E) \oplus K_\beta,$$

The Cauchy problems for the corresponding non-stationary equations

$$\mathcal{P}\Psi = \frac{1}{i}\frac{d}{dt}\Psi,\ \Psi\Big|_{t=0} = \Psi_0 \in W_2^1(R, E),$$

$$\mathcal{P}_{\beta_0}\Psi = \frac{1}{i}\frac{d}{dt}\Psi,\ \Psi\Big|_{t=0} = \Psi_0 \in W_2^1(R, E) \oplus K_0^\beta,$$

$$\mathcal{P}_\beta\Psi = \frac{1}{i}\frac{d}{dt}\Psi,\ \Psi\Big|_{t=0} = \Psi_0 \in W_2^1(R, E) \oplus K_\beta,$$

define the unitary groups in the corresponding Hilbert spaces $L_2(R, E)$, $L_2(R, E) \oplus K_0^\beta$, $L_2(R, E) \oplus K_\beta$ with equivalent incoming and outgoing subspaces

$$D_{in} = L_2(R_-, E),\ D_{out} = L_2(R_+, E).$$

Denote by P_{in}, P_{out} the orthogonal projection onto the incoming and outgoing subspaces of the dynamic $u(x) \longrightarrow u(x - t)$ and introduce the Lax-Phillips wave-operators, see [53] as:

$$\begin{array}{rcl} W_-(\mathcal{P}_0^\beta,\, \mathcal{P}_\beta) & = & s - lim_{t\to\infty} e^{i\,\mathcal{P}_0^\beta\ t} P_{in} e^{-i\mathcal{P}_\beta\ t}, \\ W_+(\mathcal{P}_0^\beta,\, \mathcal{P}_\beta) & = & s - lim_{t\to\infty} e^{-i\,\mathcal{P}_0^\beta\ t} P_{out} e^{i\mathcal{P}_\beta\ t}, \\ W_-(\mathcal{P}_0^\beta,\, \mathcal{P}_0) & = & s - lim_{t\to\infty} e^{i\,\mathcal{P}_0^\beta\ t} P_{in} e^{-i\mathcal{P}_0\ t}, \\ W_-(\mathcal{P}_0^\beta,\, \mathcal{P}_0) & = & s - lim_{t\to\infty} e^{-i\,\mathcal{P}_0^\beta\ t} P_{out} e^{i\mathcal{P}_0\ t}. \end{array} \tag{19}$$

In particular $W_{\mp}(\mathcal{P}_0^\beta,\,\mathcal{P}_0)$ coincide respectively with unity on $D_{int,out}$. The wave operators can be represented in spectral terms of the corresponding operat0rs. For instance, using the scattered waves $\mathbf{\Psi}^\beta$ constructed in previous section, see (10), we obtain

$$W_-(\mathcal{P}_\beta,\mathcal{P}_0^\beta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Psi}_\beta(p)\rangle\langle\mathbf{\Psi}_\beta P_{int}\mathbf{\Psi}_0^\beta)\rangle\langle\mathbf{\Psi}_0^\beta(p)*dp:L_2(R,E)\oplus K_0^\beta\longrightarrow L_2(R,E)\oplus K_\beta,$$

$$W_-(\mathcal{P}_0^\beta,\mathcal{P}_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Psi}_0^\beta(p)\rangle\langle\mathbf{\Psi}_0^\beta P_{int}\mathcal{P}_0)\rangle\langle\mathbf{\Psi}^0(p)*dp:L_2(R,E)\longrightarrow L_2(R,E)\oplus K_0^\beta,\quad(20)$$

with $\mathbf{\Psi}^0=e^{-ipx}$, and similar formulae for W_+:

$$W_+(\mathcal{P}_\beta,\mathcal{P}_0^\beta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Psi}_{out}^\beta(p)\rangle\,\langle\mathbf{\Psi}_0^\beta P_{out}\mathcal{P}_0)\rangle\,\langle\mathbf{\Psi}_0^\beta(p)*dp:L_2(R)\oplus K_0^\beta\longrightarrow L_2(R,E)\oplus K_\beta,$$

$$W_+(\mathcal{P}_0^\beta,\mathcal{P}_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{\Psi}_{0,out}^\beta(p)\rangle\,\langle\mathbf{\Psi}_0^\beta P_{out}\mathcal{P}_0)\rangle\,\langle\mathbf{\Psi}_0(p)*dp:L_2(R)\longrightarrow L_2(R,E)\oplus K_0^\beta.$$

The scattered waves (10) are called *incoming scattered waves* and denoted hereafter as $\mathbf{\Psi}_{in}^\beta,\mathbf{\Psi}_{in,0}^\beta$. Introducing the outgoing scattered waves $\mathbf{\Psi}_\beta(out,\nu),\mathbf{\Psi}_0^\beta(out,\nu)$ with the first components

$$\Psi_{out,0}=\left\{\begin{array}{c}e^{-ipx}S^+\nu_{out}\quad\text{if } x<0,\\ e^{-ipx}\nu_{out}\quad\text{if } x>0.\end{array}\right.,$$

we obtain for the eigenfunctions of the operators $\mathcal{P}_\beta,\ \mathcal{P}_0^\beta$

$$\Psi_{out}=\left(\begin{array}{c}\Psi_{out,0}\\ \frac{A-iI}{A-pI}\beta_{10}\left[I+\frac{i}{2}\beta_{01}P_N\frac{I+A^2}{A-pI}P_N\,\beta_{10}\right]^{-1}\ \nu_{out}\end{array}\right)=S^+\Psi_{in}.\quad(21)$$

The incoming and outgoing scattered waves of the simplest perturbation $\mathcal{P}_0^\beta$ of the momentum are parametrized by the components Ψ_0^β in $L_2(R,E)$ and by the elements (16) of the corresponding scattering matrix - a standard Blaschke factor (15), see

$$\mathbf{\Psi}_{0,in}^\beta\ \ \nu_{out}=\left(\begin{array}{c}\Psi_{0,out}^\beta\nu_{out}\\ \Psi_{0,out,1}^\beta\nu_{out},\end{array}\right),\quad(22)$$

where the first component is represented, as usual, by exponentials and the second is

$$\Psi_{0,out,1}^\beta\nu_{out}=\frac{\alpha_0-i}{\alpha_0-p-\frac{i}{2}B^2(1+\alpha^2)}e_0\rangle\langle e_0\beta_{10}\nu_0\rangle\,\langle\nu_0\nu_{out}\rangle+\frac{\alpha_0-i}{\alpha_0-p}e_0\rangle\langle e_0\beta_{10}[I-\nu_0\rangle\,\langle\nu_0]\nu_{out}\rangle.$$

In our scattering problems the Lax-Phillips wave operators exist and constitute the Lax-Phillips scattering operators for various $\mathcal{P}'$:

$$W_+^+(\mathcal{P}',\mathcal{P})W_-(\mathcal{P}',\mathcal{P})=:S^{LP}\ (\mathcal{P}',\mathcal{P})\quad(23)$$

commuting with $\mathcal{P}$ and represented, in spectral terms of $\mathcal{P}$ by the corresponding "scattering operators" calculated as:

$$\begin{array}{rcl} S^{LP}\left(\mathcal{P}_\beta, \mathcal{P}_0\right) & = & s-lim\ _{t\to\infty}\ e^{-i\,\mathcal{P}_0\ t} P_{out} e^{2i\mathcal{P}_\beta\ t} P_{in} e^{-i\,\mathcal{P}_0\ t}, \\ S^{LP}\left(\mathcal{P}_0, \mathcal{P}_0^\beta\ \right) & = & s-lim\ _{t\to\infty}\ e^{-i\,\mathcal{P}_0^\beta\ t} P_{out} e^{2i\mathcal{P}_0\ t} P_{in} e^{-i\,\mathcal{P}_0^\beta\ t}. \end{array} \tag{24}$$

Chain rule for the wave operators is true, see [27]

$$W_\pm(\mathcal{P}_\beta, \mathcal{P}_0^\beta) W_\pm(\mathcal{P}_0^\beta, \mathcal{P}_0) = W_\pm(\mathcal{P}_\beta, \mathcal{P}_0).$$

The corresponding simple relation is absent for the scattering operators, but an analog of it can be derived based on (10)

Theorem 2.3 *The Lax-Phillips Scattering operator is connected with the stationary scattering matrix defined by asymptotic of the first component* $\{e^{-ipx}\nu_{in}, e^{-ipx}S(p)\nu_{in}\}$ *of the scattered wave at* $+\infty$ *:*

$$S^{LP}(\mathcal{P}_\beta, \mathcal{P}_0) = S_\beta. \tag{25}$$

Besides the Lax-Phillips scattering operator $S^{LP}(\mathcal{P}_\beta,\, \mathcal{P}_0)$ *admits the following factorization:*

$$S_{LP}(\mathcal{P}_\beta,\, \mathcal{P}_0) = \left[W_+(\mathcal{P}_0^\beta,\, \mathcal{P}_0)\right]^+ S^{LP}(\mathcal{P}_\beta,\, \mathcal{P}_0^\beta)\ \ W_-(\mathcal{P}_0^\beta,\, \mathcal{P}_0). \tag{26}$$

Here the framing terms $W_\pm(\mathcal{P}_0^\beta,\, \mathcal{P}_0)$ *contain singularities at the resonances* $p_0(\beta), \bar{p}_0(\beta)$*, and at the eigenvalue* α_0*, and the central term* $S_{LP}(\mathcal{P}_\beta,\, \mathcal{P}_0^\beta)$ *is an analytic function of* β *at the origin, if* p *is in a small neighborhood of* α_0*. The above formula (26) can be represented in terms of incoming and outgoing scattered waves*

$$\mathbf{\Psi}_0^\beta(\nu, in),\ \mathbf{\Psi}_0^\beta(\nu', out),\ \mathbf{\Psi}_\beta(\nu', out),\ \ \mathbf{\Psi}_\beta(\nu', out)$$

in the form

$$\begin{array}{c} S^{LP}_{\nu',\nu}(\mathcal{P}_\beta,\, \mathcal{P}_0^\beta) = \\ \mathbf{\Psi}_0^\beta(\nu', out)\rangle\ \langle \mathbf{\Psi}_0^\beta(\nu', out) P_{out} \mathbf{\Psi}_\beta(\nu, out)\rangle \langle \mathbf{\Psi}_\beta(\nu', out)\ \mathbf{\Psi}_\beta(\nu, in)\rangle \\ \langle \mathbf{\Psi}_\beta(\nu, in) P_{in} \mathbf{\Psi}_0^\beta(\nu, in)\rangle\ \langle \mathbf{\Psi}_0^\beta(\nu, in) = \mathbf{\Psi}_0^\beta(\nu', out)\rangle\ S_\beta\langle\ \ \mathbf{\Psi}_0^\beta(\nu, int) = \\ \mathbf{\Psi}_0^\beta(\nu, int)\ \ (S_0^\beta)^+\ \ S_\beta \mathbf{\Psi}_0^\beta(\nu, int), \end{array} \tag{27}$$

with $\langle \mathbf{\Psi}_\beta(\nu', out)\ \mathbf{\Psi}_\beta(\nu, in)\rangle(\nu', \nu) = S_{\nu',\nu}$ *equal to the stationary scattering matrix of* $\mathcal{P}_\beta$ *with respect to* $\mathcal{P}_0$ *- the asymptotic coefficient* S_β^+ *of the outgoing scattered wave* $e^{-ipx}S_\beta^+\nu_{in}$ *in* R_+*. In particular, the matrix of the scattering operator* S^{LP} *(Lax-Phillips scattering matrix), with respect to the incoming scattered waves, is represented as:*

$$S^{LP}_\beta(\mathcal{P}_\beta, \mathcal{P}_0^\beta, in) = (S_0^\beta)^+\ \ S_\beta, \tag{28}$$

and the corresponding representation with respect to outgoing scattered waves is

$$S^{LP}_\beta(\mathcal{P}_\beta, \mathcal{P}_0^\beta, out) = S_\beta^+ S_0^\beta. \tag{29}$$

2.3 Jump-start

The last formula (27) in the previous section shows that the "total" scattering matrix $S(\mathcal{P}_\beta, \mathcal{P}_0)$ can be expressed in terms of the intermediate scattering matrix $S(\mathcal{P}_\beta, \mathcal{P}_0^\beta))$ and explicitly constructed framing factors, containing an information about resonances. These factors are calculated explicitly, see (20,29), and the intermediate term is represented, for the Friedrichs model, in rational form , or , in general case, is calculated for small β by the standard analytic perturbation procedure. It is convenient to assume that both terms are given in Blashke form:

$$S_\beta^{LP} = \left(\frac{p-p_1}{p-\bar{p_1}}P_1 + P_1^\perp\right)\prod_{s=2}^{k}\left(\frac{p-p_s}{p-\bar{p}_s}P_s + P_s^\perp\right) =: B_1(\beta,p)\prod_{s=2}^{k} B_s(\beta,p) \tag{30}$$

Each of functions $p_s(\beta)$ is an analytic function of the "small parameter" beta, see below. The structure of each Blaschke factor B_s shows that it has a zero at $p_s(\beta)$, a pole at $\bar{p}_s(\beta)$ and both of them approach to the eigenvalue α_s of the inner Hamiltonian A when $\beta \to 0$. The Blaschke factor B_1 is not analytic near $(0, \alpha_1)$ due to convergence of the zero and the pole of the factor to the same point α_1. If the eigenvalues of the inner Hamiltonian A are separated, $\min_{s,t} |\alpha_s - \alpha_t| \geq \delta$, then all factors $B_s(\beta, p)$, $s \geq 2$ are analytic with respect to (β, p) on a small neighborhood of $(0, \alpha_1)$. The whole scattering matrix (30) is not analytic with respect to the perturbation parameter β near the eigenvalue α_1 of the "inner Hamiltonian " A due to presence of the non-analytic factor B_1. One may modify the perturbation procedure *locally*, eliminating, for instance, the non-analytic factor $S_1^\beta := B_1$ via the "jump-start": by introducing of the *intermediate operator* $\mathcal{P}_1^\beta$, which is selected such that $S_1^\beta = B_1$ is the scattering matrix for the pair $\left(\mathcal{P}_1^\beta, \mathcal{P}\right)$. Then the Scattering matrix can be presented on a neighborhood of α_1 as a product of the non-analytic, but explicit factors $W_\pm(\mathcal{P}^\beta)_0)$, S_0^β, see (26) or (28) and the analytic factor $S(\mathcal{P}_\beta, \mathcal{P}_0^\beta) = \prod_{s=2}^{k} B_s(\beta, p)$. The construction of the intermediate operator $\mathcal{P}_0^\beta$, with the Scattering matrix $S(\mathcal{P}_0^\beta, \mathcal{P}_0)$ is already described above.

