LONG TIME BEHAVIOR OF THE SCHRÖDINGER GROUP ASSOCIATED WITH A POTENTIAL MAXIMUM

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ABSTRACT. We give a semiclassical expansion of the Schrödinger group in terms of the resonances created by a non-degenerate potential maximum. This formula implies that the imaginary part of the resonances gives the decay rate of states for large time of order of the logarithm of the semiclassical parameter.

1. Introduction

This report is based on a series of work concerning spectral properties of the Schrödinger operator at a maximum of its potential. The detailed proofs of the results presented here can be found in [BFRZ] and [BFRZ2].

We consider the semiclassical Schrödinger equation in $\mathbb{R}^n$

$$Pu = zu, \quad P := -h^2\Delta + V(x),$$

where $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, $h > 0$ is a small (semiclassical) parameter, $z \in \mathbb{C}$ is a complex spectral parameter and $V(x)$ is a real-valued smooth potential.

If $z = \lambda_0 \in \mathbb{R}$ is an isolated eigenvalue of P, then for any $\psi(x) \in C_0^\infty(\mathbb{R})$ supported near λ_0, one has

$$e^{-itP/h}\psi(P) = e^{-it\lambda_0/h}\Pi_{\lambda_0}\psi(\lambda_0), \tag{1}$$

where Π_{λ_0} is the orthogonal projection to the eigenspace of λ_0 generated by orthonormal eigenfunctions $\{f_j\}_j$;

$$\Pi_{\lambda_0} = \sum_j (\cdot, f_j)f_j. \tag{2}$$

Here (f, g) denotes the scalar product $\int f\bar{g}dx$.

Analogous formulae may hold also for scattering energy levels with so-called *resonances* z instead of eigenvalues. Resonances are poles of the meromorphic extension of the resolvent, and are characterized as

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complex eigenvalues of a non self-adjoint operator P_θ, called *analytic distorsion* of P (see §1).

Roughly speaking, resonances are associated with *semi-bound states.* In other words, there are no resonances near *non-trapping* energy levels (see §2). Formulae of type (1), (2) would imply in particular that a semi-bound state decays like $e^{-|\operatorname{Im} z_0|t/h}\Pi_{z_0}$ for a resonance z_0. This is why the inverse of the *width* of the resonance $|\operatorname{Im} z_0|$ is considered to describe the life span of trapped quantum particles. Remark that the projection Π_{z_0} is not orthogonal and its operator norm is no longer 1.

Formulae of type (1) have been shown by S. Nakamura, P. Stefanov and M. Zworski [NSZ] for *shape resonances* created by a *well in an island*, and by J.-F. Bony and D. Häfner [BoHä] for resonances created by a potential maximum of the wave operator in the De Sitter-Schwarzschild metric.

Here we study the global maximal level E_0 for a general multidimensional potential, that we assume to be non degenerate and attained at only one point. In the phase space, the trapped trajectories of the Hamilton flow in $p^{-1}(E_0)$ (see §3 for the terminology) consist of a unique hyperbolic fixed point. The existence and the semiclassical distribution of resonances near this level were shown by P. Briet, J.-M. Combes and P. Duclos [BCD2] and by J. Sjöstrand [Sj1] independently (Theorem 4.1). We will give a formula of type (1) in §5 (Theorem 5.1, (7), (8)), and of type (2) in §6 (Theorem 6.1, (14), (15)). These results are based on a resolvent estimate (Theorem 5.4), and a microlocal propagation theorem near a hyperbolic fixed point (Theorem 6.2).

2. Resonances

We assume the following condition (A1) on the potential V.

(A1): $V(x)$ is real on $\mathbb{R}^n$, and analytic in a domain

$$\mathcal{D} := \{x \in \mathbb{C}^n; |\operatorname{Im} x| \leq \tan\theta_0 \langle \operatorname{Re} x\rangle\}$$

for $0 < \theta_0 < \pi/2$, and $V(x) \to 0$ as $|x| \to \infty$ in $\mathcal{D}$.

