

Action-angle representation of waves in fluids and plasmas: applications to stability analysis and wave-mean field interactions

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I. INTRODUCTION

This paper provides a short review on the Hamiltonian theory of ideal fluids and plasmas from a geometric viewpoint, and then applies it to the perturbation theory for the purpose of studying waves and instabilities. The common Hamiltonian structure of the governing field equations (in the Eulerian description) is called Lie-Poisson [1–3], which is intrinsically noncanonical and originates from a Lie group structure (the Lagrangian description for fluid particles). This Hamiltonian perspective offers a remarkably succinct expression of dynamics and illuminates its topological feature. The powerful methods in classical mechanics facilitates the understanding of fluids and plasmas in terms of conservation laws, linear and nonlinear stabilities, averaged behavior and so on. However, the progress on these theories seems to be still developing in comparison with finite-dimensional mechanical systems and quantum mechanics, because of the lack of rigorous spectral theory for non-selfadjoint operators that govern perturbations, i.e., waves. Our recent works [4–7] address a part of this profound problem and propose new methods that seems to be essential.

After the short review of Lie-Poisson systems in the next section, the main content of this paper is devoted to two subjects. First, we shall discuss and clarify the notions of wave energy and wave momentum in a rigorous framework of perturbation theory (Sec. III). Second, we shall acquire more detailed knowledge, namely, the action-angle variables for linear perturbations by employing a spectral method (Sec. IV). Our approach is formally applicable even in the presence of continuous spectrum that is well-known to occur in fluids and plasmas due to inhomogeneities (gradient and shear) of mean fields [8–12].

In general, the nonlinear evolution equations, say $\partial u/\partial t = X(u)$, for ideal fluids and plasmas are not integrable, and we are often interested in behavior of perturbed solutions away from a given solution u ,

$$u_\epsilon = u + \epsilon u_1 + \frac{\epsilon^2}{2} u_2 + \frac{\epsilon^3}{3!} u_3 + \dots, \quad (1)$$

where $\epsilon \in \mathbb{R}$ is assumed to be small. The wave energy and momentum are attributed to the perturbations u_1, u_2, \dots and, in many cases, the secondary field u_2 needs to be solved to leading order. Indeed, if u is steady, all oscillatory waves in u_1 will average out to zero, whereas u_2 involves a wave-driven mean field (a wave-induced correction of the pre-existing mean field u) that directly has to do with the wave energy and momentum. This wave-driven mean field is physically important in itself as a transportation mechanism induced by wave. Moreover, it possibly makes the wave energy negative, in which case the corresponding wave is ready to release free energy of the mean fields so that an instability occurs [13]. In Sec. III, we shall consider these things by taking a Lagrangian approach. Although there is some similarity to the general Lie perturbation method and existing theories for Lagrangian description [14, 15], we will explicitly present the equations of Lagrangian perturbations

(or displacements of particle orbits) in each order of ϵ . These solutions can reproduce the Eulerian perturbations owing to the topological nature of the Lie-Poisson system, and sometimes we can dispense with the laboring derivation of the secondary field u_2 [7].

Spectral decomposition of a non-selfadjoint operator that appears in the linear analysis u_1 is highly associated with the action-angle representation of waves. In Sec. IV, we will show that, the Laplace transform approach is, in practice, useful for calculating the action-angle variables even in the presence of continuous spectrum [5]. This formulation will be demonstrated in Sec. V by using the simplest and classical model, called the Case-Van Kampen equation [8, 9]. Other continuous spectra in hydrodynamics and magnetohydrodynamics can be dealt with in the same manner [5, 6].

II. GEOMETRICAL PROPERTIES OF FLUIDS AND PLASMAS

A. Lie-Poisson equation

The Lie-Poisson bracket is known as the common Hamiltonian structure of rigid bodies, fluids, magnetohydrodynamics, the Vlasov-Poisson system and so on [1, 16–20]. It is, therefore, of great interest to study general properties of the Lie-Poisson system. Here, we briefly prepare some terminology of geometry and Lie groups, which are requisite for defining the Lie-Poisson system. The geometrical notations can handle such a variety of mechanical systems from a unified viewpoint and drastically reduce lengthy calculations in practice. We will also promote a physical interpretation by showing the example of incompressible fluids. More complete review of the Lie-Poisson system can be found in many textbooks [2, 3].

Let G be a Lie group and $\mathfrak{g}(\simeq T_{\text{Id}}G)$ its Lie algebra (the tangent space to G at the identity element $\text{Id} \in G$). According to the theory of Lie groups, we can always associate $v \in \mathfrak{g}$ with a one-parameter subgroup $\exp tv \in G$, $t \in \mathbb{R}$, through the exponential map $\exp : \mathfrak{g} \rightarrow G$. This $\exp tv$ is intuitively understood as “the flow generated by the vector field v ”, which satisfies

$$v = \left. \frac{d}{dt} \right|_{t=0} \exp tv \quad \text{and} \quad (\exp tv)|_{t=0} = \text{Id}. \quad (2)$$

For later use, we introduce the adjoint representation. The adjoint action $\text{Ad}(\varphi) : \mathfrak{g} \rightarrow \mathfrak{g}$ of $\varphi \in G$ on $v \in \mathfrak{g}$ is defined by

$$\text{Ad}(\varphi)v := \left. \frac{d}{dt} \right|_{t=0} \varphi \circ \exp tv \circ \varphi^{-1}, \quad \varphi \in G, v \in \mathfrak{g}, \quad (3)$$

and the so-called “ad” operator is

$$\text{ad}(v_1)v_2 := \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp tv_1)v_2, \quad v_{1,2} \in \mathfrak{g}. \quad (4)$$

This manipulation is known as the definition of the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$,

$$\text{ad}(v_1)v_2 = [v_1, v_2]. \quad (5)$$

Denote the pairing between \mathfrak{g} and its dual space \mathfrak{g}^* by $\langle v, u \rangle$ for $u \in \mathfrak{g}^*$ and $v \in \mathfrak{g}$. Then, the coadjoint action $\text{Ad}^*(\varphi) : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ of $\varphi \in G$ and the “ad*” operator are defined by,

respectively,

$$\langle v, \text{Ad}^*(\varphi)u \rangle = \langle \text{Ad}(\varphi)v, u \rangle, \quad (6)$$

$$\langle v_2, \text{ad}^*(v_1)u, \rangle = \langle \text{ad}(v_1)v_2, u \rangle. \quad (7)$$

The Lie-Poisson bracket (for the right-invariant case) is defined by

$$\{F_1, F_2\} := \left\langle \left[\frac{\delta F_1}{\delta u}, \frac{\delta F_2}{\delta u} \right], u \right\rangle \quad \text{for } F_{1,2} : \mathfrak{g}^* \rightarrow \mathbb{R}, \quad (8)$$

where $\delta F/\delta u \in \mathfrak{g}$ denotes the functional derivative of F . In this paper, we will restrict our consideration to this right-invariant case that involves fluids and plasmas. But, the same observation can be made about the left-invariant case (such as rigid bodies) with a slight modification of signs.

For a given Hamiltonian functional $H : \mathfrak{g}^* \rightarrow \mathbb{R}$, the Hamiltonian equation is written as usual by

$$\frac{\partial F}{\partial t} = \{F, H\}, \quad \forall F : \mathfrak{g}^* \rightarrow \mathbb{R}. \quad (9)$$

This is especially referred to as Lie-Poisson equation since the bracket $\{, \}$ is not the canonical one but given by (8). Since $F(u)$ is arbitrary, the Lie-Poisson equation (9) is equivalent to

$$\frac{\partial u}{\partial t} = - \text{ad}^* \left(\frac{\delta H}{\delta u} \right) u, \quad (10)$$

and governs the evolution of $u(t) \in \mathfrak{g}^*$. Although we are mostly interested in this solution $u(t)$, it is remarkable that there exists an underlying group structure G behind it, which sometimes plays an essential role especially when studying the topology.

Example: Euler equation for incompressible fluids

The Euler equation is known as a Hamiltonian system with the Lie-Poisson bracket (8). In that case, G corresponds to the group of volume-preserving diffeomorphisms $S\text{Diff}(V)$ on a domain $V \subset \mathbb{R}^3$ filled with a fluid. The Lie algebra \mathfrak{g} is the space $\mathfrak{X}(V)$ of divergence-free vector fields in V which are tangent to the boundary wall ∂V .