The approximate calculation of the resonance $p_1(\beta)$, for small β can be done based of the matrix-version of Rouche theorem, see [37], see the Appendix 2, but it is not easy to recover an exact Blaschke factor B_1. Indeed, in the case of Friedrichs model there is a temptation to calculate, see Appendix 2, the right Blaschke factor B_1 approximately based on the rational form (17) of the scattering matrix, see Appendix 2. But the approximation gives $p_1(\beta)$ only asymptotically, for $\beta \to 0$, as a finite power expansion on the perturbation parameter. The intermediate operator constructed based on the approximate data has scattering matrix which does not coincide with the exact right factor, hence can't eliminate the non-analytic exact right factor B_1 of S_β. This is a reason why the jump-start technique in the analytic perturbation procedure suggested in previous sections for Friedrichs model should be applied stepwise, depending on the selected approximation for $B_1(\beta, p)$. Only in the case when we succeeded to split of an exact Blaschke factor B_1, the remaining chain $\prod_{s\geq 2} B_s(\beta, p)$ is analytic with respect to β, p on a small neighborhood of $(0, \alpha_1)$ and hence the scattering matrix and the scattered waves can be calculated by a single pitch of the two-steps jump-start + analytic perturbation procedure. Generally a countable sequence of the pitches is needed, but the speed of convergence of the total procedure depends on selection of the approximation B_1 for every pitch.

3 Effective zero-range Hamiltonians: history and motivation.

The suggested jump-start technique is applicable to various scattering and spectral problems, providing a leading thread in the corresponding analytic perturbation analysis. While in the Friedrichs model the rational form provided basic data for the scattering matrix in form of the Krein function, in other scattering problems the corresponding data are encoded into a relevant (sometimes non-standard) Dirichlet-to-Neumann map. It appeared that the scattering matrix in most of interesting problems of resonance scattering admits a representation of the scattering matrix by the formula similar to (17) with the Krein function substituted by an appropriate Dirichlet-to Neumann map. An analog of the approximate scattering matrix S_0^β is obtained via substitution, in the formula for S_0^β, of the Dirichlet-to Neumann map by an appropriate rational approximation. The fitted solvable model of the resonance scattering system is constructed as a Hamiltonian $\mathcal{H}_0^\beta$ which corresponds to the approximate scattering matrix S_0^β taking into account only nearest resonances on a certain spectral interval.

In the end of previous section we noticed that the approximate scattering matrix can't help eliminating the non-analytic factors in the chain of Blaschke factors, but it can give a good hint for optimization of the corresponding computation procedure anyway. Indeed, only a small part of linear partial differential equations can be solved analytically, with final result represented by an elegant formula connecting directly the physical effect described by the equation with the geometrical and physical parameters of the medium. Powerful computers are able to produce numerical solutions, but optimization of the result requires expensive and resource consuming scanning over the multidimensional space of all essential parameters of the problem. To reduce the area of scanning we need, minimally, a list of essential parameters or, better, an approximate formula for the solution in terms of the essential parameters. Pure mathematicians were unable to answer this challenge. First steps in that direction were done by engineers, physicists and applied mathematicians. Only nowadays the mathematical community is able to interpret the findings of applied sciences and supply them with solid mathematical basement. Fortunately this reconstruction extends the field of application and reliability of the modern analytic perturbation procedure, so that the corresponding mathematics proves to be useful. Here are some examples presented in chronological order.

3.1 Saint-Venant principle

In 1855 French elasticity theorist Jean Claude Barre de Saint Venant published in the memoir [38] the statement ".. the strains that can be produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part". Other words: the strains obtained from the solution of the differential equations of elasticity under a localized stress on the boundary, depend, in remote area, on a small number of essential parameters. Richard von Mises proposed a mathematical interpretation of the principle, see [39], based on asymptotic of the corresponding Green function.

Question 1.*Consider a symmetric 3d Lame or 2d bi-harmonic differential operator L, on a compact domain $\Omega_3 \subset R_3$ or $\Omega_2 \subset R_2$,respectively, supplied with a self-adjoint boundary conditions and additionally restricted $L \to L_a$ by the condition of vanishing of elastic boundary data on a neighborhood of the given boundary point a, see for instance [40], and construct a self-adjoint extension of the restricted operator L_a. Is it possible to recover an explicit dependence of the eigenfunctions and eigenvalues of the extension from the extension parameters? If it is, then the parameters of the self-adjoint extension play a role of the Saint Venant parameters.* See more comments in [40].

3.2 Kirchhoff Ansatz

In 1882 Gustav Kirchhoff suggested a final version of his Ansatz for the Green function of the Helmholtz equation on domains connected by a small opening, see [41].

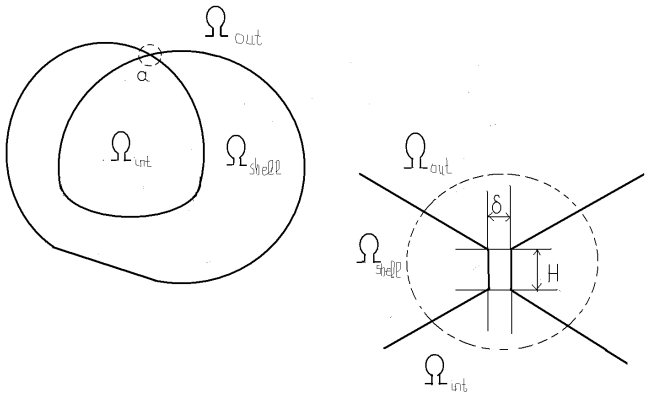

Fig. 1. Hemlholtz Resonator with a point-wise opening at the point a and an enlarged detail of the resonator with a narrow short channel, $\delta << H << \lambda^{-1/2}$.

In the case of a point-wise opening a, connecting the compact domain Ω_{int} with the non-compact domain Ω_{out} the Kitchhoff Ansatz for the Green function of the Helmholtz equation in the union $\Omega = \Omega_{int} \cup \Omega_{out}$

$$-\Delta G_\lambda(x,y) - \lambda G_\lambda(x,y) = \delta(x-y),\ \frac{\partial G}{\partial n_x}\Big|_\Omega = 0,\ x,y \in \Omega$$

is taken the form of a linear combination of the Green functions $G^{int}_\lambda(x,y)$, $G^{out}_\lambda(x,y)$ of the inner and outer problems:

$$-\Delta G^{in,out}_\lambda(x,y) - \lambda G^{in,out}_\lambda(x,y) = \delta(x-y),\ \frac{\partial G}{\partial n_x}\Big|_{\Omega_{in,out}} = 0.$$

$$G_\lambda(x,y) - \left\{ \begin{array}{ccc} G^{out}_\lambda(x,y) & + & A^{out}\, G^{out}_\lambda(x,a),\ x,y \in \Omega_{out} \\ A^{int}\, G^{in}_\lambda(x,a), & \text{if} & y \in \Omega_{out},\ x \in \Omega_{int}, \end{array} \right. ,$$

The Kirchhoff coefficients $A^{out,int}$ can be defined from a zero-range model for the Neumann Laplace equation, based on symmetric restriction of the Laplacian onto functions vanishing in a neighborhood U_a of a. Then the Kirchhoff constants are defined by the extension parameters and from the local asymptotic of the Green function with the pole at a.

Question 2 *Is it possible to define the the extension parameters such that the corresponding Kirchoff Ansatz gives a first order approximation of the Green function of the full Neumann problem for the pair of domains $\Omega_{int}, \Omega_{out}$ connected by the thin short channel ω connecting?*

3.3 Fermi zero-range potential

In 1934 Enrico Fermi, [3], proposed using an effective Hamiltonian for the scattering problem of neutrons n by nucleon S of Sulfur, constructed in form of Laplacian in $L_2(R_3)$ defined on smooth functions $u \in L_2(R_3)$ with a singularity at the origin and a boundary condition

$$u(x) = \frac{A^u}{4\pi|x|} + B^u + \dots, \ A^u = \gamma B^u, \ \gamma = \bar{\gamma}.$$

The Laplacian with this boundary condition is symmetric (even self-adjoint) and admits explicit construction of eigenfunctions - this model is "solvable". Fermi suggested to "fit" this " zero-range model" choosing $\gamma = -4\pi p_0^{-1}$, if $-p_0^2$ is a small negative eigenvalue in the system n, S. The model can be extended to the case when $\gamma > 0$, and fitted to the small purely imaginary resonance $p_0 = i\gamma$. Almost 30 years later Felix Berezin and Ludvig Faddeev suggested an elegant interpretation of the model proposed by E. Fermi. They interpret the Fermi model in terms of John von Neumann operator extension theory, see [42]. Two decades later a similar zero-range model was constructed as a coupling of the Laplacian with a finite hermitian matrix A, playing the role of an inner Hamiltonian of a joint system $n \cup S$:

$$-\Delta \oplus A \longrightarrow -\Delta_\beta$$

defined by the boundary condition β imposed onto the boundary data of elements from the domain of $-\Delta$ and A, see [43].

Question 3 *Consider a scattering amplitude of the system $n \cup S$ on a certain spectral interval Δ. Is it possible to select the hermitian matrix A and the parameters of the extension such that the scattering amplitude of the model constructed by the operator extension procedure gives a rational approximation of the scattering amplitude of the original system $n \cup S$ on given spectral interval Δ ?.* See more comments in [44]

3.4 Datta-Das Sarma boundary condition at the junction

Quantum nano-electronic networks constructed of quantum wells connected by the straight leads, width δ, have interesting transport properties which can be used for manufacturing of nano-electronic devices and, eventually, the quantum computer.

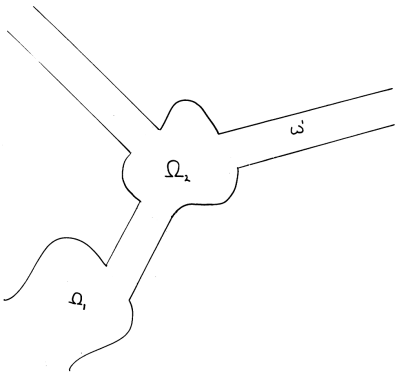

Fig. 2 Quantum Network: a detail.

A basic detail of the quantum network is a junction, see Fig. (refF:figure 3).

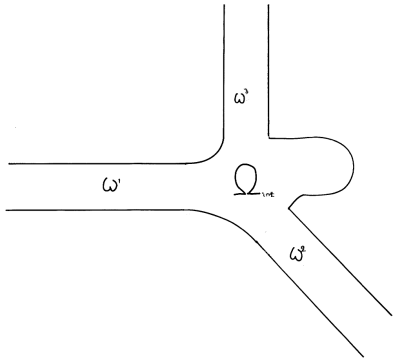

Fig. 3 Junction Ω is the most important basic detail of the quantum network. The simplest junction is composed of a quantum well and few semi-infinite quantum wires, of constant width, attached to it.

A one-body "scattering problem " on a junction is equivalent to the construction of a solution of the re-normalized Schrödinger equations $\mathcal{L}_0\psi = E\psi$ with $\mathcal{L}_0 = l^\omega \oplus L_{int}$ with the energy substituted by the spectral parameter $\lambda = 2mE\hbar^{-2}$ and

$$-\triangle u + Vu \ =: \mathcal{L}$$

$$l_l^m \ = \ -\frac{d^2}{dx^2} + \frac{\pi^2 l^2}{\delta^2} + V_\delta, \quad l \geq 1 \ ,$$

satisfying the zero boundary conditions on the border and the matching condition on the common boundary of the vertex domain and the leads.