Then P is a self-adjoint operator on $L^2(\mathbb{R}^n)$ with $\sigma_{ess}(P) = \mathbb{R}_+$. To this operator, we associate a *distorted* operator

$$\tilde{P}_\mu = U_\mu P U_{-\mu}, \quad (U_\mu f)(x) := |\det(\mathrm{Id} + \mu dF)|^{1/2} f(x + \mu F(x))$$

for small real μ and $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with

$$F(x) = 0 \text{ on } |x| < R \text{ and } F(x) = x \text{ on } |x| > R + 1$$

for large R. This operator $\tilde{P}_\mu$ is analytic of type-A with respect to μ, and, taking R large enough, $P_\theta := \tilde{P}_{i\theta}$ is well-defined for θ small enough. Then $\sigma_{ess}(P_\theta) = e^{-2i\theta}\mathbb{R}_+$, and the spectrum of P_θ in $C_\theta := \{z \in \mathbb{C}\backslash\{0\}; -2\theta < \arg z < 0\}$ is discrete.

Definition 1. *Resonances* are the eigenvalues of P_θ in C_θ. The multiplicity of a resonance z_0 is the rank of the spectral projection

$$\Pi_{z_0} = \frac{1}{2\pi i}\int_\gamma (z - P_\theta)^{-1} dz, \tag{3}$$

where γ is a small circle centered at z_0 and we choose θ with $z_0 \in C_\theta$. Resonances are independent of θ in the sense that $\sigma(P_{\theta'}) \cap C_\theta = \sigma(P_\theta) \cap C_\theta$ for $\theta < \theta'$ taking the multiplicity into account. Moreover, the resonances are also independent of F. Hence we will denote the set of resonances by $\Gamma(h)$ without indicating θ and F.

Equivalently, we can define the resonances of P by showing that the resolvent $(z-P)^{-1} : L^2_{comp}(\mathbb{R}^n) \to L^2_{loc}(\mathbb{R}^n)$ has a meromorphic extension $R_+(z)$ from the upper half plane to C_θ across $(0,\infty)$. We have

$$\chi R_+(z)\chi = \chi(z-P_\theta)^{-1}\chi.$$

for any cut-off function χ whose support is in $|x| < R$. The poles are the resonances and the multiplicity of a resonance is also given by rank $\frac{1}{2\pi i}\int_\gamma R_+(z)dz$.

3. Resonance free domain

To the Schrödinger operator P corresponds the classical Hamitonian

$$p(x,\xi) = \xi^2 + V(x),$$

where $\xi = (\xi_1, \dots, \xi_n)$ denotes the momentum, which is the dual variable of the position x, and $\xi^2 = \xi_1^2 + \dots + \xi_n^2$. In the phase space $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, the Hamilton vector field is defined by

$$H_p = \nabla_\xi p \cdot \nabla_x - \nabla_x p \cdot \nabla_\xi.$$

We denote by $\exp tH_p(x_0,\xi_0)$ the integral curve of H_p starting from the point (x_0,ξ_0), and we call it a *Hamiltonian curve.*

The classical Hamiltonian p is invariant along any Hamitonian curve:

$$\frac{d}{dt}p(x(t),\xi(t)) = 0,$$

i.e. any Hamitonian curve is contained in an energy surface $p^{-1}(\lambda)$ for some real λ.

A *trapped trajectory* is a trajectory which is confined to some bounded set. Consider the following outgoing and incoming set

$$\Gamma_\pm(\lambda) := \{(x_0,\xi_0) \in p^{-1}(\lambda); |\exp tH_p(x_0,\xi_0)| \not\to \infty \text{ as } t \to \mp\infty\}.$$

Then $K(\lambda) := \Gamma_+(\lambda) \cap \Gamma_-(\lambda)$ is the union of the trapped trajectories in $p^{-1}(\lambda)$ and it is a compact set.

The following result suggests a close relationship between the semiclassical distribution of resonances near the real positive axis and the geometry of the corresponding classical dynamics. It is implicit in

[HeSj2] and was also proved in [BCD1] under a stronger hypothesis called the virial assumption. For the C^∞ potential case, see [Ma].

Theorem 3.1. *Let $E_0 > 0$ be such that there are no trapped trajectories in $p^{-1}(E_0)$. Then there exist $\delta > 0$ and $h_0 > 0$ such that for any $0 < h < h_0$, one has*

$$\Gamma(h) \cap D(E_0, \delta) = \emptyset,$$

where $D(E_0, \delta)$ denotes the complex disc centered at E_0 with radius δ.

