On some properties of a discrete hungry Lotka-Volterra system of multiplicative type

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Abstract. Two kinds of discrete hungry Lotka-Volterra systems (dhLV) are known as discretizations of the additive type hungry Lotka-Volterra system and the multiplicative one. By associating the dhLV of additive type (dhLVI) and the discrete hungry Toda equation (dhToda) with LR transformations, some of the authors give a Bäcklund transformation between these two systems. In this paper, from the dhLV of multiplicative type (dhLVII), we first derive the qd-type dhLVII. Through finding the positivity of the qd-type dhLVII and the LR transformation associated with the dhLVII, we present Bäcklund transformations among the dhLVI, the dhLVII and the dhToda. Moreover, by using one of the Bäcklund transformations, we show asymptotic convergence of the qd-type dhLVII.

Keywords. Bäcklund transformation, LR transformation, asymptotic convergence, discrete hungry Toda equation, discrete hungry Lotka-Volterra system

1. INTRODUCTION

The integrable Lotka-Volterra system (LV) is known as one of the ordinary differential equations that describe predator-prey dynamics in mathematical biology. In [1, 2, 3], one of extended LV is presented as

\[
\begin{align*}
\frac{du_k(t)}{dt} &= u_k(t) \left( \sum_{p=1}^{M-1} u_{k-p}(t) - \sum_{p=1}^{M} u_{k-p}(t) \right), \\
&= 1, 2, \ldots, M, \ t \geq 0, \\
&u_0(t) \equiv 0, \ldots, u_0(t) \equiv 0, \\
u_{M+1}(t) \equiv 0, \ldots, u_{M+t}(t) \equiv 0,
\end{align*}
\]

and another extended LV is given in [2, 3] as

\[
\begin{align*}
\frac{dv_k(t)}{dt} &= v_k(t) \left( \prod_{p=1}^{M} v_{k+p}(t) - \prod_{p=1}^{M} v_{k-p}(t) \right), \\
&= 1, 2, \ldots, M = 0, \ t \geq 0, \\
v_{M+1}(t) \equiv 0, \ldots, v_{M+t}(t) \equiv 0, \\
v_{M+M}(t) \equiv 0, \ldots, v_{M+M+(M-1)}(t) \equiv 0,
\end{align*}
\]

where \( M \) is a positive integer, \( M := (M+1)k - M \), and \( u_k(t) \) and \( v_k(t) \) denote the populations of the \( k \)th species at the continuous time \( t \). Eqs. (1) and (2) describe the competition such that the \( k \)th species is predator of the \((k+1)\)th, the \((k+2)\)th, \ldots, the \((k+M)\)th species and is prey of the \((k-1)\)th, the \((k-2)\)th, \ldots, the \((k-M)\)th species. In the case of \( M = 1 \), both (1) and (2) become the original LV. As \( M \) grows larger, for the \( k \)th species, the number of species of both the preys and the predators increase. So, (1) and (2) are called the hungry LV (hLV) of additive type and multiplicative type, respectively. Sometimes, (1) and (2) are referred to as the Bogoyavlensky lattices. The hLV (1) and (2) are also derived from a spatial discretization of the Korteweg-de Vries equation [4].

The discretized version of (1) is presented in [5, 6] as

\[
\begin{align*}
u_k^{(n+1)}(t) &= \prod_{p=1}^{M} (1 + \delta^{(n+1)}(k)p) \equiv v_k^{(n)}(t), \\
&= 1, 2, \ldots, M, \ n \geq 0, \ t \geq 0, \\
u_0^{(n)}(t) \equiv 0, \ldots, u_0^{(n)}(t) \equiv 0, \\
u_{M+1}^{(n)}(t) \equiv 0, \ldots, u_{M+M}^{(n)}(t) \equiv 0,
\end{align*}
\]

and that of (2) is given in [6] as

\[
\begin{align*}
v_k^{(n)}(t) &= \prod_{p=1}^{M} (1 + \delta^{(n+1)}(k)p) \\
&= 1, 2, \ldots, M = 0, \ n \geq 0, \ t \geq 0, \\
v_0^{(n)}(t) \equiv 0, \ldots, v_0^{(n)}(t) \equiv 0, \\
v_{M+1}^{(n)}(t) \equiv 0, \ldots, v_{M+M+(M-1)}^{(n)}(t) \equiv 0,
\end{align*}
\]

respectively. Both (3) and (4) are called the discrete hLV (dhLV). In this paper, in order to distinguish two kinds of the dhLVs, we simply refer to (3) and (4) as the dhLVI associated with the continuous hLV of additive type (1) and the dhLVII associated with the continuous hLV of multiplicative one (2), respectively. In (3) and (4), \( \delta^{(n)} \) represents the step size at the discrete time \( n \). The variables \( u_k^{(n)} \) and \( v_k^{(n)} \) denote the population of the \( k \)th species at the
discrete time $n$. The dhLV$_I$ (3) is shown in [7] to have an application for computing complex eigenvalues of a certain band matrix.