The oscillating solutions in the wires are

$$\chi_\pm^{m,l} \ = e^{\pm i\,\sqrt{\lambda-\lambda_l}\ x}\ e_l^m(y) := \ e^{\pm iK_+^{m,l}\ x}\, e_l^m(y), \quad \lambda > \lambda_l = \pi^2\, l^2 \delta^{-2}, \tag{31}$$

and the exponentially decreasing solutions (evanescent modes) are

$$\xi_-^{m,s} \;=\; e_s^m(y)\; e^{-\;\sqrt{\lambda_s-\lambda}\;x} := \; e^{-K_-^{m,s}\,x}\, e_s^m(y), \quad \lambda<\lambda_s. \tag{32}$$

The linear combination of them constitutes a scattering Ansatz $\psi_l^m(x) =:$

$$\left\{\begin{array}{l} \chi_+^l(x)+\sum_{\pi^2r^2/\delta^2<\lambda} S_{l,r}^{m,m}\chi_-^r(x)+\sum_{\pi^2r^2/\delta^2>\lambda} s_{l,r}^{m,m}\xi^r(x), x\in\omega^m \\ \sum_{\pi^2r^2/\delta^2<\lambda} S_{l,r}^{m,n}\chi_-^r(x)+\sum_{\pi^2r^2/\delta^2>\lambda} s_{l,r}^{m,n}\xi^r(x), x\in\omega^n,\; n\neq m\;, \end{array}\right. \tag{33}$$

with coefficients which should be calculated from the smooth matching conditions on the common boundary of the wires and the vertex domain of the junction. The result of matching is the scattered wave satisfying the homogeneous equation $\mathcal{L}\psi=\lambda\psi$.

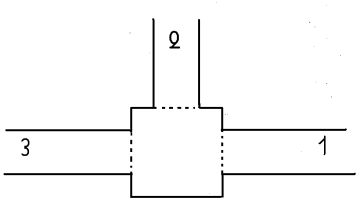

Fig. 4. Symmetric junction is invariant with respect left-right reflection.

The coefficients in front of the oscillating terms of the scattered wave are interpreted as transmission/reflection coefficients which define the transport properties of the junction.

Exponential solutions of the time-dependent equation $\frac{1}{i}\frac{\partial\psi}{\partial t}+\mathcal{L}\psi=0$ are obtained as products of the scattered wave with the time-dependent exponential $e^{i\lambda t}$.

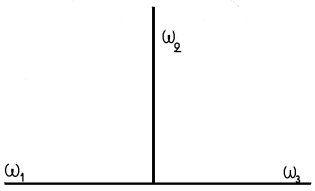

Fig. 5 Model 1d junction.

There is a good reason to substitute 2d junction by it's 1d analog, with one-dimensional Schrödinger equation on the wires $-\psi''+V\psi=\lambda\psi$ on the wires and an appropriate boundary condition at the vertex.

The boundary conditions for the model T-junction suggested by Supriyo Datta and Sankar Das Sarma in [45] is presented in terms of limit values of the wave-function on the 1d wires $\{\psi_i\}_{i=1}^3 := \vec{\psi}$ and the values of the corresponding outward derivatives (boundary currents) $\{\psi_i'\}_{i=1}^3 := \vec{\psi'}$ at the vertex:

$$\psi_1 = \beta^{-1}\psi_2 = \psi_3\,, \quad \psi_1' + \beta\psi_2' + \psi_3' = 0, \tag{34}$$

or in the form

$$P_\beta^\perp \vec{\psi} = 0\,, \quad P_\beta \vec{\psi'} = 0, \tag{35}$$

with the projection

$$P_\beta = \frac{1}{\beta^2+2}\begin{pmatrix} 1 & \beta & 1 \\ \beta & \beta^2 & \beta \\ 1 & \beta & 1 \end{pmatrix}. \tag{36}$$

The corresponding scattering matrix is constant $S_\beta = I - 2P_\beta$, β is responsible for the connection between the "bar" and the "leg" of T. The following questions are highly important in computational nano-electronics, see more comments in [19, 46, 47]

Question 4 *Derive the boundary condition suggested by Datta-Das Sarma for 1d model junction from the first principles.*

Question 5 *Interpret the phenomenological parameter β, P_β in spectral terms of the junction.*

The marked questions concerning the listed models are now objects of thorough investigation by extended group of researchers. Similar questions for other scattering systems are waiting for enthusiasts who are able to dedicate their time and efforts to this branch of mathematical physics.

4 Spectral meaning of resonances: basic examples

In all above examples the answers to the formulated questions can be found based on universality of the zero-range interaction defined by the operator extension constructions. The strongest confirmation of the universality of this interaction comes from the possibility of fitting of the parameters of the extension procedure, see [47]. The universality of the zero-range interaction also permits to construct self-adjoint dilatations of the dissipative operators, and to develop the spectral theory of the dissipative operators based on the dilatation, see [50]

4.1 Discrete shift: invariant subspaces.

Consider the unilateral shift operator $\mathbf{s}$ in $l_2 = \{\mathbf{x} = (x_0, x_1, x_3, \dots x_l, \dots)\}$:

$$\mathbf{s} : (x_0, x_1, x_3, \dots x_l, \dots) \to (0, x_0, x_1, x_3, \dots x_l, \dots). \tag{37}$$

It is obvious that the subspace of all square-summable sequences with a certain number of zeros in front is an invariant subspace of $\mathbf{s}$. What are other invariant subspaces? To make the first step toward the answer of the question one should translate the problem into the language of analysis in Hardy class H^2_+ of analytic functions on the unit disc D with square-integrable boundary values on the unit circle:

$$\mathbf{x} \longrightarrow \sum_{l\geq 0} x_l \; z^l \equiv \mathbf{x}(z), \; |z|<1, \; l_2 \longrightarrow H^2_+.$$

Simplest examples of invariant subspaces D_B obtained by direct calculations are generated by finite Blaschke products

$$B_N \; (z) = \prod_{s=1}^{N} \frac{z_s - z}{1-\bar{z}_s \, z}\theta_s, \theta_s = \bar{z}_s \, |z_s|^{-1},$$

in form $D_{out} = B_N H^2_+$. All other invariant subspaces are parametrized by the uniform limits of the finite Blaschke products on the unit disc, for instance infinite Blaschke products $B(z)$ with convergent $\prod_s |z_s|$ or so-called singular functions parametrized by singular measures $\mu : \frac{d\mu}{d\varphi} = 0$ a.e.$0 < \varphi \leq 2\pi$:

$$\Theta_\mu(z) = e^{-\int \frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\mu}, \; b_\beta(z) = \frac{\Theta_\mu - \beta}{1-\Theta_\mu \beta} \longrightarrow \Theta_\mu, \; \text{if } \beta \to +0.$$

The products $S \equiv B\,\Theta$ of Blaschke products and the singular functions form the class of so-called *inner functions* on the unit disc and parametrize the corresponding invariant subspaces as

$$D_S = SH^2_+, \; H^2_+ = D_S \oplus K, \tag{38}$$

4.2 Scattering problem for the discrete shift. Characteristic function

Consider the decomposition of the space L_2 of all square-integrable functions on the unit circle into an orthogonal sum of the Hardy class H^2_+, the class H^2_- of analytic functions on the complement tending to zero at infinity and more detailed decomposition:

$$L_2 = H^2_+ \oplus H^2_- = SH^2_+ \oplus K \oplus H^2_-. \tag{39}$$

Denote by P_K an orthogonal projection in L_2 onto K. The positive semigroup $\left\{z^l\right\}^+, l = \ldots 0, 1, 2, \ldots$ of the unitary *shift group* $U^l \equiv \left\{z^l\right\}, l = \ldots -2, -1, 0, 1, 2, \ldots$ has an invariant subspace $D_{out}SH^2_+$, and the negative semigroup $\left\{z^l\right\}, l = \ldots -2, -1$ has the invariant subspace $D_{in} = H^2_-$. These subspaces are called outgoing and incoming subspaces of the unitary group $\left\{z^l\right\}$. The following very simple statement forms the basement of the theory of the Nagy-Foias functional models for contractions with discrete and singular spectrum :

Theorem 4.1 *Let $S = B\Theta$ be an inner function with simple zeros z_s. Then the family of operators*

$$Z_l^+ = P_K z^l P_K, \quad l = 0, 1, 2 \dots \tag{40}$$

forms a semigroup of contractions $Z_l = T^l$, $l = 0, 1, 2 \dots$ and the zeros z_s of the Blaschke product B are the eigenvalues of the generator T. The corresponding eigenfunctions are

$$\Psi_s(z) = \frac{S(z)}{z - z_s} \in K, \quad |\Psi_s|_K = [1 - |z_s|^2]^{-1/2}. \tag{41}$$

The adjoint family

$$Z_l^- = P_K z^{-l} P_K, \ l = 0, 1, 2 \dots \tag{42}$$

forms a semigroup $[T^+]^l$, with eigenfunctions

$$\Phi_s(z) = \frac{I}{1 - z\bar{z}_s} \in K, \quad |\Phi_s|_K = [1 - |z_s|^2]^{-1/2}. \tag{43}$$

and eigenvalues $\bar{z}_s$. The families Φ_s, Ψ_s are bi-orthogonal

$$\langle \Phi_s, \, \Phi_t \rangle = \frac{1}{2\pi} \int \Phi_s(e^\varphi), \bar{\Psi}_s(e^\varphi) d\varphi = \delta_{st} \Theta z_s B_s(z_s) \frac{1}{1 - |z_s|^2}, \tag{44}$$

where $B_s(z) = \prod_{l \neq s} \frac{z_l - z}{1 - \bar{z}_l z} \theta_l$. The systems of eigenfunctions of T, T^+ are complete in K if and only if the singular factor is trivial $\Theta \equiv Const$.

One can see from the theorem, that the inner function defines all essential spectral properties of the operator T. There exist also a simple formula connecting the inner function S with the resolvent of the operator T^+. In fact the function S can be calculated for any contraction T which only slightly deviates from an unitary operator, e.g. $I - T^+T$ is one-dimensional. Complete theory can be developed for operators deviating from a unitary operator by trace - class perturbation.

Compare the unitary group $\left\{z^l\right\} \equiv \left\{U^l\right\}$ with the group $\left\{U_S^l\right\}$ restricted onto $D_{out} \oplus D_{int}$. Consider the decomposition of elements $u \in D_{in} \oplus D_{out}, \ u = u_{in} \oplus u_{out}$

$$U_S \left[u_{out} + u_{in}\right] \equiv z u_{out} + S P_K z u_{in} + P_{in} z u_{in}. \tag{45}$$

Theorem 4.2 *The restricted group $\left\{U_S^l\right\}$ is unitary in $D_{out} \oplus D_{int}$ and unitary equivalent to $\left\{U^l\right\}$. The equivalence is defined by the wave operators $W_{in,out}$ as*

$$s - lim_{l \to \infty} U^{-l} P_{out} U_S^l = W_{out} = P_{out} + S P_{in},$$

$$s - lim_{l \to -\infty} U^{-l} P_{in} U_S^l = W_{in} = P_{in} + S^+ P_{out},$$

and

$$W_{out}^+ W_{in} = S^{LP}. \tag{46}$$

This theorem shows that the function S can be obtained via comparison of dynamics.

The nearest example of the scattering system with a continuous evolution group which posses the described properties is provided by the Friedrichs model $\mathcal{P}_\beta$, see previous sections 1,2. Indeed, the role of incoming and outgoing subspaces of the evolution group is played by $L_2(R,E)$ and $L_2(R,E)$, the scattering matrix defined by the formula (46) is obtained as an adjoint of the stationary scattering matrix $S^{LP} = S^+_\beta$. It is analytic the the upper half-plane $\Im p > 0$. Of course the analog of the theorem 4.1 is true. A convenient and representative example of the perturbed shift is considered in section 2 when discussing the Lax-Phillips scattering for the Friedrichs model. In this section we turn to an equivalent classical example of the string with a point mass:

4.3 A string with a point mass.

The simplest realistic resonance scattering problem may be obtained via attachment to the homogeneous semi-infinite string $[0,\infty)$ a point mass M at $x = a$. This would not change the tension, but results in additional term $|u_t(t,a)|^2 M/2$ in the kinetic energy. From our everyday experience we know that the solution of the wave equation with the point mass is still a continuous function of x,t, but it is not smooth anymore: it has a jump of the first derivative at the point a such that the resulting component of the tension gives exactly an additional negative acceleration of the point mass returning that to the equilibrium position:

$$2T\left[u_x(a+0) - u_x(a-0)\right] = Mu_{tt}(a) \tag{47}$$

To give the reader a pleasure of independent (but jet supervised !) study of this simple classical object, we arrange the text in a form of a sequence of problem, so that each problem lays a basement for the solution of the following problem. We do not care about maximal generality or minimal assumption business, but wish to provide an appropriate pool where the reader - supposedly a graduate student of Physics or Mathematics - could "swim independently"[1].