4. Barrier top resonances

According to the previous theorem, there may be resonances near E_0 only if $K(E_0)$ is non-empty. Trapped trajectories may have various type of geometrical structure: fixed points, periodic orbits, homoclinic and heteroclinic orbits or more complicated structures. Here, we shall study the case where $K(E_0)$ reduces to a point $\{(0,0)\}$. We assume that it is a hyperbolic fixed point.

We assume the following conditions (A2) and (A3) besides (A1):

(A2): $V(0) = E_0 > 0$, $V'(0) = 0$, $V''(0) < 0$, i.e. for suitable coordinates,

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \text{as } x \to 0,$$

for some positive constants $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(A3): $K(E_0) = \{(0,0)\}$.

The assumption (A2) implies that the origin $(x,\xi) = (0,0)$ is a hyperbolic fixed point of the Hamilton vector field H_p. Let us consider the canonical system of p:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} \nabla_\xi p\big(x(t),\xi(t)\big) \\ -\nabla_x p\big(x(t),\xi(t)\big) \end{pmatrix} = \begin{pmatrix} 2\xi(t) \\ -\nabla_x V\big(x(t)\big) \end{pmatrix}. \tag{4}$$

The linearization at the origin is

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = F_p \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}, \tag{5}$$

where F_p is the fundamental matrix

$$F_p := \begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \xi} & \frac{\partial^2 p}{\partial \xi^2} \\ -\frac{\partial^2 p}{\partial x^2} & -\frac{\partial^2 p}{\partial \xi \partial x} \end{pmatrix}\Bigg|_{(x,\xi)=(0,0)} = \begin{pmatrix} 0 & 2\,\mathrm{Id} \\ \frac{1}{2}\mathrm{diag}\,(\lambda_j^2) & 0 \end{pmatrix}.$$

This matrix has n positive eigenvalues $\{\lambda_j\}_{j=1}^n$ and n negative eigenvalues $\{-\lambda_j\}_{j=1}^n$. The eigenspaces $\Lambda_\pm^0$ corresponding to these positive and negative eigenvalues are respectively outgoing and incoming stable manifolds for the quadratic part p_0 of p:

$$\begin{aligned}\Lambda^0_\pm = &\ \big\{(x,\xi)\in\mathbb{R}^{2n};\ \exp tH_{p_0}(x,\xi)\to(0,0)\quad\text{as}\quad t\to\mp\infty\big\}\\ = &\ \big\{(x,\xi)\in\mathbb{R}^{2n};\ \xi_j=\pm\tfrac{\lambda_j}{2}x_j,\ j=1,\cdots,n\big\}.\end{aligned}$$

By the stable manifold theorem, we also have outgoing and incoming stable manifolds for p:

$$\Lambda_\pm=\big\{(x,\xi)\in\mathbb{R}^{2n};\ \exp tH_p(x,\xi)\to(0,0)\quad\text{as}\quad t\to\mp\infty\big\},$$

which are tangent to $\Lambda^0_\pm$ at the origin. They are Lagrangian manifolds and can be written near $(0,0)$ as

$$\Lambda_\pm=\Big\{(x,\xi)\in\mathbb{R}^{2n};\ \xi=\frac{\partial\phi_\pm}{\partial x}(x)\Big\},$$

with the generating functions $\phi_\pm$ behaving like

$$\phi_\pm(x)=\pm\sum_{j=1}^n\frac{\lambda_j}{4}x_j^2+\mathcal{O}(|x|^3)\quad\text{as}\quad x\to0. \tag{6}$$

The assumption (A3) implies that E_0 is the global maximum of V, and it is attained only at $x=0$.

Under the assumptions (A1)-(A3), the semiclassical distribution of resonances is known near the barrier top energy E_0 (in [BCD2], the virial condition is assumed):

Theorem 4.1. ([BCD2], [Sj1]) *Let $\Gamma_0(h)$ be the discrete set*

$$\Gamma_0(h):=\left\{z^0_\alpha:=E_0-ih\sum_{j=1}^n\lambda_j\left(\alpha_j+\frac12\right);\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\right\}.$$

and let C be an h-independent positive constant such that $C\neq\sum_{j=1}^n\lambda_j$ $(\alpha_j+\frac12)$ for any $\alpha\in\mathbb{N}^n$. Then, in $D(E_0,Ch)$, there exists a bijection

$$b_h:\Gamma_0(h)\cap D(E_0,Ch)\to\Gamma(h)\cap D(E_0,Ch)$$

such that $b_h(z)=z+o(h)$.