The discrete Toda equation

$$
\begin{align*}
\tilde{q}_i^{(n+1)} + q_i^{(n)} - q_{i-1}^{(n)} &= \delta^{(n)}(i), \\
q_i^{(n+1)} - q_{i+1}^{(n)} &= \gamma^{(n)}(i), \\
\xi_0^{(n)} &= 0, & n = 0, 1, \ldots,
\end{align*}
$$

is also a famous integrable system. Here, the superscript $n$ is the time variable, as in (3) and (4), and the subscript $i$ denotes the spatial variable. A study on box and ball systems in [8] leads to an extended version of the discrete Toda equation (5),

$$
\begin{align*}
Q_i^{(n+M)} + E_i^{(n+1)} &= Q_i^{(n)} + E_i^{(n)}, & i = 1, 2, \ldots, m, \\
Q_i^{(n+M)} E_i^{(n+1)} &= Q_i^{(n)} E_i^{(n)}, & i = 1, 2, \ldots, m-1, \\
E_0^{(n)} &= 0, & E_m^{(n)} = 0, & n = 0, 1, \ldots,
\end{align*}
$$

with positive integer $M$, which is named the discrete hungry Toda equation. In this paper, for the simplicity, we call (6) the dhToda. In [9], a new algorithm for computing matrix eigenvalues is designed based on the dhToda (6).

Some of the authors in [10] found a relationship of dependent variables, namely, a Bäcklund transformation, between the dhLV$_I$ (3) and the dhToda (6) through associating these integrable systems with a sequence of LR transformations of matrices. Bäcklund transformation is originally derived from the study of differential geometry. Explicit form of the Bäcklund transformation helps us to understand intrinsic features of an integrable system such as the solutions and symmetry and its relationship with another integrable system [11].

Here, for a nonsingular matrix $A$, the LR transformation [12] is defined as

$$
A = LR, \quad \hat{A} = RL.
$$

The 1st equation of (7) represents the LR decomposition of $A$ where $L$ is a lower triangular and $R$ is a unit upper triangular matrix. It is to be noted that the LR decomposition where $R$ has unit diagonal entries is uniquely given. The 2nd equation generates $\hat{A}$ as the matrix product $RL$. Let $A = LR$ be the LR decomposition of $A$. From (7), we get $\hat{A} = RL$. This type of equation appears in the matrix representation of some discrete integrable systems, and is called the Lax representation of them. The eigenvalues of $\hat{A}$ coincide with those of $A$. So, the LR transformation (7) yields a similarity transformation from $A$ to $\hat{A}$, namely, $A = RAR^{-1}$. For example, in order to compute the eigenvalues of a symmetric tridiagonal matrix, the quotient difference (qd) algorithm employs a sequence of LR transformations. It is interesting that the recursion formula of the qd algorithm is just equal to the discrete Toda equation (5).

However, there is no observation that the dhLV$_II$ (4) is associated with LR transformations. In this paper, we first associate the dhLV$_II$ (4) with a sequence of LR transformations. Based on this result, we present a Bäcklund transformation between the dhLV$_{II}$ (4) and the dhToda (6). Additionally, a Bäcklund transformation between the dhLV$_I$ (3) and the dhLV$_{II}$ (4) is also presented for the case of $\delta^{(n)} \to \infty$.

With the help of the relationship among the dhLV$_I$ (3), the dhLV$_{II}$ (4) and the dhToda (6), we next show the asymptotic behavior of the dhLV$_{II}$ (4) as $n \to \infty$, by using the convergence property of the dhToda (6) given in [9]. The dhLV$_{II}$ (4) is also shown to be applicable for matrix eigenvalue computation.

This paper is organized as follows. In Section 2, we derive a system called the qd-type dhLV$_{II}$ from the original dhLV$_{II}$ (4) through variable transformation. We also show the positivity of the qd-type dhLV$_{II}$ under suitable conditions. In Section 3, we give a Lax representation for the dhLV$_{II}$ (4), and then relate it to the LR transformation of a band matrix. We also review the Lax representation for the dhToda (6) and the LR transformation associated with it. In Section 4, by comparing two LR transformations in Section 2, we derive a Bäcklund transformation between the dhLV$_{II}$ (4) and the dhToda (6). By taking account of the Bäcklund transformation between the dhLV$_I$ (3) and the dhToda (6) given in [10], we also derive a Bäcklund transformation between the dhLV$_I$ (3) and the dhLV$_{II}$ (4) for the case of $\delta^{(n)} \to \infty$. We investigate the asymptotic behaviour of the dhLV$_{II}$ variables through the Bäcklund transformation between the dhLV$_{II}$ (4) and the dhToda (6). The asymptotic behaviour of the dhToda variables is already shown in [9]. In Section 5, we give numerical examples in order to demonstrate some theorems in the previous sections. Finally, in Section 6, conclusion is presented.