Problem* **1** Derive the above boundary condition (47) from Lagrange principle for the string with the additional kinetic term (due to the point mass) taken into account, assuming that a possible jump of the derivative $[u_x] = u_x(a+0) - u_x(a-0) \neq 0$ may occur at the point a.

Consider the semi-infinite string with constant density ρ, zero boundary condition at the origin, and a point mass attached at the point a. We are looking for a continuous solution $u(t,x)$ of the following equation,

$$\rho u_{tt} = Tu_{xx},\ x \neq a,$$
$$2T\left[u_x(a+0) - u_x(a-0)\right] = Mu_{tt}(a),\ u(0) = 0, \tag{48}$$

which is twice differentiable with respect t,x for $x \neq a$. The eigen-modes of the equation - the solutions represented as products of functions depending on t and x respectively- have a form

[1]Some problems below are supplied my starlets. These problems may require deeper insight.

of exponentials $e^{i\omega t}\Psi_p(x)$, where

$$\rho\omega^2 = T\ p^2, \text{ or } \omega = \pm c\, p.$$

The differential equation for the amplitudes Ψ_p is

$$-\frac{d^2\Psi_p}{dx^2} = p^2\ \Psi_p, \tag{49}$$

and the boundary conditions are

$$\Psi(0) = 0,\ 2\left[\Psi_x\right]\Big|_{x=a} = \frac{M}{\rho}c^2\ p^2\Psi(a) := 2\ \kappa^2\ p^2\ \Psi(a), \tag{50}$$

with $2\kappa^2 = c^2 M/\rho$. In fact (49, 50) is a Sturm-Liouvilles problem for the corresponding differential operator l_a on the semi-infinite interval with the point mass M attached at $x = a$. It has the purely continuous spectrum $\sigma = [0, \infty)$, and the system of eigenfunctions $\left\{\Psi_p^a\right\}$ is complete and orthogonal in $L_2(0, \infty)$. In our case these eigenfunctions can be chosen as "scattered waves", see the Problem 11 below.

Problem 2

The eigenfunctions of the problem (49, 50) can be constructed in form of "scattered waves"

$$\Psi_p^a(x) = \begin{cases} \alpha \sin px, & 0 < x < a, \\ e^{ip(x-a)} - e^{-ip(x-a)}\ S_a(p), & a < x < \infty \end{cases} \tag{51}$$

where

$$S_a(p) = \frac{ip - p\cot pa - \kappa}{ip + p\cot pa + \kappa},\ \alpha = \frac{1 - S_a(p)}{\sin pa}. \tag{52}$$

Then the spectral representation of the operator l_a is defined by the following pair of formulae similar to the standard for the Fourier transform

$$\tilde{u}(p) = \frac{1}{\sqrt{2\pi}}\int_0^\infty u(x)\bar{\Psi}_p^a(x)dx,$$

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \tilde{u}(p)\Psi_p^a(p)dp. \tag{53}$$

Note that for zero-mass we obtain $S_a = e^{-2\,i\,p\,a}, \alpha = -i$, and Ψ_p coincides with the corresponding eigenfunction of the operator l.

Problem 3 Explore the limit behavior of the eigenfunctions Ψ_p when the mass becomes large, $\kappa \to \infty$. Is the result compatible with your expectations based on "common sense" ?

Now we explore the dynamics of the string with a mass attached assuming that the incident wave has a form:

$$u(x+ct) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ip(x-a+ct)}\ \tilde{h}_{-}(p)\, dp = h(x-a+ct), \tag{54}$$

with a density $\tilde{h}_{-}(p)$ analytic in the lower half-plane $\Im p < 0$, and rapidly decreasing at infinity. Then $u(x+ct) = 0$ if $x - a + ct)0$. One can prove by standard methods of operator theory, that the spectrum of the operator

Problem 12** Using the spectral representation defined by the formulae (53), construct the solution of the problem (48) defined by the above incident wave (54) and calculate the asymptotic of it when $t \to \infty$.

The straightforward calculation of the dynamics defined by the equations (48) suggested in Problem 12 is an elementary (but not a simple) problem. We will do this calculation based on reduction of the wave equation to the Lax-Phillips form, see [53].

Introduce Cauchy data of the wave equation (48) as

$$U(t,x) = \begin{pmatrix} u(t,x) \\ \frac{1}{c} u_t(t,x) \end{pmatrix} := \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}.$$

Then the equations (48) take the Schrödinger form suggested by Peter Lax and Ralph Phillips:

$$\frac{1}{ic}\frac{dU}{dt} = i \begin{pmatrix} 0 & -1 \\ -\frac{d^2}{dx^2} & 0 \end{pmatrix} U := \mathcal{L}_a U \tag{55}$$

. The initial condition defined by the incoming incident wave (54) takes the form:

$$U(0,x) = \begin{pmatrix} h(x-a) \\ h_x(x-a) \end{pmatrix},\ x > a,$$

because for incoming waves depending on $(x+ct)$ the differential operations $\frac{\partial}{\partial x}$ and $c^{-1}\frac{\partial}{\partial t}$ are equivalent. It appeared that the matrix differential operator $\mathcal{L}_a$ defined by (55) is self-adjoint in the space of Cauchy data U supplied with the "energy metric"

$$\mathcal{E}_a(U) = \frac{1}{2}\left[\int_0^\infty \left\{c^{-2}|u_t|^2 + |u_x|^2\right\} dx + M\ |u_t(a)|^2\right]. \tag{56}$$

Problem 4 Define the dot-product which corresponds to the energy norm.

The enegry-normed Hilbert space $\mathcal{E}$ of Cauchy data plays the central role in further study of scattering on the string.

Problem 5 Prove that the transformation of smooth compactly supported Cauchy data defined by the wave equation is unitary with respect to the energy norm (56).

Problem 6 The Cauchy data of the outgoing waves $u(x-ct)$ have the form:

$$U(0,x)=\left(\begin{array}{c} h(x-a) \\ -h_{x}(x-a) \end{array}\right),\ x>a.$$

Prove that Cauchy data of outgoing waves are orthogonal in Energy metric to Cauchy data of incoming waves.

Introduce the closure $\mathcal{D}_{in}$ in energy norm of all Cauchy data of incoming waves supported by the interval $(0,\infty)$, and the closure $\mathcal{D}_{out}$ in energy norm of all Cauchy data of the outgoing waves supported by the interval $(0,\infty)$. We call them incoming and outgoing subspaces of the unitary evolution group $\mathcal{U}(t)$ of Cauchy data defined by the wave equation. It follows from the previous problem that the incoming and the outgoing subspaces are orthogonal in energy norm. Introduce the orthogonal complement $\mathcal{K}$ of $\mathcal{D}_{in}\oplus\mathcal{D}_{out}$ in the energy-normed space $\mathcal{E}$ of Cauchy data:

$$\mathcal{K}:=\mathcal{E}\ominus\left[\mathcal{D}_{in}\oplus\mathcal{D}_{out}\right].$$

The subspace $\mathcal{K}$ is called the *co-invariant subspace* of the evolution group $\mathcal{U}(t)$.

Problem 7 First components of elements from K are continuous and constant on (a,∞). The second components of elements from K vanish on (a,∞).

The following statement serves a bridge between scattering theory and theory of non-self-adjoint (dissipative) operators;

Theorem (P.Lax - R. Phillips) Denote by $\mathcal{P}_{\mathcal{K}}$ the orthogonal projection onto $\mathcal{K}$. Consider the family of bounded operators in $\mathcal{K}$ defined as

$$\mathcal{P}_{\mathcal{K}}\mathcal{U}(t)\ \Big|_{\mathcal{K}}:=\mathcal{Z}(t),\ t\geq 0.$$

The family $\mathcal{Z}(t)$ is a strongly continuous semigroup. It has a dissipative generator

$$\mathcal{B}=\text{s-lim}_{t\to 0}\frac{\mathcal{Z}(t)-I}{it},$$

$(2i)^{-1}\left[\mathcal{B}-\mathcal{B}^{+}\right]\geq 0.$

Proof We just make sure that the family $\mathcal{Z}(t)$ is a semigroup. Really, consider the sum of orthogonal projections onto $\mathcal{D}_{in},\mathcal{K}\,\mathcal{D}_{out},\ \left[\mathcal{P}_{\rangle\backslash}+\mathcal{P}_{\mathcal{K}}+\mathcal{P}_{out}\right]=I$ and insert it between the unitary operators $\mathcal{U}(t_1)\mathcal{U}(t_2)$ in the formula for $\mathcal{Z}\left(t_1+t_2\right)$:

$$\mathcal{P}_{\mathcal{K}}\mathcal{U}(t_1)\ \left[\mathcal{P}_{\rangle\backslash}+\mathcal{P}_{\mathcal{K}}+\mathcal{P}_{out}\right]\mathcal{U}(t_2)\ \Big|_{\mathcal{K}}.$$

Due to the invariance of $\mathcal{D}_{out}$ with respect $\mathcal{U}(t_1)$ we have for $t_1>0$:

$$\mathcal{P}_{\mathcal{K}}\mathcal{U}(t_1)\,\mathcal{P}_{out}=0.$$

Similarly, due to invariance of $\mathcal{D}_{in}$ with respect to $\mathcal{U}^{+}(t_2)$ we have for $t_2 > 0$:

$$\mathcal{P}_{\mathcal{K}}\mathcal{U}^{+}(t_2)\,\mathcal{P}_{out} = 0,$$

hence $\mathcal{P}_{\backslash\backslash}\mathcal{U}(t_2)\,\Big|_{\mathcal{K}} = 0$. *End of the proof.*

The semigroup $\mathcal{Z}(t) = e^{i\mathcal{B}t},\ t \geq 0$ is called Lax-Phillips semigroup associated with the scattering problem. We will re-write the semigroup in spectral representation of the unitary evolution group.

Problem 8 The spectral representation of the unitary group $\mathcal{U}(t)$ associated with scattered waves

$$\vec{\Psi}(x,p) = \begin{pmatrix} (ip)^{-1}\,\Psi(x,p) \\ \Psi(x,p) \end{pmatrix},\ \mathcal{L}\vec{\Psi} = p\vec{\Psi},$$

is given by the formula

$$\mathcal{J}_a : U\ \to \tilde{U} = \langle U,\,\vec{\Psi}\rangle_{\mathcal{E}} = \frac{1}{2}\int_0^{\infty}\left[\frac{\partial u_0}{\partial x}(-ip)\bar{\Psi}_{x} + u_1\bar{\Psi}\right]dx.$$

In particular, the spectral image of the incoming data $(u_0,\,\frac{\partial u_0}{\partial x})$ is calculated as:

$$\frac{1}{2}\int_a^{\infty}\frac{\partial u_0}{\partial x}\,e^{-ipx}\,dx,$$

and the spectral image of outgoing elements $(u_0,\,-\frac{\partial u_0}{\partial x})$ is calculated as

$$\frac{1}{2}\,\bar{S}_a(p)\int_a^{\infty}\frac{\partial u_0}{\partial x}\,e^{ipx}\,dx.$$

The incoming and outgoing subspaces are closed. Hence, due to Paley-Wiener theorem, the spectral image of $\mathcal{D}_{in}$ coincides with the Hardy class H^2_- of all square integrable functions on the real axis p which can be analytically continued into the lower half-plane $\Im p < 0$. Similarly, the spectral image of $\mathcal{D}_{out}$ coincides with $\bar{S}_a H^2_+$, where H^2_+ is the Hardy class of all square integrable functions on the real axis p which can be analytically continued into the upper half-plane $\Im p > 0$. Hence the spectral image of the co-invariant subspace $\mathcal{K}$ coincides with $H^2_+ \ominus \bar{S}_a H^2_+ := K_S$, and the Lax-Phillips semigroup is presented as

$$P_K\,e^{ipt}\Big|_{K_S} := Z(t),\ t \geq 0.$$

Note that $\bar{S}_a(p)$ is a bounded analytic function in the upper half-plane and the zeros of $\bar{S}_a(p)$ are situated in the upper half-plane.

Problem 9 Calculate the asymptotic of zeros p_n of $\bar{S}_a(p)$:

$$p_n \approx n\pi + (n\pi - 1) + in\pi(n\pi - 1)^{-1},\ n \to \infty.$$

Problem 10 Make sure that the eigenvalues of the Lax-Phillips semigroup coincide with $e^{ip_n t}$, $t > 0$, and the corresponding eigenfunctions in spectral representation are equal to

$$u_n(p) = \frac{\bar{S}_a(p)}{p_n - p}.$$

Problem 11 Make sure that the eigenvalues of the adjoint Lax-Phillips semigroup coincide with $e^{-i\bar{p}_n t}$, $t > 0$, and the corresponding eigenfunctions in spectral representation are equal to

$$v_n(p) = \frac{1}{\bar{p}_n - p}.$$

Problem 12 Make sure that the the system $\{u_n, v_n\}$ is bi-orthogonal:

$$\langle u_l, v_m \rangle_{L_2} = 2\pi i \frac{d\bar{S}_a}{dp}(p_l)\, \delta_{lm}.$$

We are able to answer also important questions on completeness of the eigenfunctions $\{u_n, v_n\}$ in K and the corresponding spectral expansion. But previously we should observe few basic facts on the spectral theory of analytic functions.