In coordinates $\{x^i, i = 1, 2, 3\}$, we introduce the *global* inner product as $\langle v, u \rangle = \int_V u(v) d^3x = \int_V u_i v^i d^3x$ between a vector field $v = v^i \partial/\partial x^i \in \mathfrak{X}(V)$ and a 1-form $u = u_i dx^i$. The dual space $\mathfrak{X}^*(V)$ of $\mathfrak{X}(V)$ is then the quotient space of all 1-forms on V , modulo all exact 1-forms on V , namely, $u \simeq u + df \in \mathfrak{X}^*(V)$ for any 0-form f [2].

The adjoint action of $\varphi \in S\text{Diff}(V)$ is identified as the push-forward φ_* of any vector field $v \in \mathfrak{X}(V)$,

$$\text{Ad}(\varphi)v = \varphi_* v := \frac{\partial \varphi^i}{\partial x^j} (\varphi^{-1}(x)) v^j (\varphi^{-1}(x)) \frac{\partial}{\partial x^i} \quad (11)$$

and hence the ‘‘ad’’ operator (and simultaneously the Lie bracket) becomes

$$\text{ad}(\xi)v = [\xi, v] = (v \cdot \nabla)\xi - (\xi \cdot \nabla)v, \quad \forall \xi, v \in \mathfrak{X}(V). \quad (12)$$

The coadjoint action is associated with the pull-back of 1-form $u \in \mathfrak{X}^*(V)$,

$$\text{Ad}^*(\varphi)u = \varphi^*u + df = u_j(\varphi(x))\frac{\partial\varphi^j}{\partial x^i}(x)dx^i + \frac{\partial f}{\partial x^i}(x)dx^i, \quad (13)$$

where $f(x)$ is arbitrary, and we get

$$\text{ad}^*(\xi)u = \left(u_j \frac{\partial \xi^j}{\partial x^i} + \xi^j \frac{\partial u_i}{\partial x^j} + \frac{\partial f}{\partial x^i} \right) dx^i, \quad (14)$$

$$= \xi^j \left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right) dx^i + \frac{\partial f}{\partial x^i} dx^i. \quad (15)$$

By introducing a Riemannian metric to the manifold V , we can associate a 1-form $u = u_i dx^i + df \in \mathfrak{X}^*(V)$ with a vector field $\mathbf{u} = u^i \partial / \partial x^i \in \mathfrak{X}(V)$ by the relation $u^i = g^{ij} u_j + g^{ij} \partial f / \partial x^j$ where g^{ij} is the metric tensor and f is now chosen such that \mathbf{u} becomes divergence-free. The Riemannian metric is usually prescribed by the Hamiltonian function, which is given by

$$H(u) = \frac{1}{2} \int_V |\mathbf{u}|^2 d^3x = \frac{1}{2} \int_V u_i g^{ij} u_j d^3x. \quad (16)$$

In terms of this \mathbf{u} , we can write

$$\text{ad}^*(\xi)u = [-\boldsymbol{\xi} \times (\nabla \times \mathbf{u}) + \nabla f]_i dx^i, \quad (17)$$

where $[]_i$ represents the i th covariant component of the parentetic vector field.

Then, the Lie-Poisson equation (10) indeed coincides with the Euler equation.

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times (\nabla \times \mathbf{u}) + \nabla f, \quad (18)$$

$$= -(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla f, \quad (19)$$

where f is always chosen such that the right hand side becomes divergence-free. In the last expression, $-f$ is physically equivalent to the pressure field.

B. Coadjoint orbits

The topological aspect of the Lie-Poisson equation is viewed as follows. Now, suppose that a solution $u(t)$ of the Lie-Poisson equation (10) is obtained under a Hamiltonian H and an initial value $u(0)$. We regard $\delta H / \delta u(t)$ as a time-dependent vector field $v(t)$,

$$v(t) = \frac{\delta H}{\delta u}(t) \in \mathfrak{g}. \quad (20)$$

This $v(t)$ uniquely generates a one-parameter subgroup $\varphi_t \in G$ that satisfies

$$v(t_0) = \frac{d}{dt} \Big|_{t=t_0} (\varphi_t \circ \varphi_{t_0}^{-1}), \quad \varphi_0 = \text{Id}. \quad (21)$$

The mapping φ_t is again said to be the flow generated by the vector field $v(t)$, but it differs from the exponential map, $\varphi_t \neq \exp tv(t)$, since the vector field is now time-dependent. Note that the φ_t constructed in this way satisfies

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_{t=t_0} \text{Ad}^*(\varphi_t^{-1}) &= - \left. \frac{\partial}{\partial t} \right|_{t=t_0} \text{Ad}^*(\varphi_{t_0}^{-1} \circ \varphi_t \circ \varphi_{t_0}^{-1}) \\ &= - \left. \frac{\partial}{\partial t} \right|_{t=t_0} \text{Ad}^*(\varphi_t \circ \varphi_{t_0}^{-1}) \text{Ad}^*(\varphi_{t_0}^{-1}) \\ &= - \text{ad}^*(v(t_0)) \text{Ad}^*(\varphi_{t_0}^{-1}). \end{aligned}$$

Therefore, the solution $u(t)$ turns out to be written in the form of

$$u(t) = \text{Ad}^*(\varphi_t^{-1})u(0). \quad (22)$$

Even if the Hamiltonian $H(u)$ were replaced by any functional of u , the same expression would hold with a different flow φ_t . Hence, regardless of the choice of Hamiltonian function, any solution $u(t)$ must belong to an invariant subspace of \mathfrak{g}^* characterized by

$$\text{Orb}(u(0)) := \{\text{Ad}^*(\varphi)u(0) \in \mathfrak{g}^* | \varphi \in G\}, \quad (23)$$

which is called the coadjoint orbits through $u(0)$ [3]. The Lie-Poisson structure thus restricts the evolution of $u(t)$ to some extent, which can be understood as the topological (or kinematical) constraint built into the system. Since all states in $\text{Orb}(u(0))$ are accessible from $u(0)$ via a continuous map φ , they will have the same topology as $u(0)$. In other words, they are accessible from $u(0)$ without changing the topology.

Example: Euler equation for incompressible fluids (part 2)

As for the incompressible fluids, the vector field $v(x, t) \in \mathfrak{X}(V)$ corresponds to the Eulerian velocity field. The flow $\varphi_t \in \text{SDiff}(V)$, which is generated by

$$\frac{d\varphi_t}{dt}(x) = v(\varphi_t(x), t), \quad \varphi_0(x) = x, \quad (24)$$

represents the motion of fluid particles, namely, the Lagrangian description of the fluid.

The coadjoint action of φ_t is what is called the Weber transformation in the fluid mechanics [21],

$$(22) \quad \Leftrightarrow \quad \varphi_t^* u(t) + df = u(0). \quad (25)$$

This implies that the vorticity field $\mathbf{w} = \nabla \times \mathbf{u}$ must be *frozen* to the fluid motion φ_t (the Kelvin's circulation theorem). Indeed, the vorticity field $\mathbf{w} = \nabla \times \mathbf{u}$ is identified as the 2-form $w = du$ and the exterior differentiation of $\varphi_t^* u(t) + df = u(0)$ leads to $\varphi_t^* w(t) = w(0)$. Let S_t be any arbitrary surface moving with the fluid, $S_t = \varphi_t(S_0)$. Then, the vorticity flux passing through the surface will be conserved; $\int_{S_0} w(0) = \int_{S_0} \varphi_t^* w(t) = \int_{S_t} w(t)$. This topological aspect of ideal fluids was pointed out by Arnold, who call $\text{Orb}(u(0))$ the isovortical sheet [1, 2]. The phase space $\mathfrak{X}^*(V)$ is (at least locally) foliated by such isovortical sheets, and this fact becomes indispensable when developing the perturbation theory in a rigorous manner.