Remark 4.2. The discrete set $\{h\sum_{j=1}^n\lambda_j(\alpha_j+1/2);\alpha\in\mathbb{N}^n\}$ is the set of eigenvalues of the harmonic oscillator $-h^2\Delta+\displaystyle\sum_{j=1}^n\lambda_j^2x_j^2/4$.

Let us denote $z_\alpha=b_h(z^0_\alpha)$. We call z^0_α *pseudo-resonance* (see [Sj2]). We say that a pseudo-resonance z^0_α is *simple* if $z^0_\alpha=z^0_{\alpha'}$ implies $\alpha=\alpha'$. If a pseudo-resonance z^0_α is simple, then the corresponding resonance z_α is simple, i.e. its multiplicity is one, and has an asymptotic expansion in powers of h whose leading term is z^0_α.

5. Representation formula of the propagator

Let us consider the Cauchy problem for the time-dependent Schrödinger equation

$$\begin{cases} ih\frac{\partial}{\partial t}\psi(t,x) = P\psi(t,x), \\ \psi(0,x) = \psi_0(x). \end{cases}$$

We denote the solution $\psi(t,x)$ by $e^{-itP/h}\psi_0$. The operator $e^{-itP/h}$ is unitary on $L^2(\mathbb{R}^n)$.

Theorem 5.1. *Assume (A1)-(A3). Let C be any positive constant such that $C \neq \sum_{j=1}^n(\beta_j+\frac{1}{2})\lambda_j$ for all $\beta\in\mathbb{N}^n$. Then, for any $\chi\in C_0^\infty(\mathbb{R}^n)$ and any $\psi\in C_0^\infty(\mathbb{R})$ supported in a sufficiently small neighborhood of E_0, there exists $K>0$ such that for any t, one has as $h\to 0$,*

$$\chi e^{-itP/h}\chi\psi(P) = \sum_{z_\alpha\in\Gamma(h)\cap D(E_0,Ch)} \chi \operatorname{Res}_{z_\alpha}\left(e^{-itz/h}R_+(z)\right)\chi\,\psi(P) + \mathcal{O}(h^\infty)+\mathcal{O}(e^{-Ct}h^{-K}). \tag{7}$$

If, in particular, all the pseudo-resonances in $D(E_0,Ch)$ are simple, one has, for any t, and as $h\to 0$,

$$\chi e^{-itP/h}\chi\psi(P) = \sum_{z_\alpha\in\Gamma(h)\cap D(E_0,Ch)} e^{-itz_\alpha/h}\,\chi\Pi_{z_\alpha}\chi\,\psi(P) + \mathcal{O}(h^\infty)+\mathcal{O}(e^{-Ct}h^{-K}). \tag{8}$$

Here, Π_{z_α} is the spectral projection given by (3).

Remark 5.2. We will see in §6 Theorem 6.1 (14), (15) that $\chi\Pi_{z_\alpha}\chi\sim h^{-|\alpha|-n/2}$ when z_α^0 is simple. Since, on the other hand, $|e^{-itz_\alpha/h}| = e^{-t|\operatorname{Im} z_\alpha|/h}\sim e^{-t\sum_{j=1}^n\lambda_j(\alpha_j+\frac{1}{2})}$ for $z_\alpha\in\Gamma(h)\cap D(E_0,Ch)$, the α-th term of the RHS of (8) is greater than the errors for

$$t \geq \frac{K-\frac{n}{2}-|\alpha|}{C-\sum_{j=1}^n\lambda_j(\alpha_j+\frac{1}{2})}\ln\frac{1}{h} + \text{Cte}. \tag{9}$$

Remark 5.3. If $\{\lambda_j\}_{j=1}^n$ are $\mathbb{Z}$-independent, all the pseudo-resonances are simple and (8) holds for any C.