5 Selfadjoint dilation and a Functional model of a dissipative operator.

It is well known that the standard model of a selfadjoint operator in form of the multiplication operator in the space $L_2(\sigma)$ with an appropriate spectral measure σ. All attempts to construct a model of non-self-adjoint operators in similar form failed, because of non limited growth of the resolvent of the nonselfadjoint operator at the spectrum. In 1948 Arno Beurling suggested a remarkable description of invariant subspaces of the shift operator, based on inner functions, see [51]. That paper by A. Beurling provided a generic example which can be fitted to most important class of completely non-unitary contractions or completely-nonselfadjoint dissipative operators. Modern theory of dissipative operators and contractions is reduced to the construction of the functional model, which is based again on the operator extension technique. In 60-ties a series of papers by B. Szökefalvy-Nagy and C. Foias appeared, see the book [52]. In these papers the spectral theory of contractions and dissipative operators was considered as a chapter of the theory of commutative unitary groups, intertwined with the theory of analytic functions- the spectral theory of function. The role of the spectral representation in this approach was played by some union of facts of the spectral theory of analytic function united under the name of functional model. The functional model of the non-selfadjoint operator is prametrized by the characteristic function, introduced by Mihail Samoilovich Livshits [55].

Simultaneously and independently from [52] a new version of the Scattering Theory appeared see [53], which provided a profound connection between the functional model and the resonance

scattering. Soon the whole spectral theory of the dissipative operators was re-written from the point of view of the spectral theory of functions and functional model see [49, 59, 60, 56]. The field of applications of this elegant theory the resonance scattering in acoustics was complemented soon by the resonance scattering on the fundamental domain of the modular group SL_2R on the Lobachevsky plane in an interplay with the critical zeros of ζ-function, see [58, 62]. The union of the resonance scattering theory and the functional model form now a solid base for the spectral theory of non-selfadjoint operators. In this section we supply a review of basic achievements in this area since the middle of previous century.

The theory of unitary groups with incoming and outgoing subspaces can be applied to much broader area of spectral analysis. In particular it may serve a basement for construction of convenient functional models of dissipative operators with continuous spectrum. First steps in this direction were dine by C. Foias and B.S-Nagy, see [52]. It appeared that the model suggested in [52] may be essentially simplified when selecting a natural class of eigenfunction of the self-adjoint dilation- the scattered waves, see [54]. Find below a compressed version of the text.

Consider a dissipative operator in a Hilbert space K

$$L = A + \frac{i}{2}\Gamma^+\Gamma$$

with a real part $A = A^+$ and a finite-dimensional [2] positive imaginary part. Assuming that $\Gamma K = E$ contains a generating subspace of the operator A, consider the extended space $\mathcal{E} = L_2(R_-, E) \oplus K \oplus L_2(R_+, E)$ of vector-functions

$$\vec{u} = \begin{pmatrix} u_- \\ u \\ u_+, \end{pmatrix}$$

and the operator $\mathcal{L}$ in $\mathcal{E}$ defined on vector functions $\vec{u}, u \in D(A), u_\pm \in W_2^1(R_\pm, E)$ submitted to the condition $u_-(0) - u_+(0) = i\Gamma u$:

$$\mathcal{L}\vec{u} = \begin{pmatrix} i\frac{du_-}{dx} \\ Au + \frac{\Gamma^+}{2}[u_-(0) + u_+(0)] \\ i\frac{du_+}{dx} \end{pmatrix}.$$

Without loss of generality we can assume that the operator Γ is selfadjoint $\Gamma = \Gamma^+$ and positive, but still we will use both Γ, Γ^+ in futher formulae, if not specified otherwise. The elements from the domain of $\mathcal{L}$, which vanish on either of semi-axes $R_\pm$, satisfy the corresponding homogeneous boundary conditions $u_-(0) = i\Gamma u$, or $-u_+(0) = i\Gamma u$. These conditions are fulfilled for the absorbing and radiating eigenvectors of the dilation , see (60) below.

Theorem 5.1 *The operator $\mathcal{L}$ is a self-adjoint operator in $\mathcal{E}$. Moreover, the compression of the resolvent $\mathcal{R}_\lambda$, $\Im\lambda < 0$ of the operator $\mathcal{L}$ onto the subspace K coincides with the resolvent of the operator L:*

$$P_{_K}\left[\mathcal{L} - \lambda I\right]^{-1} P_{_K} = \left(L - \lambda I\right)^{^{-1}}, \ \Im\lambda < 0. \tag{57}$$

[2] this condition can be essentially relaxed, see the remark after the theorem 3.2

Proof of the corresponding statement for the dissipative Schrödinger operator with complex potential is given in [64]. Proof of the announced abstract result follows the same pattern: to obtain the first statement we have to verify the symmetry of $\mathcal{L}$ and the symmetry of the adjoint operator; to prove the second statement we may use a simple algebra and the basic fact of existence of limits of R-function on the real axis from the upper half-plane. Then using Riesz-integral of the resolvent one may derive from (57) that for any bounded analytic function $\Phi(\lambda)$ in upper half-plane $\Im\lambda > -2\epsilon$

$$P_{_K}\Phi(\mathcal{L})P_{_K} =$$

$$-\frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} P_{_K}\Phi(\lambda)\left[\mathcal{L}-\lambda I\right]^{-1}P_{_K}d\lambda =$$

$$-\frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\Phi(\lambda)[L-\lambda I]^{-1}d\lambda = \Phi(L)$$

which means, in particular, that for $t > 0$: $e^{iLt} = P_{_K}e^{i\mathcal{L}t}P_{_K}$.

□

The unitary group $e^{i\mathcal{L}t}$ is a *unitary dilation* [52] of the contracting semigroup e^{iLt} and the operator $\mathcal{L}$ is the *self-adjoint dilation* of the dissipative operator L. The constructed dilation is *minimal* - i.e. it does not have proper self-adjoint parts - if the subspace E is a generating subspace of the operator A.

We can construct the symmetric spectral representation for the original dissipative operator following the pattern of the previous section. We begin with description of eigenfunctions of the dilation.

The space $\mathcal{E}$ of the dilation $\mathcal{L}$ may be decomposed into orthogonal sum of invariant subspaces generated by incoming and outgoing waves and corresponding complementary ("radiating" and "absorbing") components $\mathcal{E} = \mathcal{E}_- \oplus \mathcal{E}^< = \mathcal{E}_+ \oplus \mathcal{E}^>$. The spectrum of $\mathcal{L}_\pm$ in each of components $\mathcal{E}_\pm$ is absolutely continuous with the constant multiplicity $\dim\Gamma$ on the whole real axis R. The spectrum of the "absorbing" and "radiating" components $\mathcal{L}^>$, $\mathcal{L}^<$ in subspaces $\mathcal{E}^>$, $\mathcal{E}^<$ consists of intervals of real axis where nonzero generalized solutions of the homogeneous equation $\mathcal{L}\psi - \lambda\psi = 0$ exist which vanish on $L_2(R_+)$ (for $\psi^>$) or vanish on $L_2(R_-)$ (for $\psi^<$). The corresponding eigen-functions of the dilation in each component $\mathcal{E}_\pm$ can be found, according to philosophy developed in [68, 67, 66] as elements of some rigged space constructed with a help of some Hilbert-Schmidt operator $\mathcal{T}$ which has a dense range i.e. with all non-zero eigenvalues. This general statement can be specified in our case by selection of a special class of eigenfunctions which play a role of Scattered waves. This result can be obtained via selection of a special rigging (i.e. the operator $\mathcal{T}$) correlated with the imaginary part of the considered dissipative operator. Without loss of generality we can assume that the operator $\Gamma = \Gamma^+$ is a part of the positive Hilbert-Schmidt operator $\mathcal{T}$ acting in K, $\langle Ku, u\rangle > 0$. Moreover we can assume that the operator $\mathcal{T}^2 > 0$ is presented as a product of an operator of the trace class and an operator from Matsaev class, so that it's eigenvalues are $s_n(\mathcal{T}^2) = O(\alpha_n\,\beta_n)$ with $\sum_n|\alpha_n| < \infty$, $\sum_n \beta_n/n < \infty$ and β_n tend to zero monotonically. Consider the Gelfand triple [68] associated with the operator $\mathcal{T}$ as $\mathcal{T}K = \mathcal{K}_1 \subset K \subset \mathcal{K}^1 = \mathcal{T}^{-1}K$. Then the following statement is true:

Theorem 5.2 *The incoming and outgoing eigen-functions of the ditation $\mathcal{L}$ can be presented as generalized solutions of the corresponding homogeneous equation with exponential behavior in $L_2(R_\pm, E)$:*

$$\psi_-(e) = \begin{cases} e^{-ikx}e, & e \in E, \quad x \in R_-, \\ u_-(e) & in \qquad \mathcal{K}^1, \\ e^{-ikx}\mathbf{S}^+e, & e \in E, \quad x \in R_+, \end{cases}$$

$$\psi_+(e) = \begin{cases} e^{-ikx}\mathbf{S}e, & e \in E, \quad x \in R_-, \\ u_+(e) & in \qquad \mathcal{K}^1, \\ e^{-ikx}e, & e \in E, \quad x \in R_+. \end{cases} \tag{58}$$

These eigenfunctions are labeled by the "direction vectors" [3] *$e \in E$. The mid components $u_\mp$ are generalized solutions of the non-homogeneous equation in complex plane and are uniquely defined by the direction vectors $e \in E$, see (60) below, as $\mathcal{T}^{-1}$ images of strong limits of properly framed resolvent of the self-adjoint operator A or the resolvent of L, L^+ on the real axis from the lower (upper) half-planes. The transmission coefficients $\mathbf{S}$, $\mathbf{S}^+$ are also uniquely defined from the homogeneous equation. In particular, $\mathbf{S}$, $\mathbf{S}^+$ are analytic matrix-function in upper and lower half-planes $\Im k > 0$, $\Im k < 0$*

$$\mathbf{S}^+(k - i0) = I - i \lim_{\lambda \to k - i0} \Gamma \frac{I}{L - \lambda I} \Gamma^+ = \lim_{\lambda \to k - i0} \frac{I - \frac{i}{2}\Gamma \frac{I}{A - \lambda I}\Gamma^+}{I + \frac{i}{2}\Gamma \frac{I}{A - \lambda I}\Gamma^+},$$

$$\mathbf{S}(k + i0) = I + i \lim_{\lambda \to k + i0} \Gamma \frac{I}{L^+ - \lambda I} \Gamma^+. \tag{59}$$

$$u_-(e) = -\frac{1}{2}\frac{1}{A - (k - i0)}\left(I + \mathbf{S}^+(k - i0)\right)e = -\frac{I}{L - (k - i0)}\Gamma^+ e, \; e \in E.$$

$$u_+(e) = -\frac{1}{2}\frac{I}{A - (k + i0)}\left(I + \mathbf{S}(k + i0)\right)e = -\frac{I}{L^+ - (k + i0)}\Gamma^+ e, \; e \in E. \tag{60}$$

The eigenfunctions $\psi^>$, $\psi^<$ of components of the dilation in complementary subspaces $\mathcal{E} \ominus \mathcal{E}_- = \mathcal{E}^<$ and $\mathcal{E} \ominus \mathcal{E}_+ = \mathcal{E}^>$ have a form:

$$\psi^< = \begin{pmatrix} 0 \\ u^< \\ e^{-ikx}e^< \end{pmatrix}, \; \psi^> = \begin{pmatrix} e^{-ikx}e^> \\ u^> \\ 0 \end{pmatrix}, \tag{61}$$

when choosing vectors $e^>$, $e^<$ as eigenvectors of operators $\Delta^> = I - \mathbf{S}^+\mathbf{S}$, $\Delta^< = I - \mathbf{S}\mathbf{S}^+$ with non-zero eigenvalues $\delta^>$, $\delta^<$ respectively, we obtain:

$$u^>(e^>) = \frac{1}{\delta^>}\left[u_-(e^>) - u_+(\mathbf{S}^+e^>)\right],$$

$$u^<(e^<) = \frac{1}{\delta^<}\left[u_+(e^<) - u_-(\mathbf{S}e^<)\right].$$

[3] The term is borrowed from [9]

Proof. It is easy to verify the above formulae for the eigenfunctions on a formal level, see for instance [64]. Note that analysis of the absolutely- continuous spectrum of the Symmetric model in the rigged space is presented in [59, 60, 56]. We suggest below only the sketch of the proof of existence of the scattered wave of the dilation (actually - the proof of existence of their mid- components) based on the Theorem 7 from [61]. It is proved in that theorem, in particular, that the non-tangential limits exist on the real axis for the operator-valued R-function presented by the properly framed resolvent of a self-adjoint operator. The remark attached to the theorem shows that the statement remains true for the limit of the R-function

$$\mathcal{T}\ P_{_K} \frac{I}{\mathcal{L} - \lambda I} P_{_K}\, \mathcal{T} =$$

$$\mathcal{T}\ \frac{I}{L^+ - \lambda I}\, \mathcal{T}$$

from the upper half-plane $\lambda \to k + i0$ and for the adjoint function $\mathcal{T}\ \frac{I}{L-\lambda I}\, \mathcal{T}$ from the lower half-plane, $\lambda \to k - i0$. Now it is easy to verify the existence of the mid-component u_+ of the outgoing scattered wave. Really, the limit

$$\lim_{\lambda \to k - i0}\ \mathcal{T} \frac{I}{L^+ - \lambda I}\, \mathcal{T}$$

exists in the trace class, hence the mid-component can be presented as

$$u_+ = \mathcal{T}^{^{-1}} \lim_{\lambda \to k - i0}\ \mathcal{T} \frac{I}{L^+ - \lambda I}\, \mathcal{T} e \in \mathcal{K}^{^1}.$$

Similarly the mid-component of the scattered wave ψ_- can be obtained.The mid-components $u^<$, $u^>$ of the eigenfunctions in the complementary subspaces can be obtained as linear combinations of them with proper coefficients and properly chosen direction vectors.