III. LAGRANGIAN APPROACH TO NONLINEAR PERTURBATIONS

A. Difficulty in naive perturbation expansion

Suppose that we have a particular solution $u(t)$ of the Hamiltonian equation, which might be one of trivial or steady states. In order to study behavior of solutions near the basic one $u(t)$, it is conventional to expand the unknown solution, say $u_\epsilon(t)$, around $u(t)$ as follows.

$$u_\epsilon(t) = u(t) + \epsilon u_1(t) + \frac{\epsilon^2}{2} u_2(t) + \frac{\epsilon^3}{3!} u_3(t) + \dots \quad (26)$$

where a parameter ϵ is assumed to be small and orders the magnitude of perturbations.

Substituting u_ϵ into the Hamiltonian function, it is also expanded as

$$H(u_\epsilon) = H(u) + \epsilon H_1 + \frac{\epsilon^2}{2} H_2 + \frac{\epsilon^3}{3!} H_3 + \dots, \quad (27)$$

where

$$H_1 = \left\langle \frac{\delta H}{\delta u}, u_1 \right\rangle, \quad (28)$$

$$H_2 = \left\langle \frac{\delta H}{\delta u}, u_2 \right\rangle + \left\langle \frac{\delta^2 H}{\delta u^2} u_1, u_1 \right\rangle, \quad (29)$$

$$H_3 = \left\langle \frac{\delta H}{\delta u}, u_3 \right\rangle + 2 \left\langle \frac{\delta^2 H}{\delta u^2} u_1, u_2 \right\rangle + \left\langle \frac{\delta^3 H}{\delta u^3} (u_1, u_1), u_1 \right\rangle, \quad (30)$$

and so on. We have naturally introduced the higher-order functional derivatives, $\delta^2 H / \delta u^2 : \mathfrak{g}^* \rightarrow \mathfrak{g}$ and $\delta^3 H / \delta u^3 : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}$, which are all evaluated at the given solution $u(t)$.

The Hamiltonian equation is similarly expanded with respect to ϵ , which gives a series of equations for u_1, u_2, \dots as follows,

$$\frac{\partial u_1}{\partial t} = -\text{ad}^* \left(\frac{\delta H}{\delta u} \right) u_1 - \text{ad}^* \left(\frac{\delta^2 H}{\delta u^2} u_1 \right) u, \quad (31)$$

$$\frac{\partial u_2}{\partial t} = -\text{ad}^* \left(\frac{\delta H}{\delta u} \right) u_2 - 2\text{ad}^* \left(\frac{\delta^2 H}{\delta u^2} u_1 \right) u_1 - \text{ad}^* \left(\frac{\delta^2 H}{\delta u^2} u_2 + \frac{\delta^3 H}{\delta u^3} (u_1, u_1) \right) u, \quad (32)$$

...

One can solve these equations in order. But, the solution u_2 stands as an obstacle, because u_2 often diverges secularly and the assumed series expansion loses its validity. The various techniques have been devised to remedy this failure, most of which exploit some renormalization theory (the weakly nonlinear analysis). Putting aside the mathematical justification, these techniques require involved calculations.

Note that $H_n, n = 1, 2, \dots$, are all constant in time and may take either positive or negative values. At this stage, there is no reason to neglect H_1 . But, it will be shown that, if the basic solution $u(t)$ was a steady state, H_1 would vanish for *most* linear perturbation u_1 . In that case, the wave energy rests on H_2 to leading order, and the secondary field u_2 is actually needed for evaluating it. For the same reason, such as the wave momentum and the wave-driven mean fields are often associated with u_2 .

B. Equations for Lagrangian perturbations

In terms of the basic solution $u(t)$, we can construct the corresponding vector field $v(t)$ and flow φ_t as shown in the previous section and obtain the expression (22). Now, suppose that the perturbed initial condition $u_\epsilon(0)$ has the same topology as $u(0)$, namely, there exists a mapping $\varphi_{\epsilon,0} \in G$ such that

$$u_\epsilon(0) = \text{Ad}^*(\varphi_{\epsilon,0}^{-1})u(0), \quad \varphi_{0,0} = \text{Id}, \quad (33)$$

where a parameter ϵ measures the amplitude of this perturbation. Then, the subsequent perturbed solution $u_\epsilon(t)$ must have the same topology $u_\epsilon(t) \in \text{Orb}(u(0))$, and hence there must exist a two-parameter mapping $\varphi_{\epsilon,t} \in G$ such that the perturbed flow is expressed by $\varphi_{\epsilon,t} \circ \varphi_t$. This $\varphi_{\epsilon,t}$ represents the deviation from the unperturbed orbit as follows.

$$u_\epsilon(t) = \text{Ad}^*((\varphi_{\epsilon,t} \circ \varphi_t)^{-1})u(0) = \text{Ad}^*(\varphi_{\epsilon,t}^{-1})\text{Ad}^*(\varphi_t^{-1})u(0) = \text{Ad}^*(\varphi_{\epsilon,t}^{-1})u(t). \quad (34)$$

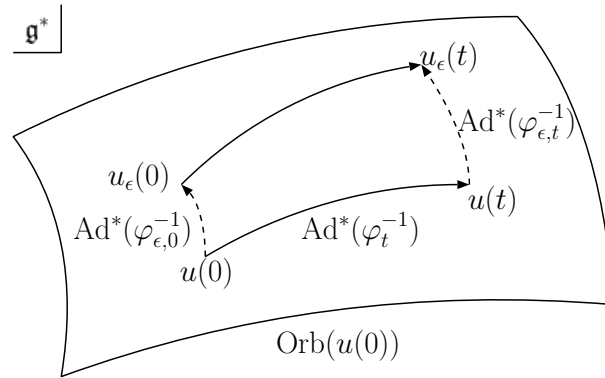


Fig. Unperturbed and perturbed orbits on $\text{Orb}(u(0))$

Moreover, for rather small ϵ , this near-identity map $\varphi_{\epsilon,t}$ is generally represented by an exponential map;

$$\exists \xi_\epsilon(t) \in \mathfrak{g} \quad \text{s.t.} \quad \varphi_{\epsilon,t} = \exp \xi_\epsilon(t). \quad (35)$$

Thus, any perturbed solution $u_\epsilon(t)$ having the same topology as $u(t)$ is fully characterized by a vector field $\xi_\epsilon(t) \in \mathfrak{g}$, which can be physically thought of as the displacement of the Lagrangian variables (particle orbits). Now, we are interested in what kind of equation governs this $\xi_\epsilon(t)$ in place of $u_\epsilon(t)$. It can be constructed in a perturbative treatment as follows.

First, due to the change in the orbits from φ_t to $\varphi_{\epsilon,t} \circ \varphi_t$, the corresponding $v(t)$ and $u(t)$ are perturbed into

$$v_\epsilon(t_0) = \frac{\partial}{\partial t} \Big|_{t=t_0} (\varphi_{\epsilon,t} \circ \varphi_t \circ \varphi_{t_0}^{-1} \circ \varphi_{\epsilon,t_0}^{-1}) = v + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} [\text{ad}(\xi_\epsilon)]^n \frac{D\xi_\epsilon}{Dt}, \quad (36)$$

$$u_\epsilon(t) = \text{Ad}^*(\varphi_{\epsilon,t}^{-1})u(t) = \sum_{n=0}^{\infty} \frac{1}{n!} [-\text{ad}^*(\xi_\epsilon)]^n u, \quad (37)$$

where

$$\frac{D}{Dt} := \frac{\partial}{\partial t} - \text{ad}(v) = \frac{\partial}{\partial t} + \mathcal{L}_v \quad (38)$$

denotes the total Lie-derivative along the given vector field $v(t)$.