To prove Theorem 5.1, we need the following resolvent estimate (see [BFRZ2]). For $a,b>0$, let us denote by $\Omega(a,b)$ the complex rectangular domain

$$\Omega(a,b) = \{z\in\mathbb{C};\,|\operatorname{Re} z - E_0| < a,\ -b<\operatorname{Im} z<b\}.$$

Theorem 5.4. *Assume (A1)-(A3). Let $\epsilon>0$ be sufficiently small. Then for any $C,C'>0$ and for any $\chi\in C_0^\infty(\mathbb{R}^n)$, there exists $h_0>0$ such that for $0<h<h_0$ there is no resonance in $\Omega(\epsilon,C'h)\backslash\Omega(Ch,C'h)$. Moreover, there exists $K>0$ such that, for $z\in\Omega(\epsilon,C'h)$,*

$$\|\chi R_+(z)\chi\| \lesssim h^{-K}\prod_{z_\beta\in\Gamma(h)\cap\Omega(Ch,2C'h)}|z-z_\beta|^{-1}. \tag{10}$$

Sketch of the proof of Theorem 5.1: Here we sketch the proof of Theorem 5.1 using Theorem 5.4.

We assume that $\psi \in C_0^\infty([E_0 - \frac{\epsilon}{2}, E_0 + \frac{\epsilon}{2}])$, $\psi \equiv 1$ on $[E_0 - \frac{\epsilon}{4}, E_0 + \frac{\epsilon}{4}]$ for a sufficiently small $\epsilon > 0$, and we calculate $I := \chi e^{-itP/h}\chi\psi(P)$. By the standard theory of pseudo-differential operators (see (13)), we have

$$I = \chi e^{-itP/h} f(P)\chi\psi(P) + \mathcal{O}(h^\infty),$$

for any cut-off function $f \in C_0^\infty(\mathbb{R})$ such that $f \equiv 1$ on $[E_0 - 2\epsilon, E_0 + 2\epsilon]$. Let E_λ be the spectral decomposition associated with P. Then

$$\chi e^{-itP/h} f(P)\chi\psi(P) = \int_{\mathbb{R}} e^{-it\lambda/h} f(z)\chi dE_\lambda \chi\psi(P).$$

By Stone's formula,

$$dE_\lambda = \frac{1}{2i\pi}\left((P - (\lambda + i0))^{-1} - (P - (\lambda - i0))^{-1}\right) d\lambda,$$

this can be rewritten as

$$\chi e^{-itP/h} f(P)\chi\psi(P) = -\frac{1}{2i\pi}\int_{\mathbb{R}} e^{-it\lambda/h} f(\lambda)\chi\big(R_+(\lambda) - R_-(\lambda)\big)\chi\, d\lambda\psi(P),$$

where $R_\pm(z) = (P - z)^{-1}$ is analytic for $\pm \operatorname{Im} z > 0$.

Let us modify the integral contour $\mathbb{R}$ to the union of the following intervals:

$$\Gamma_1 = (-\infty, E_0 - \epsilon], \quad \Gamma_5 = [E_0 + \epsilon, +\infty),$$
$$\Gamma_2 = E_0 - \epsilon + i[0, -Ch], \quad \Gamma_4 = E_0 + \epsilon + i[-Ch, 0],$$

and

$$\Gamma_3 = [E_0 - \epsilon, E_0 + \epsilon] - iCh.$$

We define

$$I_j = \frac{1}{2i\pi}\int_{\Gamma_j} e^{-itz/h} f(z)\chi\left(R_+(z) - R_-(z)\right)\chi dz\psi(P) \quad (j = 1, 5),$$

$$I_j = \frac{1}{2i\pi}\int_{\Gamma_j} e^{-itz/h}\chi\left(R_+(z) - R_-(z)\right)\chi dz\psi(P) \quad (j = 2, 3, 4).$$

Since $R_-(z)$ is holomorphic in $\Omega(\epsilon, Ch) \cap \{z \in \mathbb{C};\ \operatorname{Im} z \le 0\}$, one has, by the residue formula,

$$I = \sum_{z_\alpha \in \Gamma(h)\cap\Omega(\epsilon, Ch)} \chi \operatorname{Res}_{z_\alpha}\left(e^{-itz/h} R_+(z)\right)\chi\,\psi(P) - \sum_{j=1}^{5} I_j + \mathcal{O}(h^\infty).$$

The first term of the RHS coincides with that of the formula (7) thanks to Theorem 5.4.