□

Remark 1 One can see that the above calculation can be applied to the situation when the imaginary part of the dissipative operator is a positive operator presented as a part in E of the positive operator $\mathcal{T}$ such that the square $\mathcal{T}^{^2}$ of it is product of an operator with a finite trace and an operator from Matsaev class. This is actually the natural class of dissipative operators for which the symmetric functional model may be obtained by the procedure described above. This class can be extended via considering the corresponding relative classes with the imaginary part subordinated to the real part. It will be done elsewhere.

Based on the explicit formulae for the eigenfunctions one can prove that the characteristic function obtained above as a stationary transmission coefficient can be also interpreted in non-stationary terms.

Considering the non-perturbed shift generator in the space $D_{in} \oplus D_{out} = L_2(R_-, E) \oplus L_2(R_+, E)$:

$$\mathcal{L}_0 = i \frac{d}{dx}.$$

Then the characteristic function of the original dissipative operator as Lax-Phillips-Adamjan-Arov scattering matrix, see [63] for the pair $\mathcal{L}$, $\mathcal{L}_0$ is

$$s - \lim_{t\to\infty} \mathcal{J}_\prime e^{-i\mathcal{L}_0 t} P_+ e^{2i\mathcal{L}t} P_- e^{-i\mathcal{L}_0 t} \mathcal{J}_0^+.$$

Theorem 5.3 *The Lax-Phillips-Adamjan-Arov scattering matrix for the pair $\mathcal{L}$, $\mathcal{L}_0$ coincides with the transmission coefficient* [4] *:*

$$\mathbf{S}^+(k-i0) = I - i \lim_{\lambda\to k-i0} \Gamma \frac{I}{L-\lambda I}\Gamma^+ =$$

$$= \lim_{\lambda\to k-i0} \frac{I - \frac{i}{2}\Gamma\frac{I}{A-\lambda I}\Gamma^+}{I + \frac{i}{2}\Gamma\frac{I}{A-\lambda I}\Gamma^+}, \tag{62}$$

Proof is obtained by the straightforward calculation using the fact that the spectral representation $\mathcal{J}_0$ for the non-perturbed operator is defined by Fourier transform. Hence the scattering matrix coincides with the transmission coefficient $\mathbf{S}$ in front of the exponential e^{-ikx} in the formula for the scattered wave ψ_- in the outgoing subspace.

□

We construct now the symmetric functional model for the original operator L based on eigenfunctions $\psi_\pm$ of it's self-adjoint dilation , see (58).

Theorem 5.4 *Consider the maps $\mathcal{J}_\pm$ of the spaces $L_2(E)$ into $\mathcal{E}_\pm$:*

$$\mathcal{J}_+ h_+ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_+ \left(h_+(p)\right) dp,\ h_+ \in L_2(E),$$

$$\mathcal{J}_- h_- = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_- \left(h_-(p)\right) dp,\ h_- \in L_2(E),$$

and the map $\mathcal{J}$ of the column $\begin{pmatrix} h_+ \\ h_- \end{pmatrix} := \mathbf{h}$ *into $\mathcal{E}$:*

$$\mathcal{J}\mathbf{h} = \mathcal{J}_+ h_+ + \mathcal{J}_- h_-.$$

Then the following Parseval identity is true:

$$\langle \mathcal{J}\mathbf{f},\, \mathcal{J}\mathbf{g}\rangle = \langle f_+,\, g_+\rangle + \langle \mathbf{S}^+ f_-,\, g_+\rangle + \langle \mathbf{S} f_+,\, g_-\rangle + \langle f_-,\, g_-\rangle =$$

$$\int \langle \begin{pmatrix} \mathbf{I} & \mathbf{S}^+ \\ \mathbf{S} & \mathbf{I} \end{pmatrix} \mathbf{f},\ \mathbf{g}\rangle_{E\oplus E} dk \tag{63}$$

[4] The numerator and denominator of the the announced representation for the Scattering matrix are commuting, so the order of them is not important

Proof. Note that for $h_- \in H^2_-(E)$

$$\mathcal{J}\begin{pmatrix} h_- \\ 0 \end{pmatrix} = \begin{pmatrix} h_-(x) \\ 0 \\ 0 \end{pmatrix}$$

with $h_-(x) = \int_{-\infty}^{\infty} e^{-ikx} h_-(k)dk$ non equal to zero identically if $x \in R_-$. Similarly for $h_+ \in H^2_+(E)$

$$\mathcal{J}\begin{pmatrix} 0 \\ h_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h_+(x) \end{pmatrix}$$

with $h_+(x) = \int_{-\infty}^{\infty} e^{-ikx} h_+(k)dk \neq 0$ if $x \in R_+$. Hence incoming and outgoing subspaces$\mathcal{D}_{in,out} = L_2(R_\pm)$ are mutually orthogonal. The invariant subspaces $\mathcal{E}_\pm \in \mathcal{E}$ of the dilation developed from the incoming and outgoing subspaces,

$$\mathcal{E}_\pm = \bigvee_{t=-\infty}^{t=\infty} \int_{-\infty}^{\infty} e^{ikt} \psi_\pm (h_\pm)\, dk,\ h_\pm \in H^2_\pm,$$

are represented as

$$\mathcal{E}_+ = \mathcal{J}\begin{pmatrix} 0 \\ L_2 \end{pmatrix},\ \mathcal{E}_- = \mathcal{J}\begin{pmatrix} L_2 \\ 0 \end{pmatrix}.$$

Then for $f_+ \in H^2_+$, $g_- \in H^2_-$ we obtain :

$$\langle \mathcal{J}\begin{pmatrix} 0 \\ f_+ \end{pmatrix}, \mathcal{J}\begin{pmatrix} e^{ikt} g_- \\ 0 \end{pmatrix} \rangle =$$

$$\int_{-\infty}^{0} \langle \mathbf{S} f_+, e^{ikt} g_- \rangle_{L_2} dx$$

for any finite t.Following the pattern of the previous section one may derive from it that for any $f_+, g_- \in L_2$

$$\langle \mathcal{J}\begin{pmatrix} 0 \\ f_+ \end{pmatrix}, \mathcal{J}\begin{pmatrix} g_- \\ 0 \end{pmatrix} \rangle =$$

$$\frac{1}{2\pi} \int_{-\infty}^{0} dx \int dk \int d\hat{k} e^{-ikx} e^{i\hat{k}x} \mathbf{S}(k) f_+ \bar{g}_- +$$

$$\frac{1}{2\pi} \int_{0}^{\infty} dx \int dk \int d\hat{k} e^{-ikx} e^{i\hat{k}x} f_+ \overline{\mathbf{S}^+(\hat{k}) g_-} +$$

$$\langle u_+(f_+), u_-(g_-) \rangle_K = \langle \mathbf{S} f_+,\, g_- \rangle_{L_2},$$

since $\langle u_+(f_+), u_-(g_-) \rangle_K = 0$, and remaining integrals over semi-axes shuld be combined to the delta-function $\delta(k - \hat{k})$. This way the announced statement is verified for special elements

$$\begin{pmatrix} 0 \\ f_+ \end{pmatrix},\ \begin{pmatrix} g_- \\ 0 \end{pmatrix}.$$

The proof is accomplished based on similar arguments for various choice of special elements and linearity of the map $\mathcal{J}$.

□

Consider the non-perturbed operator $\mathcal{E}_0 = \mathcal{L}_0 \oplus \mathcal{L}_0$ in $L_2(R,E) \oplus L_2(R,E)$. The corresponding evolution group $U_t : u(x) \to u(x-t)$ has unilateral invariant subspaces $L_2(R_-,E) := \mathcal{D}_{in}$, $L_2(R_+,E) := \mathcal{D}_{out}$. We denote by $P_{in,out}$ the orthogonal projections onto $\mathcal{D}_{in,out}$ respectively. Similarly to the above reasoning in section 2 we calculate the symmetric spectral representation via Lax-Phillips wave operators:

Theorem 5.5 *The wave-operators*

$$W_- = s - \lim_{t \to -\infty} e^{-i\mathcal{L}_0 t} P_{in} e^{i\mathcal{L}t}$$

$$W_+ = s - \lim_{t \to \infty} e^{-i\mathcal{L}_0 t} P_{out} e^{i\mathcal{L}t}$$

exist as strong limits and are isometric operators from the invariant subspaces $\mathcal{E}_{in,\,out} \subset \mathcal{E}$ obtained by development of the incoming and outgoing subspaces $L_2(R_-,E)$ and $L_2(R_+,E)$ with evolution generated by $\mathcal{L}$. The column

$$\begin{pmatrix} f_{in} \\ f_{out} \end{pmatrix} := \mathbf{f}$$

defines the symmetric spectral map as

$$\mathcal{J}\mathbf{f} = \mathcal{J}_- f_{in} + \mathcal{J}_+ f_{out}$$

which is calculated from the column of Cauchy data as $\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \mathbf{f}(0)$ *as*

$$\begin{pmatrix} f_{in} \\ f_{out} \end{pmatrix} = \begin{pmatrix} \mathcal{F}W_-\mathbf{f}(0) \\ \mathcal{F}W_+\mathbf{f}(0) \end{pmatrix}$$

where $\mathcal{F}$ is the standard Fourier transform in L_2:

$$f(x) \to \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(x) dx = f(k).$$

Note that we suggested a unique recipe of construction of coordinates of the symmetric spectral representation of the dilation, but , once constructed, the column of coordinates may be a subject to change within proper limits caused by possible presence of intervals where the Scattering matrix is unitary, see the discussion in [64].

Note that the eigenfunctions of the complementary component are found uniquely, up to the parametrization with the direction vectors. Their mid-components $u^<$, $u^>$ $\mathcal{E}^<$, $\mathcal{E}^>$ may serve

a *canonic system of eigenfunctions* of the absolutely continuous spectrum of the original dissipative operator and adjoint operator, respectively. The corresponding spectral expansion

$$u = \frac{1}{2\pi}\int_{\sigma_a} \frac{|\mathbf{S}(k)|^2 - 1}{\mathbf{S}^+(k)} u^<(k)\langle u,\, u^>(k)\rangle dk, \tag{64}$$

is converging for elements u represented as orthogonal projections of elements of the complementary subspace $\mathcal{E}^<$ onto K. This set is dense in the absolutely-continuous subspace of the operator L, see [52] and the detailed discussion of the eigen-function expansion of the dissipative Schrödinger Operator with complex potential in [64, 60]. Thus the incoming-outgoing eigenfunctions of the dilation and eigenfunctions in the complementary subspaces $\mathcal{E}^<$, $\mathcal{E}^>$ play essentially different roles in spectral problem for the dissipative operator. The above formula (64) shows that the problem of proper choice of the *canonic system* of eigenfunction of the absolutely-continuous spectrum for dissipative operators is naturally resolved. Note that similar question about a *canonic system* of eigenfunctions of absolutely continuous spectrum of a self-adjoint operator remains obscure. The only bridge between the General Spectral Theorem for self-adjoint operators and the *expansion theorem* is formed by classical results of I.Gelfand-A.Kostyuchenko [66] on differentiation of the spectral measure of a self-adjoint operator in properly rigged spaces. We hope to discuss the important question of the construction of the canonic system of eigenvectors of the abstract self-adjoint operator somewhere. For discussion of choice of the canonic system of eigenfunctions of the absolutely-continuous spectrum in case of spectral multiplicity one for a unitary operator and a canonic system of eigenfunction of it's contracting perturbation see [57].

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7 Appendix 1: Blaschke-product.

If the scattering matrix $S_\beta(p)$ is unitary on the real axis, then the right Blaschke factors are constructed at each zero of $S_\beta(p)$:

$$S_\beta(p) = B^s(p)\ \mathbf{B}_s(p),\ \mathbf{B}_s(p) = \frac{p - p_s(\varepsilon)}{p - \bar{p}_s(\varepsilon)}\mathbf{P}_s + [I - \mathbf{P}_s]\,. \tag{65}$$

The right Blaschke factors $\mathbf{B}_s(p)$ do not coincide with the corresponding Blaschke factors $B_s(p)$ in the above product (18), due to the non-commutativity of the factors. We suggest here the procedure of construction of the factors $B_s(p)$ once the factors $\mathbf{B}_s(p)$ are given.