Now, let us expand $\xi_\epsilon(t)$ with respect to ϵ as follows.

$$\xi_\epsilon(t) = \epsilon \xi_1(t) + \frac{\epsilon^2}{2} \xi_2(t) + \frac{\epsilon^3}{3!} \xi_3(t) + \dots \quad (39)$$

Then, $v_\epsilon(t)$ and $u_\epsilon(t)$ are expanded as

$$v_\epsilon = v + \epsilon v_1 + \frac{\epsilon^2}{2} v_2 + \dots, \quad \text{with} \quad v_1 = \frac{D\xi_1}{Dt}, \quad (40)$$

$$v_2 = \frac{D\xi_2}{Dt} + \text{ad}(\xi_1) \frac{D\xi_1}{Dt}. \quad (41)$$

$$u_\epsilon = u + \epsilon u_1 + \frac{\epsilon^2}{2} u_2 + \dots, \quad \text{with} \quad u_1 = -\text{ad}^*(\xi_1)u, \quad (42)$$

$$u_2 = -\text{ad}^*(\xi_2)u + \text{ad}^*(\xi_1)\text{ad}^*(\xi_1)u. \quad (43)$$

Recall that these $v_\epsilon(t)$ and $u_\epsilon(t)$ must satisfy the relation

$$v_\epsilon(t) = \left. \frac{\delta H}{\delta u} \right|_\epsilon(t), \quad (44)$$

where $\delta H/\delta u|_\epsilon$ denotes the functional derivative of H evaluated at u_ϵ . Both the left and right hand sides of this relation are now related to $\xi_\epsilon(t)$. By equating them in each order of ϵ , we obtain the equations of $\{\xi_n, n = 1, 2, \dots\}$. This can be summarized as follows;

Theorem: Let $u(t)$ be a solution of (10). If $u_\epsilon(0) \in \text{Orb}(u(0))$ at $t = 0$, the perturbed solution $u_\epsilon(t)$ takes the form of (42), where the evolutions of ξ_1, ξ_2, \dots are determined by solving

$$\frac{D\xi_1}{Dt} = \frac{\delta^2 H}{\delta u^2} u_1, \quad (45)$$

$$\frac{D\xi_2}{Dt} + \text{ad}(\xi_1) \frac{\delta^2 H}{\delta u^2} u_1 = \frac{\delta^2 H}{\delta u^2} u_2 + \frac{\delta^3 H}{\delta u^3}(u_1, u_1), \quad (46)$$

...

with u_1, u_2, \dots being expressed by (42). ■

As for perturbed solutions having the same topology as the unperturbed one, this theorem allows us to solve (45) and (46) instead of (31) and (32). While the calculation cost does not seem to be reduced, these variables ξ_1, ξ_2, \dots are more suitable for describing wave energy and momentum as shown in the next section. Note that it is easy to generate u_1, u_2, \dots from ξ_1, ξ_2, \dots via (42), whereas the converse mapping is generally difficult.

Example: Euler equation for incompressible fluids (part 3)

The equations (45) and (46) can be read as

$$\frac{\partial \xi_1}{\partial t} + (\mathbf{u} \cdot \nabla) \xi_1 - (\xi_1 \cdot \nabla) \mathbf{u} = \mathbf{u}_1, \quad (47)$$

$$\frac{\partial \xi_2}{\partial t} + (\mathbf{u} \cdot \nabla) \xi_2 - (\xi_2 \cdot \nabla) \mathbf{u} + (\mathbf{u}_1 \cdot \nabla) \xi_1 - (\xi_1 \cdot \nabla) \mathbf{u}_1 = \mathbf{u}_2, \quad (48)$$

in the conventional vector representation $\boldsymbol{\xi}, \mathbf{u} \in \mathfrak{X}(V)$. Note that $x_\epsilon(t) = \varphi_{\epsilon,t} \circ \varphi_t(x(0)) \in V$ corresponds to the perturbed motion of a fluid particle that leaves $x_\epsilon(0) = \varphi_{\epsilon,0}(x(0)) \in V$ at $t = 0$, where $\varphi_{\epsilon,t}$ indicates the deviation from the unperturbed orbit $\varphi_t(x)$.

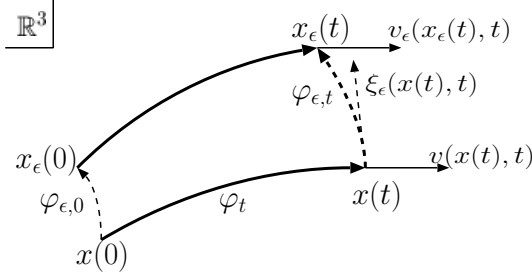


Fig. Unperturbed and perturbed orbits of a fluid particle

It is interesting to note that the relation (47) recovers the definition of the Lagrangian displacement field $\boldsymbol{\xi}_1(x, t)$ [22]. Therefore, we may call $\boldsymbol{\xi}_2(x, t)$ the “second-order” Lagrangian displacement field, which specifies the fluid motion to second order as follows.

$$\varphi_{\epsilon,t}(x) = x + \epsilon \boldsymbol{\xi}_1 + \frac{\epsilon^2}{2} [(\boldsymbol{\xi}_1 \cdot \nabla) \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2] + O(\epsilon^3). \quad (49)$$

The isovortical perturbation, namely (42), is written in the form of

$$\mathbf{u}_1 = \mathcal{P}[\boldsymbol{\xi}_1 \times \mathbf{w}], \quad (50)$$

$$\mathbf{u}_2 = \mathcal{P}[\boldsymbol{\xi}_1 \times (\nabla \times (\boldsymbol{\xi}_1 \times \mathbf{w})) + \boldsymbol{\xi}_2 \times \mathbf{w}], \quad (51)$$

where $\mathbf{w} = \nabla \times \mathbf{u}$ is the vorticity and \mathcal{P} denotes the projection to the space $\mathfrak{X}(V)$ of divergence-free vector fields. This expression completes the equations of $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ as follows.

$$\frac{\partial \boldsymbol{\xi}_1}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\xi}_1) + \mathcal{P}(\boldsymbol{\xi}_1 \times \mathbf{w}), \quad (52)$$

$$\begin{aligned} \frac{\partial \boldsymbol{\xi}_2}{\partial t} = & \nabla \times (\mathbf{u} \times \boldsymbol{\xi}_2) + \mathcal{P}(\boldsymbol{\xi}_2 \times \mathbf{w}) \\ & + \nabla \times (\mathcal{P}(\boldsymbol{\xi}_1 \times \mathbf{w}) \times \boldsymbol{\xi}_1) + \mathcal{P}[\boldsymbol{\xi}_1 \times (\nabla \times (\boldsymbol{\xi}_1 \times \mathbf{w}))]. \end{aligned} \quad (53)$$

In contrast to the Arnold’s isovortical *variation*, these Lagrangian displacements evolve temporally and determine an actual fluid motion, and the corresponding velocity field as well.

C. Wave energy and wave momentum

We have seen that any perturbed solution $u_\epsilon(t)$ having the same topology as the basic one $u(t)$ is expressed by the “displacement” vector field $\xi_\epsilon(t)$. Use of the ξ_1, ξ_2, \dots (Lagrangian description) instead of u_1, u_2, \dots (Eulerian description) reduces the wave energy, (28) and (29), into

$$H_1 = - \left\langle \xi_1, \frac{\partial u}{\partial t} \right\rangle, \quad (54)$$

$$H_2 = - \left\langle \xi_2, \frac{\partial u}{\partial t} \right\rangle - \left\langle \xi_1, \frac{\partial u_1}{\partial t} \right\rangle, \quad (55)$$

where we have used the relation (42) and the fact that $\xi_{1,2}$ are the solution of (45) and (46).

If the basic solution $u(t)$ was independent of time, H_1 would vanish automatically, which implies that any steady state must be an extremum of $H(u)$ *within* the coadjoint orbits $\text{Orb}(u)$ (or iso-topological surface). In addition, H_2 does not require ξ_2 and can be evaluated only by the linear solution ξ_1 .

This favorable feature of the displacement field is also applicable to the wave momentum. To show this, we first invoke the Hamiltonian version of the Noether's theorem [3];

Theorem: For the Hamiltonian equation (9), suppose that there exists a special vector field $\eta \in \mathfrak{g}$ that satisfies

$$\{\langle \eta, u \rangle, H\} = \left\langle \left[\eta, \frac{\delta H}{\delta u} \right], u \right\rangle = \left\langle \frac{\delta H}{\delta u}, \text{ad}^*(\eta)u \right\rangle = 0 \quad \text{for all } u \in \mathfrak{g}^*, \quad (56)$$

or, equivalently,

$$H(\text{Ad}^*(\exp -\tau\eta)u) = H(u) \quad \text{for all } u \in \mathfrak{g}^*, \tau \in \mathbb{R}. \quad (57)$$

Then, the momentum defined by $J = \langle \eta, u \rangle$ is a constant of motion. ■

Under this assumption, let us expand the momentum $J_\epsilon = \langle \eta, u_\epsilon \rangle$ of the perturbed solution $u_\epsilon(t)$ around the basic one $J = \langle \eta, u \rangle$;

$$J_\epsilon = \langle \eta, u \rangle + \epsilon \langle \eta, u_1 \rangle + \frac{\epsilon^2}{2} \langle \eta, u_2 \rangle + O(\epsilon^3), \quad (58)$$

$$=: J + \epsilon J_1 + \frac{\epsilon^2}{2} J_2 + O(\epsilon^3). \quad (59)$$

The perturbation terms $\epsilon J_1 + \epsilon^2 J_2/2 + \dots$ can be thought of as the wave momentum.