Hence it suffices to estimate each I_j $(j = 1, \dots, 5)$. We can show by pseudodifferential calculus that

$$I_1, I_5 = \mathcal{O}(h^\infty) \quad \text{and} \quad I_2, I_4 = \mathcal{O}(h^\infty), \tag{11}$$

and

$$\|I_3\| = \int_{\Gamma_3} |e^{-itz/h}| \|\chi \left(R_+(z) - R_-(z)\right) \chi\| dz = \mathcal{O}(e^{-Ct} h^{-K}), \tag{12}$$

as bounded operator on $L^2(\mathbb{R}^n)$.

Here the resolvent estimate (10) is relevant. It ensures that the resolvent $R_+(z)$, as well as $R_-(z)$, stay at most of polynomial order with respect to h on the contour (of course near E_0).

The estimates (11) follow from the fact that the support of ψ is at positive distance from the real part of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \cup \Gamma_5$. In fact, for two functions $f, g \in C_0^\infty(\mathbb{R})$, it holds that

$$\operatorname{supp} f \cap \operatorname{supp} g = \emptyset \quad \Rightarrow \quad f(P)\chi g(P) = \mathcal{O}(h^\infty), \tag{13}$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$. □

6. Projection

In this final section, we give a representation formula of the projection Π_{z_α} in the case when z_α^0 is simple.

Theorem 6.1. *Assume (A1)-(A3) and suppose $z_\alpha^0 \in \Gamma_0(h)$ is simple. Then, as operator from $L^2_{comp}(\mathbb{R}^n)$ in $L^2_{loc}(\mathbb{R}^n)$, one has*

$$\Pi_{z_\alpha} = c(h)(\cdot, \overline{f_\alpha}) f_\alpha, \tag{14}$$

with

$$c(h) = h^{-|\alpha| - \frac{n}{2}} \frac{e^{-i\frac{\pi}{2}(|\alpha| + \frac{n}{2})}}{(2\pi)^{\frac{n}{2}} \alpha!} \prod_{j=1}^{n} \lambda_j^{\alpha_j + \frac{1}{2}}, \tag{15}$$

where $f_\alpha = f_\alpha(x,h)$ is a solution to $P f_\alpha = z_\alpha f_\alpha$, locally L^2 uniformly in h, vanishes in the incoming region (in the microlocal sense) and has an asymptotic expansion as $h \to 0$ for x near the origin

$$f_\alpha = d_\alpha(x,h) e^{i\phi_+(x)/h}, \tag{16}$$

with

$$d_\alpha(x,h) \sim \sum d_{\alpha,j}(x) h^j \ \text{ as } \ h \to 0, \tag{17}$$

$$d_{\alpha,0}(x) = x^\alpha + \mathcal{O}(|x|^{|\alpha|+1}) \ \text{ as } \ x \to 0. \tag{18}$$

Sketch of the proof: First, it is obvious that the projection Π_{z_α} can be written in the form (14), since it is a rank one operator to the space generated by a resonant state f_α associated with the simple resonance z_α.

Then, again thanks to the simplicity of z_α, it suffices to calculate $\Pi_{z_\alpha} v$ for a certain non trivial function $v(x)$. We write it as

$$\Pi_{z_\alpha} v = c_1(h) f_\alpha, \tag{19}$$

and we have

(20) $$c(h) = c_1(h)/(v, \overline{f_\alpha}).$$

To compute (19), we will construct $(z-P_\theta)^{-1}v$ for non-resonant energies z satisfying $|z-z_\alpha| = \epsilon h$ (see (3)). We do this microlocally as follows:

We take for v a function whose microsupport is contained in a small neighborhood of a point (x_0, ξ_0) on the incoming stable manifold Λ_-. Then we see, by the standard theory of propagation of microsupport and a result in [BoMi], that, on Λ_-, the microsupport of $(z-P_\theta)^{-1}v$ is the evolution of the microsupport of v (denoted by MS$[v]$) by the Hamilton flow:

$$\mathrm{MS}[(z-P_\theta)^{-1}v] \cap \Lambda_- \subset \bigcup_{t\geq 0} \exp tH_p(\mathrm{MS}[v]).$$

Furthermore, this microsupport propagates to the outgoing stable manifold Λ_+ through the fixed point $(0,0)$ (see [BFRZ]). As we will see (see also (16)), the microsupport of the singular part of $(z-P_\theta)^{-1}v$ with respect to z, (hence also that of f_α), reduces only to Λ_+, as is expected since the resonant state is "outgoing".