Assume that the factors B_1, B_2, ... are ordered from the right to the left such that B_1 is the first factor from the right, B_2 is the second factor from the right and so on. Denoting by $\mathbf{N}_s$, N_s

the ranges of $\mathbf{P}_s$, P_s respectively and by ν_s, ν_s any vectors from $\mathbf{N}_s$, N_s, we can write down the following chain of equations

$$B_1(p_1) = \mathbf{B}_1(p_1),\ N_1 = \mathbf{N}_1,\ P_1 = \mathbf{P}_1,$$

$$B_2(p_2)B_1(p_2)\nu_2 = 0, \text{ or } N_2 = B_1(p_2)\mathbf{N}_2,$$

$$\dots\ \dots\ \dots\ \dots\ \dots$$

$$B_l(p_l)\dots B_2(p_l)B_1(p_l)\nu_l = 0, \text{ or } N_l = B_{l-1}(p_l)\dots B_1(p_l)\mathbf{N}_l, \tag{66}$$

We obtain the chain of one-dimensional subspaces if each product $B_{l-1}(p_l)\dots B_1(p_l)B_0(p_l)$ of the Blaschke factors does not degenerate on the corresponding subspace $\mathbf{N}_1$:

$$B_{l-1}(p_l)\dots B_1(p_l)\nu_l \neq 0. \tag{67}$$

Theorem 7.1 *The condition (67) of transformation of the rational form of the scattering matrix (17) into the Blashke-product (18) is fulfilled for small values of the perturbation parameter β.*

Proof For small values of the perturbation parameter the imaginary parts $\Im p_s$ of resonances are small, hence each term in the previous chain of equations can be re-written in the form :

$$N_2 = \left[I - i\frac{2\Im p_1}{p_2 - \bar{p}_1}P_1\right]\mathbf{N}_2$$

$$\dots\ \dots\ \dots\ \dots\ \dots$$

$$N_l = \left[I - i\frac{2\Im p_{l-1}}{p_l - \bar{p}_{l-1}}P_{l-1}\right]\left[I - i\frac{2\Im p_{l-2}}{p_l - \bar{p}_{l-2}}P_{l-2}\right]\mathbf{N}_l.$$

Then due to small $\Im p_1$ the operator $\left[I - i\frac{2\Im p_1}{p_2 - \bar{p}_1}P_1\right]$ is invertible and hence N_1 has the same dimension as $\mathbf{N}_1$. The projection P_1 exists. Then the above argument can be applied to the second equation, to find P_2, and so on until all projections P_l are defined.

The end of the proof.

8 Appendix 2: Rouche theorem

Remark 2 Note that the scattering matrix (9) is an analytic function of the small parameter β_{01} on the complement of the discrete spectrum of the operator A, but is not an analytic function of β_{01} on the hermitian neighborhood of the origin $\beta_{01} = 0$. The poles of the scattering matrix - the resonances - are found as vector-zeros (p_β, μ_β) of the denominator

$$m^\beta(p) = 2i - \beta_{01}P\frac{I + A^2}{A - pI}P\,\beta_{10} = 2i - \sum_{s=1}^{k}\frac{1 + \alpha_s^2}{\alpha_s - p}\,B_s^2 Q_s \tag{68}$$

where $q_s = e_s\rangle\langle e_s$ is an orthogonal spectral projection of A at the eigenvalue α_s, $B_s^2 = \| \beta_{01} P e_s \|^2$ and $Q_s = \beta_{01} P q_s P \beta_{10} B_s^{-2} = \nu_s\rangle\langle\nu_s$. Consider the leading term of the denominator in a neighborhood U_1 of the eigenvalue α_1 of A:

$$m_0^\beta(p) = 2i - \frac{1+\alpha_1^2}{\alpha_1 - p} B_1^2 \nu_1\rangle\langle e_1, \tag{69}$$

and compare the analytic functions m^β, m_0^β on the neighborhood U_1, based on Gohberg-Sigal matrix Rouche theorem, see [37]. Find below a "softened" version of this extremely useful result, which we quote here for finite matrices:

Theorem 8.1 *If two finite square matrices m, $m_0 : F \to F$, dim $F < \infty$, depend analytically on the parameter p in the disc D radius δ centered at the point p_0, and m_0 has only one zero p_0 at the center of the disc, with the multiplicity M_0, and both functions have no zeros on the circle $\Sigma_0 = \{\beta : |p - p_0| = \delta\}$, and are comparable on the circle:*

$$max_{p\in\Sigma_0} \| m_0^{-1}(p)\,[m(p) - m_0(p)] \| < 1,$$

then the total multiplicity M_1 of zeros of the function m inside the circle Σ_0 is equal to the multiplicity M_0 of the zero p_0 of the function m_0.

Similar more general statement is true for poles and zeros of a pair of comparable analytic functions with poles and zeros on the disc.

Applying the above theorem 8.1 to the pair m_0^β, m^β, consider the Gohberg-Sigal ratio for $p \approx \alpha_1$:

$$\frac{m^\beta}{m_0^\beta} - I = \sum_{s=2}^{k} \frac{1+\alpha_s^2}{\alpha_s - p} B_s^2 Q_s \ \left[m_0^\beta\right]^{-1} = \sum_{s=2}^{k} \frac{1+\alpha_s^2}{\alpha_1 - \alpha_s} B_s^2\, B_1^{-2}\, \frac{\alpha_1 - p}{1+\alpha_1^2}. \tag{70}$$

We see that in δ - neighborhood U_δ of α_1

$$\delta < \left[\sum_{s=2}^{k} \frac{|1+\alpha_s^2|}{|1+\alpha_1^2|} B_s^2\, B_1^{-2}\right]^{-1} \min_{s\neq 1} |\alpha_1 - \alpha_s|$$

the Rouche condition is fulfilled. Hence, in particular, if the eigenvalue α_1 is simple , then in the disc $|p - \alpha_1^\beta| < \delta$ there exist a unique resonance - the simple vector zero of m^β, which can be calculated approximately (first order approximation):

$$p^\beta \approx p_0^\beta + \frac{i}{2}(1+\alpha^2) B_0^2,\ e^\beta \approx$$

We supply below a non-formal discussion of the above Theorem 8.1.

Let F_0 be a proper subspace of the finite-dimensional Hilbert space F, $\mathbf{P}_0$ be an orthogonal projection onto F_0 and $\mathbf{P}_0^\perp = I - \mathbf{P}_0$ be the projection onto the orthogonal complement $F_0^\perp = F \ominus F_0$. We say that the analytic matrix-function m defined on the domain D_m has a *simple*

isolated right vector zero at the point $p_0 \in D_m$, if it is represented on a neighborhood $U_0 \subset D_\mu$ of the point as a product

$$m(p) = \mu_0^r(p)\left[(p-p_0)\mathbf{P}_0 + b\mathbf{P}_0^\perp\right] \tag{71}$$

with the complementary orthogonal projection $\mathbf{P}_0$, $\mathbf{P}_0^\perp$ and invertible near p_0 analytic matrix-functions $\mu_0^r(p),\ b$

$$\mu_0^r(p) = \mu_0(p_0) + \frac{p-p_0}{1!}\mu_0'(p_0) + \ldots,\ b(p) = b(p_0) + \frac{p-p_0}{1!}b'(p_0) + \ldots.$$

Multiple zeroes are defined similarly to (71), with several right factors containing possibly different projections $P_0, P_1, P_2, \ldots$. One can define, in a similar way, the left vector zero and the corresponding left projection, based on the factorization

$$m(\beta) = \left[p-p_0)\mathbf{Q}_0 + b\mathbf{Q}_0^\perp\right]\mu_0^l(p), \tag{72}$$

with the invertible $\mu_0^l(p_0)$ and the orthogonal projections Q_0, $Q_0^\perp$. For the finite-dimensional square matrix-functions the left and right vector zeroes coincide and dim $\mathbf{P}_0$ = dim $\mathbf{P}_0^+$ due to Fredholm theorem. The vectors $\mathbf{e}_0$, $\mathbf{e}_0^+ \in \mathbf{N}_0$, $\mathbf{N}_0^+$ from the corresponding null-subspaces are called respectively right and left *root-vectors*, $m(p_0)\mathbf{e}_0 = 0$, $m^+(p_0)\mathbf{e}_0^+ = 0$. For vectors $\mathbf{e}_\perp$ from the complementary subspace $\mathbf{e}_\perp \in E_{0\perp}$ we have $m(p_0)e_\perp \neq 0$. Similarly the *simple isolated vector pole* is defined : we say that the function m has a simple isolated vector pole at the point p_0 if it is represented as

$$m(\beta) = \mu\left[\frac{\mathbf{P}_0}{p-p_0} + b\mathbf{P}_0^\perp\right] \tag{73}$$

with an invertible function b and an orthogonal projection $\mathbf{P}_0$ onto proper subspace $\mathbf{N}_0 \subset E$, the complementary projection $\mathbf{P}_0^\perp$ and an analytic invertible function μ in a neighborhood U_0 of the point $p_0 \in D_m$. Similarly the left poles are defined, which coincide with right poles in the finite-dimensional case. Both isolated poles and zeroes of analytic matrix-functions are called in [37] *characteristic values* of the argument p of the function m. The logarithmic residue of the function m at the simple isolated zero or pole is defined as an integral of the logarithmic derivative $m'(p)m^{-1}(\beta)$ on a simple smooth loop $\Gamma_0 \subset U_0$ oriented in anti-clockwise ("positive") direction around the characteristic value m_0:

$$I_{m,p_0} = \frac{1}{2\pi i}\oint_{\Gamma_0} m'(p)m^{-1}(p)dp.$$

In [37] the *period* of the logarithmic derivative $m'(p)m^{-1}(\beta)$ on the simple cycle $\Gamma_0 \subset U_0$ containing no other characteristic points (zeroes, poles) inside

$$M_{m,\eta_0} = \frac{1}{2\pi i}Trace\oint_{\Gamma_0} m'(p)m^{-1}(p)dp.$$

is called the "multiplicity" of the characteristic value. Straightforward calculation of the above integrals gives the following result:

$$I(m,p_0) = \frac{1}{2\pi i}\oint_{\Gamma_0} m'(p)m^{-1}(p)dp =$$

$$\frac{1}{2\pi i}\oint_{\Gamma_0}\mu(\beta)\mathbf{P}_{_0}\left[(p_{_0}-p)\mathbf{P}_{_0}+b\mathbf{P}_{_0}^{\perp}\right]^{^{-1}}\mu^{^{-1}}(p)dp=$$

$$\frac{1}{2\pi i}\mu(p_{_0})\oint_{\Gamma_0}\mathbf{P}_{_0}\left[(p_{_0}-p)\mathbf{P}_{_0}+b\mathbf{P}_{_0}^{\perp}\right]^{^{-1}}dp\mu^{^{-1}}(p_{_0})=\mu(p_{_0})\,\mathbf{P}_{_0}\,\mu^{^{-1}}(p_{_0}),$$

and

$$M_{m,p_{_0}}=\pm\dim\ \mathbf{P}_{_0},$$

where the sign $\pm$ is defined by the type of the characteristic value: plus for zero, minus for pole.

Remark 3 If $B=0$, then $S=1$, which corresponds to the non-perturbed operator. But *it is impossible* to construct an analytic (with respect to the perturbation parameters $\beta_{_{il}}$) branch of eigenfunctions $\Psi_{_\nu}(p,\beta)$ of the perturbed operator for any p that coincides with the eigenfunction $e^{ipx}\nu$ of the non-perturbed operator at $B=0$. In the following section we will suggest a special perturbation procedure which allows us to overcome this basic difficulty locally, near a certain point $(p_{_0},\,0)$ in the space $(p,\,\beta)$ based on introduction of an appropriate *intermediate Hamiltonian*. If the boundary parameters are selected such that $S_\beta(\infty)=I$, then the corresponding scattering matrix is represented as a finite Blashke-Potapov product, see [65], with zeroes (resonances) in the upper half-plane $\Im p>0$.

In particular, the scattering matrix tends to I at infinity if $\beta_{_{11}}=0$ and $\beta_{_{00}}+\beta_{_{01}}PAP\beta_{_{10}}=0$. Then

$$S_{_\beta}(p)=\frac{2i-\beta_{_{01}}\frac{I+A^{^2}}{A-pI}\beta_{_{10}}}{2i+\beta_{_{01}}\frac{I+A^{^2}}{A-pI}\beta_{_{10}}} \tag{74}$$

with vector zeroes $p_{_s}:\ S_{_\beta}(p_{_s})\nu_{_s}=0$ in the upper half-plane $\Im p_{_s}>0$, in form of the Blaschke-Potapov product:

$$S_{_\beta}(p)=\prod_{_s}\left[\frac{p-p_{_s}}{p-\bar{p}_{_s}}P_{_s}+P_s^{\perp}\right]=:\prod_{_s}B_s. \tag{75}$$

The orthogonal projections $P_{_s}$ in E depend on the order of factors, see the discussion in the end of next section. The expressions $\beta_{_{01}}\frac{I+A^{^2}}{A-pI}\beta_{_{10}}$ in both numerator and denominator of the scattering matrix are Nevanlinna functions which tend to 0 at infinity. To prove the Blashke-Potapov decomposition (75), we need to check, if the zero $p_{_s}$ and the pole $\bar{p}_{_s}$ can be collected into one Blashke factor in the Blashke-Potapov representation (75). The constructed model has all typical features of the Lax-Phillips scattering system, see [53], and it can serve as a simplest model of such a system.