If the perturbed state $u_\epsilon(t)$ has the same topology as $u(t)$, we get

$$J_1 = \langle \xi_1, \text{ad}^*(\eta)u \rangle, \quad (60)$$

$$J_2 = \langle \xi_2, \text{ad}^*(\eta)u \rangle + \langle \xi_1, \text{ad}^*(\eta)u_1 \rangle. \quad (61)$$

Moreover, if the basic solution $u(t)$ also had the symmetry $\text{ad}^*(\eta)u = 0$ (or $\text{Ad}^*(\exp -\tau\eta)u = u$ for all τ), we again find that J_1 vanishes identically and $J_2 = \langle \text{ad}(\eta)\xi_1, u_1 \rangle$ is determined only by the linear solution ξ_1 .

Example: Euler equation for incompressible fluids (part 4)

For a steady flow \mathbf{u} , the wave energy H_2 corresponds to the isovortical second variation ($\delta^2 H$) of energy, derived by Arnold [1]. Since our $\xi_1(t)$ evolves dynamically, it is also written as

$$H_2 = \int_V \mathbf{w} \cdot \left(\frac{\partial \xi_1}{\partial t} \times \xi_1 \right) d^3x. \quad (62)$$

The same expression was observed by Kop'ev & Chernyshev [23], but in their treatment, the existence of the secondary field ξ_2 had gone unnoticed. The wave energy H_2 is thus

well-defined in the Lagrangian approach. Since H_2 is a constant of motion, either positive or negative definiteness of H_2 implies linear stability of any isovortical disturbance \mathbf{u}_1 .

If the fluid system is invariant under a flow generated by a vector field $\boldsymbol{\eta}$, the corresponding momentum (or impulse) in the direction of $\boldsymbol{\eta}$,

$$J(\mathbf{u}) = \int_V \mathbf{u} \cdot \boldsymbol{\eta} d^3x \quad (63)$$

is a constant of motion.

If the basic solution \mathbf{u} has the symmetry $\mathcal{P}[\boldsymbol{\eta} \times \mathbf{w}] = 0$ (or $\mathcal{L}_{\boldsymbol{\eta}} \mathbf{w} = 0$), the wave momentum for perturbation is of second order and given by

$$J_2 = \int_V \mathbf{w} \cdot (\boldsymbol{\xi}_1 \times \mathcal{L}_{\boldsymbol{\eta}} \boldsymbol{\xi}_1) d^3x. \quad (64)$$

As an example, let us consider a steady flow $\mathbf{u} = (0, u^\theta(r), u^z(r))$ in the cylindrical coordinate (r, θ, z) , where the boundary wall ∂V may exist at some radius $r = r_0$ (which may be infinite $r_0 \rightarrow \infty$) and the cylinder vessel is either infinite or periodic in the z direction. Note that this basic flow as well as the configuration system are symmetric with respect to both the translation along and the rotation about the z -axis. Therefore, by regarding $\boldsymbol{\eta}$ as the unit vector $\mathbf{e}_z = (0, 0, 1)$ in the z direction, the z -component of the wave momentum is obtained as

$$J_{2z} := \int_V \mathbf{u}_2 \cdot \mathbf{e}_z d^3x = \int_V \mathbf{w} \cdot (\boldsymbol{\xi}_1 \times \partial_z \boldsymbol{\xi}_1) d^3x. \quad (65)$$

Similarly, we may choose $r\mathbf{e}_\theta = (0, r, 0)$ as $\boldsymbol{\eta}$ and obtain the wave (angular) momentum about the z -axis,

$$J_{2\theta} := \int_V \mathbf{u}_2 \cdot r\mathbf{e}_\theta d^3x = \int_V \mathbf{w} \cdot (\boldsymbol{\xi}_1 \times \partial_\theta \boldsymbol{\xi}_1) d^3x. \quad (66)$$

These constant momenta imply the presence of wave-driven mean flow in \mathbf{u}_2 , which modifies the pre-existing flow \mathbf{u} in second order. The integrands of H_2 and J_2 , respectively, resemble the pseudoenergy and pseudomomentum in the nonlinear wave theory [14, 15]. Although our approach must proceed to the higher order to explore the nonlinear regime, the perturbative analysis on a given basic flow is consistently feasible. Moreover, we have shown that the wave energy and momentum directly amount to the *genuine* excess of energy and momentum, without solving an ensemble-averaged nonlinear equation of mean fields.

If the linear perturbation has only one Fourier component such as $\boldsymbol{\xi}_1(r, \theta, z, t) = \text{Re}[\hat{\boldsymbol{\xi}}(r)e^{-i\omega t + im\theta + ikz}]$, $\omega, m, k \in \mathbb{R}$, the wave energy and the wave momentum are reduced to

$$H_2 = \omega\mu, \quad J_{2z} = k\mu, \quad J_{2\theta} = m\mu, \quad (67)$$

with a common quantity μ . One may call this μ the wave action in the context of the wave kinetic theory. In the next section, μ will be identified as the action variable for the single oscillatory mode.

IV. SPECTRAL APPROACH TO LINEAR WAVES

A. Non-selfadjointness of linearized equations

From now on, we will discuss linear waves on a given steady solution u . The linearized Lie-Poisson equation about a steady state has an interesting property in itself, and we can develop a theory for the wave action based on it [5, 6].

Since we shall dwell on linear perturbations, let us simplify the notations for them by writing \tilde{u} and ξ in place of u_1 and ξ_1 . The linear evolution equation (31) for $\tilde{u}(t)$ will be denoted by

$$\frac{\partial \tilde{u}}{\partial t} = \mathcal{K}\tilde{u}. \quad (68)$$

It is very important to note that the equation (45) for ξ is generated by its adjoint operator \mathcal{K}^* ,

$$\frac{\partial \xi}{\partial t} = -\mathcal{K}^*\xi. \quad (69)$$

We define an anti-symmetric operator $\mathcal{A} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ by $\mathcal{A}\xi := -\text{ad}^*(\xi)u$. Using the Jacobi identity, one can directly prove that $\mathcal{K}\mathcal{A} = -\mathcal{A}\mathcal{K}^*$ holds. This fact reconfirms a part of the foregoing theorem, that is, $\tilde{u}(t) = \mathcal{A}\xi(t)$ holds for all t if $\tilde{u}(0) = \mathcal{A}\xi(0)$. We restrict linear perturbations to be kinematically accessible (or iso-topological) by employing this assumption $\tilde{u}(0) = \mathcal{A}\xi(0)$.

For such perturbations $\tilde{u}(t) = \mathcal{A}(t)\xi(t)$, we need to solve only $\xi(t)$, and the following observations can be made. Suppose that we have a closed family of solutions $\xi(t, \theta_0)$ including a parameter $0 \leq \theta_0 < 2\pi$ that is periodic, $\xi(t, \theta_0) = \xi(t, \theta_0 + 2\pi)$, in terms of θ_0 . Since the evolution of ξ is deterministic, the dependence on θ_0 must originate from a family of initial values satisfying $\xi(0, \theta_0) = \xi(0, \theta_0 + 2\pi)$. Then, we claim that the Poincaré's invariant (or the action integral associated with the family of solutions) is given by the ensemble average,

$$S := \frac{1}{4\pi} \int_0^{2\pi} \left\langle \frac{\partial \xi}{\partial \theta_0}, \tilde{u} \right\rangle d\theta_0 = -\frac{1}{4\pi} \int_0^{2\pi} \left\langle \xi, \frac{\partial \tilde{u}}{\partial \theta_0} \right\rangle d\theta_0. \quad (70)$$

In fact, it is easy to confirm that $\partial S / \partial t = 0$.