More precisely, let $\gamma : t \mapsto (x(t), \xi(t))$ be the Hamiltonian curve $\exp tH_p(x_0, \xi_0)$. We first construct a WKB solution u of $(P-z)u = 0$ microlocally near γ:

(21) $$u(x,h) = b(x,h)e^{i\psi(x)/h}, \quad b(x,h) \sim \sum_{l=0}^{\infty} b_l(x)h^l,$$

namely, the phase function satisfies the eikonal equation

$$p\left(x, \frac{\partial\psi}{\partial x}\right) = E_0, \quad \xi_0 = \frac{\partial\psi}{\partial x}(x_0)$$

and the coefficients of the symbol satisfy the transport equations

$$2\frac{\partial\psi}{\partial x}\cdot\frac{\partial b_l}{\partial x} + \left(\Delta\psi - i\frac{z-E_0}{h}\right)b_l = i\Delta b_{l-1}, \quad l \in \mathbb{N}.$$

We assume moreover that the Lagrangian manifolds $\Lambda_\psi := \{(x,\xi); \xi = \nabla_x\psi\}$ and Λ_- intersect transversally along γ. Then, we define v as

$$v = [\chi, P]u,$$

for $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ identically 1 near $x = 0$. Thus $v = (z-P)(\chi u)$ microlocally near γ. Then $\chi u = (z-P)^{-1}v$, and in particular $(z-P)^{-1}v$ has the WKB form (21) near γ close to $x = 0$.

Now we have a WKB solution u along $\gamma \subset \Lambda_-$ close to $(0,0)$. We need to know the asymptotic behavior of u on Λ_+. This was the main subject of [BFRZ], and we need to recall some of its results now. Since $\gamma \subset \Lambda_-$, we have

$$x(t) \sim \sum_{k=1}^{\infty} g_{\mu_k}(t; x_0, \xi_0)e^{-\mu_k t} \text{ as } t \to +\infty,$$

where $0 < \mu_1 (= \lambda_1) < \mu_2 < \cdots$ are linear combinations of $\{\lambda_j\}_{j=1}^n$ over $\mathbb{N}$, $g_{\mu_k}(t; x_0, \xi_0)$ are polynomials in t. In particular g_{λ_j} is independent of t, if λ_j is *simple* in the sense that the only linear combination over $\mathbb{N}$ of the λ_k's equal to λ_j is the trivial one.

The following theorem is a simplified version of [BFRZ]. The formula (24) is due to [ABR] and the idea to express the solution in the integral form (22) goes back to [HeSj1].

Theorem 6.2. *Assume that* $Pu = zu$, $\quad ||u|| \leq 1$, *for* $z \in D(E_0, Ch)$ *satisfying* $\operatorname{dist}(z, \Gamma_0(h)) > \epsilon h$ *for a positive* ϵ. *Then,*
(i) If $u = 0$ *microlocally on* $\Lambda_- \backslash (0,0)$, *then* $u = 0$ *in a neighborhood of* $(0,0)$ *(and hence on* Λ_+*).*
(ii) Suppose that $g_{\lambda_1}(x_0, \xi_0) \neq 0$. *If* u *is of the form (21) near* γ, *then one has a formal integral representation of* u *in a neighborhood of* $(0,0)$*:*

$$u(x,h) = \frac{1}{\sqrt{2\pi h}} \int_0^\infty e^{i\varphi(t,x)/h} a(t,x;h) dt. \tag{22}$$

Here the phase $\varphi(t,x)$ *has an asymptotic expansion as* $t \to +\infty$*:*

$$\varphi(t,x) \sim \phi_+(x) + \sum_{k=1}^{\infty} \phi_{\mu_k}(t,x) e^{-\mu_k t}, \tag{23}$$

where $\phi_{\mu_k}(t,x)$ *are polynomial in* t. *Moreover, if* λ_j *is simple* $\phi_{\lambda_j}(x)$ *is independent of* t *and*

$$\phi_{\lambda_j}(x) \sim -\lambda_j g_{\lambda_j} x_j \quad \text{as } x \to 0. \tag{24}$$