References

[1] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations.* Part I. Second Edition Clarendon Press, Oxford 1962 vi+203 pp.

[2] J. von Neumann *Mathematical foundations of quantum mechanics.* Translated from the German and with a preface by Robert T. Beyer. Twelfth printing. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, (1996) xii+445 pp.

[3] E.Fermi *Sul motto dei neutroni nelle sostance idrogenate* (in Italian) Richerka Scientifica **7**(1936) p 13.

[4] Yu.N.Demkov, V.N.Ostrovskij, *Zero-range potentials and their applications in Atomic Physics*, Plenum Press, NY-London (1988).

[5] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable models in quantum mechanics.* Springer-Verlag, New York,1988.

[6] S.Albeverio, P. Kurasov *Singular Perturbations of Differential Operators*, London Math. Society Lecture Note Series 271. Cambridge University Press (2000)

[7] A. Mikhailova, B. Pavlov, L. Prokhorov. *Intermediate Hamiltonian via Glazman splitting and analytic perturbation for meromorphic matrix-functions.* In: Mathematishe Nachrichten, **280**, 12, (2007) pp 1376-1416

[8] B.Pavlov *A star-graph model via operator extension* Mathematical Proceedings of the Cambridge Philosophical Society, Volume 142, Issue 02, March 2007, pp 365-384 doi: 10.1017/S0305004106009820, Published online by Cambridge University Press 10 Apr 2007.

[9] N.I.Akhiezer, I.M.Glazman, *Theory of Linear Operators in Hilbert Space*, (Frederick Ungar, Publ., New-York, vol. 1, 1966) (Translated from Russian by M. Nestel)

[10] B. Pavlov *The theory of extensions and explicitly-solvable models* Russian Math. Surveys **42**,6 (1987) pp 127-168.

[11] T. Kato *Perturbation theory for linear operators* Springer Verlag, Berlin-Heidelberg-NY, Second edition (1966) 740 pp.

[12] K.O.Friedrichs *On the perturbation of continuous spectra* Communications in Pure and Applied Mathematics **1** (1948) pp 361-406.

[13] H. Poincaré *Methodes nouvelles de la mećanique celeste* Vol. 1 (1892), Dover, New York (1957)

[14] I. Prigogine, C. George, L. Rosenfeld *A Unified Formulation of Dynamics and Thermodynamics* In: Chemica Scripta **4** (1973), pp 5-32.

[15] I. Prigogine *Irreversibility as a Symmetry-breaking Process* In : Nature, **246**, 9 , November 1973.

[16] I.Prigogine *Why Irreversibility? The formulation of Classical and Quantum Mechanics for nonintegrable systems* International Journal of Quantum Chemistry, **53** (1995),p 105-118.

[17] C. George, F. Mayne and I. Prigogine *Scattering Theory in Superspace* In: Advances in Chemical Physics, **61**, John Wiley & Sons, Inc., (1985) pp 223-299.

[18] I. Antoniou, S. Tasaki Generalized Spectral Decompisitions of Mixing Dynamical Systems IN: International Journal of Quantum Chemistry,**40**(1993), pp 425-474.

[19] N.Bagraev, A.Mikhailova,B.Pavlov, L.Prokhorov, A.Yafyasov*Parameter regime of a resonance quantum switch*, Phys. Rev. B, 71, 165308 (2005), pp 1-16

[20] J.Howland. *Finite rank perturbations of singular spectra* Proc. Amer. Math. Soc. **97** (1986), no. 4, pp 634–636.

[21] B. Simon *Spectral analysis of rank one perturbations and applications*. Mathematical quantum theory. II. Schrdinger operators (Vancouver, BC, 1993), 109–149.

[22] E. Karpov, T. Petrosky, I. Prigogine, G. Pronko *Friedrichs model with virtual transitions* Journal of Mathematical Physics , **41** (2001) p 118.

[23] B. Pavlov *Nonphysical sheet for the Friedrichs model* St. Petersburg Math. Journal **4**,6,(1993), pp 1245-1256.

[24] L.Faddeev *On a model of Friedrichs in the theory of perturbations of the continuous spectrum* Trudy Mat. Inst. Steklov **73** (1964) pp 292-313.

[25] L.Faddeev, B. Pavlov *Zero-sets of operator-functions with a positive imaginary part* In: Linear and Complex Analysis Problems (199 Research Problems) Lecture Notes in Mathematics 1043, ed. V. Havin, S. Hruschev, N. Nikolskii (1984) pp 124-128.

[26] B. Pavlov, S. Petras *On singular spectrum of weakly perturbed multiplication operator* , Func. Analysis i ego prilogenia,v4, N2, 1970. Translated into English in "Funkt. Anal. Appl."

[27] T. Kato *On finite-dimensional perturbation of self-adjoint operators* Journ. Math. Soc. Japan. **9** (1957) pp 239-249.

[28] M. Rosenbloom *Perturbation of the continuous spectrum and unitary equivalence* Pacific Journ. Math. **7** (1957) pp 997-1010.

[29] M. Birman *On a test for the existence of wave operators* Russian Academy Doklady,**143** (1962) pp 506-509. Translated into English in "Soviet Math. Doklady".

[30] S.Albeverio, P.Kurasov *Singular Perturbations of Differential Operators*, London Math. Society Lecture Note Series 271. Cambridge University Press (2000) 429 p.

[31] M. Krasnosel'skij *On selfadjoint extensions of Hermitian Operators* (in Russian) Ukrainskij Mat.Journal **1**(1949) pp 21-38 (in Russian)

[32] P. Kurasov, B. Pavlov, *Operator relation and Scattering matrix* In: Operator Theory: Advances and Applications, **113**, ed. Nikolskij and all, *Complex Analysis, Operators and Related Topics, S.A.Vinogradov - in memoriam*, Birkhäuser Verlag, Basel,2000, pp 294-305.

[33] V.I.Gorbachuk, M.L. Gorbachuk *Boundary value problems for operator differential equations* Math. and Appl (Soviet series) **48**, Kluver, Doderecht (1991), 347 p.

[34] J. Sirokov *Strongly singular potentials in one-dimensional Quantum Mechanics* Teor. Mat. Fiz. **41** 3 (1979) pp 45-49

[35] J. Sirokov *Strongly singular potentials in three-dimensional Quantum Mechanics* Teor. Mat. Fiz. **42** 1 (1980) pp 291-302.

[36] D. Yafaev, Mathematical Scattering Theory: General Theory, (Amer. Math. Society: Translations of Mathematical Monographs,**105**, Providence, Rhode Island, 1996).

[37] I.S. Gohberg and E.I. Sigal. Operator extension of the theorem about logarithmic residue and Rouchet theorem. Mat. Sbornik. 84(1971),p 607. Translated into English in "Math. USSR-Sbornik".

[38] J. C. B. de Saint-Venant, *Memoire sur la Torsion des Prismes*, Mem. Divers Savants, 14, (1855) pp. 233-560.

[39] R. von Mises *On Saint-Venant's Principle* , Bull. AMS, 51 (1945) pp 555-562.

[40] L. Petrova, B. Pavlov *Tectonic plate under a localized boundary stress: fitting of a zero-range solvable model.* In: Journal of Physics A, **41** (2008) 085206 (15 pp).

[41] G.R.Kirchhoff.*Gesammelte Abhandlungen* Publ. Leipzig: Barth, 1882, 641p.

[42] F.A.Berezin, L.D.Faddeev *A remark on Schrödinger equation with a singular potential* Dokl. AN SSSR, **137** (1961) pp 1011-1014

[43] B. Pavlov *A model of zero-radius potential with internal structure* Teor.Mat.Fiz. 59,3, (1984)pp 345-354 (English translation: Theoret. and Math. Phys,59,3 (1984) pp 544-550).

[44] B. Pavlov, V. Kruglov *Operator Extension technique for resonance scattering of neutrons by nuclei* In: Hadronic Journal 28(2005) pp 259-268.

[45] S. Datta and B. Das Sarma *Electronic analog of the electro-optic modulator.* Appl. Phys. Lett. 1990. **56**. N.7. P.665-667.

[46] M. Harmer, B. Pavlov, A. Yafyasov *Boundary conditions at the junction.* International Workshop on Computational Electronics (IWCE-11), Vienna, 25 May - 29 May 2006, book of abstracts, pp 241-242.

[47] B. Pavlov *A solvable model for scattering on a junction and a modified analytic perturbation procedure* arXiv: 0902.4260vI [math-ph] 24 Feb 2009.

[48] B. Pavlov *Spectral Analysis of a Dissipative Singular Schrödinger Operator in terms of a Functional Model* in the book: Partial Differential Equations,ed.M.Shubin in series Encyclopedia of Mathematical Sciences, Springer, 65 (1995) pp 87-153.

[49] S.Khruschev, N. Nikolskij A functional model and some problems of spectral theory of functions Trudy Mat. Inst. Steklov, **176** (1987) pp 97 - 210.

[50] B. Pavlov*Spectral Analysis of a Dissipative Singular Schrödinger Operator in terms of a Functional Model* in the book: Partial Differential Equations,ed.M.Shubin in series Encyclopedia of Mathematical Sciences, Springer, 65 (1995) pp 87-153.

[51] A. Beurling *On two problems concerning linear transformations in Hilbert space.* Acta Math. 81, (1948) pp 239-255.

[52] B. Sz.-Nagy, C. Foias *Analyse harmonique des operateurs de Space de Hilbert* In French, Masson et Cie, Paris; Akadmiai Kiad, Budapest (1967) xi+373 pp.

[53] P. Lax, R. Phillips *Scattering theory* Second edition. With appendices by Cathleen S. Morawetz and Georg Schmidt. Pure and Applied Mathematics, 26. Academic Press, Inc., Boston, MA,(1989) xii+309 pp.

[54] B. Pavlov *A remark on Spectral meaning of the symmetric Functional Model* in: "Operator Theory: Advances and Applications", Birkhauser Verlag, Basel, 154(2004) pp 163-177

[55] M.S.Livshits *Operators, oscilations, waves - open systems* Published by Nauka, in Russian, Moscow (1966) 298 p.

[56] B. Pavlov Spectral Analysis of a Dissipative Schrödinger operator in terms of a functional model in the book : " Partial Differential Equations", ed. by M. Shubin in the series Encycl. Math. Sciences, **65**, Springer 1995, pp 87- 153.

[57] B.Pavlov, M.Faddeev*Spectral analysis of unitary perturbations of contractions*, Proc. LOMI, v115,1982.(English translation in :J. of Sov.Math. v28, p768-776(1985))

[58] L. Faddeev, B. Pavlov *Scattering theory for automorphic functions* Zap. Nauchn. seminarov LOMI,**27** (1972) pp 161- 193.

[59] V.A.Ryzhov *Construction of the functional model of non-selfadjoint operator with nonempty regular set* Dep. VINITI 21.12.98, 389-B90, St. Petersburg (1990) 92p

[60] V.A. Ryzhov Absolutely-continuous subspace of the hon-selfadjoint operator and Scattering theory, PhD Thesis, supervised by S.Naboko, St. Petersburg (199) 147p.

[61] S.N.Naboko *Non-tangential boundary values of operator-valued R-functions in a half-plane* Leningrad Math. Journal , Vol. 1,**5**(1990) pp 1255-1277.

[62] P. Lax, R. Phillips *Scattering Theory for Automorphic functions* Annales of Mathematics Studies, V **87**Princeton Univ. Press, Princeton (1976).

[63] V.Adamjan,D. Arov, *Unitary couplings of semi-unitary operators.* (Russian) Mat. Issled. 1. vyp. 2, (1966) pp.3–64.

[64] B. Pavlov *Selfadjoint dilation of the dissipative Schroedinger operator and its resolution in terms of eigenfunctions* Math. USSR Sbornik,v31,4,457-478 (1977).

[65] V.P.Potapov *Multiplicative structure of J - non-expanding matrix-functions.* In: Proceedings of the Moscow Mathematical Society (In Russian), **4** (1965) pp 125-236.

[66] I.M.Gelfand, A.G.Kostyuchenko *Expansion in eigenfunctions of differential operators* In: Dokl. AN SSSR **103**(1955), pp 349-352.

[67] I.M. Gelfand *Some questions of analysis and differential equations* IN: Uspekhi Mat. nauk (in Russian) **14**, 3 (87), 1959, pp 3-19.

[68] I.M.Gelfand, G.E.Shilov *generalized functions Volume 3 : Theory of differential equations* Translated from Russian by Menhard E. Mayer, Academic Press [Harcourt Brace Iovanovich Publisher] New-York- London, 1967 [1977]