If the family of solutions was attributed to a single oscillatory eigenmode such as $\xi(t, \theta_0) = 2\text{Re}[\hat{\xi}e^{-i\omega t - i\theta_0}]$, where $\hat{\xi}$ is a complex eigenfunction for an eigenvalue $\omega \in \mathbb{R}$, the Poincaré's invariant is simply written as

$$S = \frac{1}{4\pi} \int_0^{2\pi} \left\langle \hat{\xi}e^{-i\omega t - i\theta_0} + \bar{\hat{\xi}}e^{i\omega t + i\theta_0}, i\mathcal{A} \left(\hat{\xi}e^{-i\omega t - i\theta_0} - \bar{\hat{\xi}}e^{i\omega t + i\theta_0} \right) \right\rangle d\theta_0 \quad (71)$$

$$= \left\langle \bar{\hat{\xi}}, i\mathcal{A}\hat{\xi} \right\rangle, \quad (72)$$

where $\bar{\hat{\xi}}$ denotes the complex conjugate of $\hat{\xi}$. Since $\partial \xi / \partial t = \omega \partial \xi / \partial \theta_0$ holds for such eigenmode, this is related to the wave energy by

$$\omega S = \frac{1}{4\pi} \int_0^{2\pi} \left\langle \frac{\partial \xi}{\partial t}, \tilde{u} \right\rangle dt = \frac{1}{2} H_2. \quad (73)$$

Therefore, S is understood as the action variable for the periodic solution. The angle variable is obviously $\theta(t) = \omega t + \theta_0$.

B. Wave action

In fluids and plasmas, it is more likely that many oscillatory modes and continuum modes coexist. These are respectively associated with point and continuous spectra of the linear operator \mathcal{K} (or \mathcal{K}^*). In order to consider wave action (or action variables) in such general cases, we will formally perform the spectral decomposition by invoking the Fourier-Laplace analysis.

Let us naturally expand the vector and covector spaces \mathfrak{g} and \mathfrak{g}^* into complex ones; $\mathfrak{g}_c = \mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g}_c^* = \mathfrak{g}^* \times \mathfrak{g}^*$. By simply multiplying the imaginary unit i , the evolution equations can look like non-selfadjoint Schrödinger equations,

$$i \frac{\partial \tilde{u}}{\partial t} = \mathcal{L} \tilde{u}, \quad \tilde{u}(0) = \tilde{u}_0 = \mathcal{A} \xi_0, \quad (74)$$

$$i \frac{\partial \xi}{\partial t} = \mathcal{L}^* \xi, \quad \xi(0) = \xi_0 \quad (75)$$

where $\mathcal{L}^* = -i\mathcal{K}$ is adjoint operator of $\mathcal{L} = i\mathcal{K}$ with respect to the inner product $\langle \bar{\circ}, \circ \rangle$ between the complex vector and covector spaces.

The aforementioned property becomes $\mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}^*$, and hence we have

$$\mathcal{A}(\Omega - \mathcal{L}^*)^{-1} = (\Omega - \mathcal{L})^{-1} \mathcal{A} \quad \text{for all } \Omega \in \mathbb{C}. \quad (76)$$

We technically define a subset of the spectrum of \mathcal{L} by

$$\sigma := \{\omega \in \mathbb{C} | (\Omega - \mathcal{L})^{-1} \mathcal{A} \xi_0 \text{ is not regular at } \Omega = \omega\}. \quad (77)$$

Then, this σ has the symmetry of $\sigma = \bar{\sigma} = -\sigma = -\bar{\sigma}$, which is known as a property of Hamiltonian spectrum [5].

The solution can be written as the Dunford-Taylor integral [24],

$$\xi(t) = \frac{1}{2\pi i} \oint_{\Gamma(\sigma)} (\Omega - \mathcal{L}^*)^{-1} \xi_0 e^{-i\Omega t} d\Omega, \quad (78)$$

where the integral path $\Gamma(\sigma)$ positively encircles the whole spectrum σ . Since \mathcal{L} is usually a differential operator, the spectral set σ may be unbounded. While the path $\Gamma(\sigma)$ then goes to infinity, the integration (78) would converge in many physical situations by assuming sufficiently smooth initial data ξ_0 . Suppose that σ does not overlap the imaginary axis for simplicity and decompose $\sigma = \sigma_+ \cup \sigma_-$ such that σ_+ is inside the half plane $\text{Re}(\Omega) > 0$. Due to the reality condition, (78) is rewritten as

$$\xi(t) = 2\text{Re} \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L}^*)^{-1} \xi_0 e^{-i\Omega t} d\Omega. \quad (79)$$

By replacing the initial condition ξ_0 by $\xi_0 e^{-i\theta_0}$, we can generate a family of solutions,

$$\xi(t, \theta_0) = 2\text{Re} \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L}^*)^{-1} \xi_0 e^{-i\Omega t - i\theta_0} d\Omega. \quad (80)$$

The corresponding Poincaré's invariant (70) is then transformed into

$$S = \text{Re} \left\langle \frac{1}{2\pi i} \oint_{\Gamma'(\sigma_+)} (\Omega' - \mathcal{L}^*)^{-1} \xi_0 e^{-i\Omega' t} d\Omega', i \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 e^{-i\Omega t} d\Omega \right\rangle, \quad (81)$$

$$= \text{Re} \frac{1}{(2\pi i)^2} \oint_{\Gamma'(\sigma_+)} \oint_{\Gamma(\sigma_+)} \langle \bar{\xi}_0, i(\Omega' - \mathcal{L})^{-1} (\Omega - \mathcal{L})^{-1} \tilde{u}_0 \rangle e^{i(\Omega' - \Omega)t} d\Omega d\Omega', \quad (82)$$

$$= \text{Re} \frac{1}{2\pi i} \oint_{\Gamma(\sigma_+)} D(\Omega) d\Omega, \quad (83)$$

where a function $D : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$D(\Omega) := \langle \bar{\xi}_0, i(\Omega - \mathcal{L})^{-1} \tilde{u}_0 \rangle = \langle \bar{\xi}_0, i(\Omega - \mathcal{L})^{-1} \mathcal{A} \xi_0 \rangle. \quad (84)$$

We naturally regard this S as the wave action, where the integration over the phase angle $0 < \theta_0 \leq 2\pi$ is converted to an integral path in \mathbb{C} enclosing the spectrum σ_+ . Here, the notion of the action integral has been extended to general solutions $\xi(t)$ that are not necessarily periodic in time, since σ_+ may include complex eigenvalues and continuous spectrum.

C. Spectral decomposition of wave action

We can analytically deform the integral path $\Gamma(\sigma_+)$ such that it consists of many closed paths that individually enclose each singularity of $D(\Omega)$.

Let us introduce a notation $\mathbf{U}(\Omega) = (\Omega - \mathcal{L})^{-1} \tilde{u}_0$, which is essentially equivalent to the Laplace transform of $\tilde{u}(t)$. If there are semi-simple eigenvalues ω_n , $n = 1, 2, 3, \dots$, the resolvent operator must have poles in the Ω plane, so that

$$\mathbf{U}(\Omega) = \frac{\hat{u}_n}{\Omega - \omega_n} + \dots \quad (85)$$

where \hat{u}_n is the projection of \tilde{u}_0 onto the eigenspace for ω_n . An integral path $\Gamma(\omega_n)$ around ω_n yields the action variable for the eigenmode,

$$\mu_n = \text{Re} \frac{1}{2\pi i} \oint_{\Gamma(\omega_n)} D(\Omega) d\Omega = \text{Re} \langle \bar{\xi}_0, i\hat{u}_n \rangle = \text{Re} \langle \bar{\xi}_0, i\mathcal{A}\hat{\xi}_n \rangle. \quad (86)$$

Due to the symmetry $\bar{\sigma}_+ = \sigma_+$ of spectra, there must exist an eigenvalue $\bar{\omega}_n$, and let $\hat{u}_{\bar{n}} = \mathcal{A}\hat{\xi}_{\bar{n}}$ be the corresponding projection. Using the well-known property of the projection [24], we obtain

$$\mu_n = \text{Re} \langle \hat{\xi}_{\bar{n}}, i\mathcal{A}\hat{\xi}_n \rangle = \text{Re} \langle \hat{\xi}_n, i\mathcal{A}\hat{\xi}_{\bar{n}} \rangle = \mu_{\bar{n}} \quad (87)$$

$$\text{or } \mu_n + \mu_{\bar{n}} = \langle \hat{\xi}_{\bar{n}}, i\mathcal{A}\hat{\xi}_n \rangle + \langle \hat{\xi}_n, i\mathcal{A}\hat{\xi}_{\bar{n}} \rangle \in \mathbb{R}. \quad (88)$$

When ω_n is a real eigenvalue, there is no distinction between ω_n and $\bar{\omega}_n$, and μ_n agrees with the previous result (72).