The symbol $a(t,x;h)$ *has a classical expansion in* h*:*

$$a(t,x;h) \sim \sum_{l=0}^{\infty} a_l(t,x) h^l,$$

whose coefficients have expansion as $t \to +\infty$*:*

$$a_l(t,x) \sim \sum_{k=0}^{\infty} a_{lk}(t,x) e^{-(S+\mu_k)t}, \tag{25}$$

with

$$S(z) = \frac{1}{2} \sum_{j=1}^{n} \lambda_j - i \frac{z - E_0}{h}, \tag{26}$$

where $a_{lk}(t,x)$ *are polynomial in* t. *In particular, the first term* $a_{0,0}(x)$ *is independent of* t *and* $a_{0,0}(0)$ *is given by*

$$a_{0,0}(0) = e^{-\pi i/4} \lambda_1^{3/2} |g_{\lambda_1}| e^{-\int_0^\infty \{\Delta\psi(x(s)) - \frac{1}{2}\sum_{j=1}^n \lambda_j + \lambda_1\} ds} b_0(x_0). \tag{27}$$

Remark 6.3. It is always possible to choose (x_0, ξ_0) on Λ_- such that $g_{\lambda_1}(x_0, \xi_0) \neq 0$.

Remark 6.4. The representation (22) is formal in the sense that the integral does not always converge depending on $\operatorname{Im} z$ (see (26)). For the rigorous expression, see [BFRZ].

Using this theorem, let us calculate $\Pi_{z_\alpha} v$ to obtain the constant $c_1(h)$ and the resonant state f_α. Recall that the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is fixed such that z_α^0 is simple.

By (23), (25) and the Taylor expansion of $e^{i(\varphi-\phi_+)/h}$, the integrand of (22) can be developed as

$$e^{i\varphi/h} a = e^{i\phi_+/h} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{i}{h}\right)^m (\varphi - \phi_+)^m \sum_{l=0}^{\infty} h^l a_l(t,x)$$

$$= e^{i\phi_+/h} e^{-St} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k'=0}^{\infty} \frac{i^m}{m!} h^{l-m} \left(\sum_{k=1}^{\infty} \phi_{\mu_k} e^{-\mu_k t} \right)^m a_{lk'} e^{-\mu_{k'} t}.$$

Each term of the last sum is of the form $e^{i\phi_+/h} c_\beta(t,x;h) e^{-(S+\lambda\cdot\beta)t}$, where $c_\beta(t,x;h)$ has an expansion in powers of h whose coefficients are polynomials in t. Since

$$S + \lambda \cdot \beta = -\frac{i}{h}(z - z_\beta^0),$$

this term produces a pole at $z = z_\beta^0$ after integration with respect to t. Hence we have only to look at $c_\alpha(t,x;h)$ for the study of Π_{z_α}.

Since z_α^0 is simple by assumption, the principal term in h of $c_\alpha(t,x;h)$ comes from $l = 0$, $k' = 0$, $m = |\alpha|$, more precisely

$$c_\alpha(t,x;h) = e^{i\phi_+(x)/h} \left\{ \frac{i^{|\alpha|}}{|\alpha|!} \left(\prod_{j=1}^{n} \phi_{\lambda_j}(x)^{\alpha_j} \right) a_{0,0}(x) h^{-|\alpha|} + \mathcal{O}(h^{-|\alpha|+1}) \right\}.$$

Notice that this principal term is independent of t (see the assertion before (24)), and hence it gives a simple pole after integration in t. The residue of u, which is $\Pi_{z_\alpha} v$, is then

$$e^{i\phi_+(x)/h} \left\{ \frac{1}{\sqrt{2\pi}} \frac{i^{|\alpha|-1}}{|\alpha|!} \left(\prod_{j=1}^{n} \phi_{\lambda_j}(x)^{\alpha_j} \right) a_{0,0}(x) + \mathcal{O}(h) \right\} h^{-|\alpha|+1/2}.$$

This means, by (24) and (27), that the resonant state f_α is of the form (16), (17), (18) and that $c_1(h)$ in (19) is given by

$$c_1(h) = \frac{1}{\sqrt{2\pi}} \frac{i^{|\alpha|-1}}{|\alpha|!} \left(\prod_{j=1}^{n} (-\lambda_j g_{\lambda_j})^{\alpha_j} \right) a_{0,0}(0) h^{-|\alpha|+1/2}.$$

To finish the proof of Theorem 6.1, it remains to compute the asymptotic expansion of $(v, \bar{f}_\alpha)$ (see (20)), which can be obtained by a stationary phase argument thanks to the assumption that Λ_ψ and Λ_- intersect transversally. □

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