As for the continuous spectrum $\sigma_c \subset \mathbb{R}$ on the real axis, the path of integration is deformed into the two paths that run parallel to σ_c at the slightly upper and lower sides;

$$\frac{1}{2\pi i} \oint_{\Gamma(\sigma_c)} \mathbf{U}(\Omega) d\Omega = \lim_{\varepsilon \rightarrow 0} \frac{i}{2\pi} \int_{\sigma_c} [\mathbf{U}(\omega + i\varepsilon) - \mathbf{U}(\omega - i\varepsilon)] d\omega.$$

Hence, it is reasonable to define the generalized eigenfunction for $\omega \in \sigma_c$ by

$$\hat{u}(\omega) := \frac{i}{2\pi} [\mathbf{U}(\omega + i0) - \mathbf{U}(\omega - i0)]. \quad (89)$$

This definition of $\hat{u}(\omega)$ agrees with the Fourier transform of $\tilde{u}(t)$ according to Sato's hyperfunction theory [25] (see also the Appendix of Ref. 4). The eigenfunction $\hat{u}(\omega)$ for the continuous spectrum $\omega \in \sigma_c$ is therefore a generalized (or singular) function. This fact has been pointed out in many literatures; for example, see Case [9, 10], Sedláček [11] and Tataronis [12].

The wave action for the continuous spectrum is then given as a function of ω ;

$$\mu(\omega) = \text{Re} \frac{i}{2\pi} [D(\omega + i0) - D(\omega - i0)] = \left\langle \overline{\xi_0}, i\hat{u}(\omega) \right\rangle. \quad (90)$$

If the spectrum σ is composed of such semi-simple point spectra $\sigma_p = \{\omega_n \in \mathbb{C} : n = 1, 2, \dots\}$ and a real continuous spectrum $\sigma_c \subset \mathbb{R}$, the solution is represented by

$$\tilde{u}(t) = \sum_n \hat{u}_n e^{-i\omega_n t} + \int_{\sigma_c} \hat{u}(\omega) e^{-i\omega t} d\omega, \quad (91)$$

and the wave action is decomposed into

$$S = \sum_n \mu_n + \int_{\sigma_c} \mu(\omega) d\omega. \quad (92)$$

The action variable for continuous spectrum was already derived in several problems [26, 27]. Nevertheless, our consideration shown above is not only applicable to any Lie-Poisson system, but also suggesting an efficient way of derivation utilizing the Laplace transform. In the next section, we revisit the problem tackled by Morrison & Pfirsch [27] as a demonstration of our method.

V. EXAMPLE: VLASOV-POISSON EQUATION AND LANDAU DAMPING

A. Governing equations

Consider a collisionless plasma consisting of electrons and ions with charges $q_{e,i}$ and masses $m_{e,i}$, whereas the ions are assumed to be immobile and form a uniform background with a charge density $q_i n_i = \text{const}$. Let $\mathbf{x}, \mathbf{v} \in \mathbb{R}^3$ denote the position and velocity of particles and $f(\mathbf{x}, \mathbf{v}, t)$ be the distribution function of the electrons. The Vlasov-Poisson equations for electrons are

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q_e}{m_e} \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (93)$$

$$\text{div} \mathbf{E} = \frac{1}{\epsilon_0} \left(q_e \int f d^3v + q_i n_i \right). \quad (94)$$

This is also known as a Lie-Poisson Hamiltonian system [17, 18],

$$\frac{\partial F}{\partial t} = \{F, H\}, \quad (95)$$

$$= \iint \left[\frac{\delta F}{\delta f}, \frac{\delta H}{\delta f} \right] f d^3x d^3v, \quad (96)$$

where the Hamiltonian function is given by

$$H(f) = \iint m_e \frac{|\mathbf{v}|^2}{2} f d^3x d^3v + \epsilon_0 \int \frac{|\mathbf{E}|^2}{2} d^3x, \quad (97)$$

and the Lie bracket is defined by

$$[\zeta_1, \zeta_2] := \frac{1}{m_e} \left(\frac{\partial \zeta_1}{\partial \mathbf{x}} \cdot \frac{\partial \zeta_2}{\partial \mathbf{v}} - \frac{\partial \zeta_1}{\partial \mathbf{v}} \cdot \frac{\partial \zeta_2}{\partial \mathbf{x}} \right), \quad \forall \zeta_{1,2}(\mathbf{x}, \mathbf{v}). \quad (98)$$

Our theories developed in the previous sections are all applicable to this system. For instance, the linearization $f + \tilde{f}$ results in

$$\frac{\partial \tilde{f}}{\partial t} = \left[\frac{\delta H}{\delta f}, \tilde{f} \right] + \left[\frac{\delta^2 H}{\delta f^2} \tilde{f}, f \right]. \quad (99)$$

As for the Lie perturbation $\tilde{f} = [\zeta, f]$ generated by some $\zeta(\mathbf{x}, \mathbf{v}, t)$, one may solve the adjoint equation for ζ ,

$$\frac{\partial \zeta}{\partial t} = \left[\frac{\delta H}{\delta f}, \zeta \right] + \frac{\delta^2 H}{\delta f^2} [\zeta, f]. \quad (100)$$

In order to demonstrate the action-angle representation as shortly as possible, let us restrict our consideration to spatially uniform steady states $f(\mathbf{v})$, and Fourier-transform \tilde{f} in space

$$\tilde{f}(\mathbf{x}, \mathbf{v}, t) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(\mathbf{k}, \mathbf{v}, t) e^{i\mathbf{k} \cdot \mathbf{x}} d^3k. \quad (101)$$

For fixed \mathbf{k} , the linearized equations are greatly simplified into a 1D problem along the \mathbf{k} vector ($v := \mathbf{k} \cdot \mathbf{v}/k$). We further introduce a normalization $q_e = m_e = \epsilon_0 = 1$ and finally obtain

$$i \frac{\partial \tilde{f}}{\partial t} = kv \tilde{f} + k\eta(v) \int_{\mathbb{R}} \tilde{f} dv, \quad \tilde{f}(v, 0) = \tilde{f}_0(v) \quad (102)$$

$$i \frac{\partial \zeta}{\partial t} = kv \zeta + \int_{\mathbb{R}} k\eta(v) \zeta dv, \quad \zeta(v, 0) = \zeta_0(v) \quad (103)$$

where $\eta(v)$ is a given function associated with the steady state $f(v)$ by

$$\eta(v) = - \frac{1}{k^2} \frac{\partial f}{\partial v}. \quad (104)$$

For a detailed derivation of this equation, see Case [9], Van Kampen [8]. The relation $\tilde{f} = [\zeta, f]$ is now reduced to $\tilde{f} = -ik^3 \eta \zeta$ between $\tilde{f}(v, t)$ and $\zeta(v, t)$.

B. Laplace transform analysis

Let $F(v, \Omega)$ and $Z(v, \Omega)$ be the solutions of

$$(\Omega - kv)F = k\eta(v) \int_{\mathbb{R}} F dv + \tilde{f}_0 \quad (\tilde{f}_0 = -ik^3\eta\zeta_0), \quad (105)$$

$$(\Omega - kv)Z = \int_{\mathbb{R}} k\eta(v)Z dv + \zeta_0. \quad (106)$$

The relation $F(v, \Omega) = -ik^3\eta(v)Z(v, \Omega)$ follows immediately, and $Z(v, \Omega)$ is explicitly solved as follows.

$$Z(v, \Omega) = \frac{\zeta_0(v) - \Phi(\Omega)}{\Omega - kv} \quad (107)$$

where we have put

$$\Phi(\Omega) = -k \int_{\mathbb{R}} \eta(v)Z(v, \Omega) dv = -\frac{1}{\pi D(\Omega)} \int_{\mathbb{R}} \frac{k\eta(v)\zeta_0(v)}{\Omega - kv} dv, \quad (108)$$

$$\pi D(\Omega) = 1 - \int_{\mathbb{R}} \frac{k\eta(v)}{\Omega - kv} dv. \quad (109)$$

• Point spectra

Some eigenvalues $\sigma_p = \{\omega_n \in \mathbb{C}, n = 1, 2, \dots\}$ may arise from the zeros of $D(\Omega)$.

$$\pi D(\omega_n) = 1 - \int_{\mathbb{R}} \frac{k\eta(v)}{\omega_n - kv} dv = 0. \quad (110)$$

Using the residue theorem, the corresponding eigenfunctions are given by

$$\hat{\zeta}_n(v) = -\frac{\hat{\phi}_n}{\omega_n - kv} \quad \text{and} \quad \hat{f}_n(v) = -ik^3\eta(v)\hat{\zeta}_n(v) \quad (111)$$

where $\hat{\phi}_n \in \mathbb{C}$ is the residue of $\Phi(\Omega)$ at $\Omega = \omega_n$. If the eigenmode is neutrally stable $\omega_n \in \mathbb{R}$, the wave action is given by

$$\mu_n = \int_{\mathbb{R}} \overline{\hat{\zeta}_n(v)} i \hat{f}_n(v) dv = |\hat{\phi}_n|^2 \int_{\mathbb{R}} \frac{k^3\eta(v)}{(\omega_n - kv)^2} dv. \quad (112)$$

• Continuous spectrum

On the real axis of Ω , there is a continuous spectrum

$$\sigma_c = \{\omega \in \mathbb{R} \text{ s.t. } \eta(\omega/k) \neq 0\} \quad (113)$$

at which $Z(v, \Omega)$ is not analytic with respect to Ω . Let us introduce the following shorthand notations,

$$\eta^{\natural}(\omega) = \int_{\mathbb{R}} \eta(v)\delta(\omega - kv)k dv = \eta(\omega/k), \quad (114)$$

$$\eta^{\dagger}(\omega) = -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\eta(v)}{\omega - kv} k dv, \quad (115)$$

where \dagger corresponds to the Hilbert transform. Using the well-known formula

$$\frac{1}{\omega - kv \pm i0} = \text{p.v.} \frac{1}{\omega - kv} \mp \pi i \delta(\omega - kv), \quad (116)$$

we can evaluate $D(\Omega)$ and $\Phi(\Omega)$ in the limit of $\Omega \rightarrow \pm i0$ as

$$D(\omega \pm i0) = \lambda(\omega) \pm i\eta^{\natural}(\omega), \quad (117)$$

$$\Phi(\omega \pm i0) = \zeta_0^{\natural}(\omega) - \alpha(\omega)\lambda(\omega) \pm i\alpha(\omega)\eta^{\natural}(\omega), \quad (118)$$

where $\lambda(\omega) = \frac{1}{\pi} + \eta^{\dagger}(\omega)$ and

$$\alpha(\omega) = - \frac{(\eta\zeta_0)^{\dagger}(\omega) - \lambda(\omega)\zeta_0^{\natural}(\omega)}{\lambda^2(\omega) + \eta^{\natural 2}(\omega)}. \quad (119)$$

Therefore, the singular eigenfunctions (called the Van Kampen modes) are obtained as

$$\hat{\zeta}(v, \omega) = \frac{i}{2\pi} [Z(v, \omega + i0) - Z(v, \omega - i0)] \quad (120)$$

$$= \alpha(\omega) \left[\frac{1}{\pi} \text{p.v.} \frac{\eta^{\natural}(\omega)}{\omega - kv} + \lambda(\omega)\delta(\omega - kv) \right], \quad (121)$$

$$\hat{f}(v, \omega) = -ik^3 \alpha(\omega) \eta^{\natural}(\omega) \left[\frac{1}{\pi} \text{p.v.} \frac{\eta(v)}{\omega - kv} + \lambda(\omega)\delta(\omega - kv) \right]. \quad (122)$$

Note that we have also derived the appropriate ‘‘amplitude’’ $\alpha(\omega)$ of the singular eigenmode, since we are originally considering the projection onto the eigenspaces. The wave action for the continuous spectrum $\omega \in \sigma_c$ is found to be

$$\mu(\omega) = \int_{\mathbb{R}} \overline{\zeta_0(v)} i \hat{f}(v, \omega) dv \quad (123)$$

$$= k^2 \eta^{\natural}(\omega) [\lambda^2(\omega) + \eta^{\natural 2}(\omega)] |\alpha(\omega)|^2. \quad (124)$$

C. Landau damping

The signs of the action variables are generally important for mode couplings in Hamiltonian system [28–30]. The same is true of wave-wave interactions in fluids and plasmas. Here, by using the above result, let us consider an interaction between waves, one of which is a neutral eigenmode and the other is a continuum.

Let $\omega_n \in \mathbb{R}$ be a real eigenvalue that is isolated from the continuous spectrum, $\omega_n \notin \sigma_c$. From (110), note that $\eta(v) \equiv 0$ must hold in the neighborhood of $v = \omega_n/k$ so that such a ω_n is possible.

Now, suppose that a small change occurs in the steady state $f(v)$ so that $\eta(v) \rightarrow \eta(v) + \delta\eta(v)$ and $\delta\eta(\omega_n/k)$ becomes no longer zero. We denote the resultant movement of the eigenvalue by $\omega_n \in \mathbb{R} \rightarrow \omega_n + \delta\omega \in \mathbb{C}$. Since the complex eigenvalue $\omega_n + \delta\omega$ will be accompanied by its complex conjugate $\omega_n + \overline{\delta\omega}$, one may assume $\text{Im}(\delta\omega) > 0$ without loss of generality. The Taylor expansion of (110) around ω_n gives, to first order,

$$\delta\omega \int_{\mathbb{R}} \frac{\eta(v)}{(\omega_n - kv)^2} dv - \int_{\mathbb{R}} \frac{\delta\eta(v)}{\omega_n + i0 - kv} dv = 0, \quad (125)$$

where the limit of $\text{Im}(\delta\omega) \rightarrow +0$ is taken since $\delta\eta(\omega_n/k)$ is not zero. By taking the imaginary part of this expression, we obtain

$$\text{Im}(\delta\omega) = -\frac{\frac{\pi}{k}\delta\eta(\omega_n/k)}{\int_{\mathbb{R}} \frac{\eta(v)}{(\omega_n - kv)^2} dv} = -\frac{\pi k^2 \delta\eta(\omega_n/k) |\hat{\phi}_n|^2}{\mu_n}. \quad (126)$$

Note that the sign of $\delta\eta(\omega_n/k)$ is the same as the sign of $\mu(\omega_n)$, which is the action (124) for the continuous spectrum at $\omega = \omega_n$. If the signs of μ_n and $\mu(\omega_n)$ were opposite, there would be a complex eigenvalue $\omega + \delta\omega$ with $\text{Im}(\delta\omega) > 0$, and its conjugate $\omega + \overline{\delta\omega}$ as well. Thus, the eigenfrequency ω_n is split into a pair of complex-conjugated frequencies, one of which is a growing mode and the other is a damping mode.

If the signs of μ_n and $\mu(\omega_n)$ were the same, the obtained result $\text{Im}(\delta\omega) < 0$ contradicts our assumption $\text{Im}(\delta\omega) > 0$, which implies that the eigenvalue ω_n would disappear and be absorbed into the continuous spectrum (the resonant absorption). This phenomenon is famous as the Landau damping [31].

Such an interaction between point and continuous spectra frequently occurs in fluids and plasmas [23, 32–34] in more complicated ways. However, if the signs of the action variables were provided in advance, we could predict which resonance would cause an exponential instability or an Landau damping.

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