2. The qd-type dhLV$_{II}$ and positivity of its variables

In this section, we introduce the qd-type dhLV$_{II}$ which is derived from the dhLV$_{II}$ (4) and show the positivity of its variables. For the simplicity, we employ the notations $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and $\Phi_5$ defined as

$$
\Phi_1 := \{1, 2, \ldots, M_m + M - 1\}, \\
\Phi_2 := \{1, 2, \ldots, M_{m-1} + M\}, \\
\Phi_3 := \{1, 2, \ldots, m - 1\}, \\
\Phi_4 := \{1, 2, \ldots, m\}, \\
\Phi_5 := \{0, 1, \ldots, M - 1\}.
$$

These index sets appear throughout this paper frequently.

2.1. The qd-type dhLV$_{II}$

Let us introduce new variables

$$
\omega_k^{(n)} := \nu_k^{(n)} \left(1 + \delta^{(n)} \prod_{p=1}^{\nu_k^{(n)}} \nu_k^{(n)}\right), \quad \forall k \in \Phi_1,
$$

$$
\gamma_k^{(n)} := \delta^{(n)} \prod_{p=0}^{M} \nu_k^{(n)}, \quad \forall k \in \Phi_2,
$$

where
from the dhLV II variable $v^{(n)}$ and the discrete step size $\delta^{(n)}$. From the boundary condition of $v^{(n)}_k$, we have

$$\omega^{(n)}_k = v^{(n)}_k, \quad \forall k \in \Phi_5 \cup \{M\} \setminus \{0\},$$

$$\gamma^{(n)} = 0, \quad \forall k \in \Phi_5,$$

$$\gamma^{(n)}_{M_n + j} = 0, \quad \forall j \in \Phi_5.$$  

Then these variables satisfy the recursion formula

$$\begin{cases}
\omega^{(n+1)}_k + \gamma^{(n)}_k = \omega^{(n)}_k + \gamma^{(n)}_k, \quad \forall k \in \Phi_1, \\
\omega^{(n+1)}_k - \omega^{(n)}_k = \omega^{(n)}_{k+1 + \gamma^{(n)}_k} = \omega^{(n)}_{k+M+1 + \gamma^{(n)}_k}, \quad \forall k \in \Phi_2 \setminus \{M_m + M\}.
\end{cases}$$ (13)

This is easily checked as follows.

$$\begin{align*}
\omega^{(n+1)}_k + \gamma^{(n)}_k &= \omega^{(n)}_k + \gamma^{(n)}_k + \delta^{(n)} \prod_{p=0}^{M_1} v^{(n)}_{k-M+p} \\
&= \omega^{(n)}_k + \gamma^{(n)}_k \prod_{p=1}^{M} v^{(n)}_{k-p} \\
&= \omega^{(n)}_k + \gamma^{(n)}_k \\
\end{align*}$$

$$\begin{align*}
\omega^{(n+1)}_{k+1} + \gamma^{(n)}_{k+1} &= \omega^{(n)}_{k+1} + \gamma^{(n)}_{k+1} + \delta^{(n)} \prod_{p=1}^{M} v^{(n)}_{k+M+1-p} \\
&= \omega^{(n)}_{k+1} + \gamma^{(n)}_{k+1} \prod_{p=1}^{M} v^{(n)}_{k+1+M+1-p} \\
&= \omega^{(n)}_{k+1} + \gamma^{(n)}_{k+1}.
\end{align*}$$

Eq. (13) has the form similar to the recursion formula of the qd algorithm (5). In order to distinguish (13) from (11), we hereinafter call (13) the qd-type dhLV II. Also, we can rewrite the qd-type dhLV II as

$$\begin{align*}
\omega^{(n+1)}_k &= \omega^{(n)}_k + \gamma^{(n)}_k - \delta^{(n)}_{k-M}, \\
\gamma^{(n)}_k &= \frac{\omega^{(n)}_{k+1 + \gamma^{(n)}_k}}{\omega^{(n+1)}_k - \gamma^{(n)}_k}.
\end{align*}$$ (14)

If $\omega^{(n)}_k$ for $\forall k \in \Phi_1$ and $\gamma^{(n)}_k$ are given, we can obtain $\omega^{(n+1)}_k$ for $\forall k \in \Phi_1$ and $\gamma^{(n)}_k$ for $\forall k \in \Phi_2 \setminus \{1\}$ by using (14). Let us assume that $\omega^{(n)}_k > 0$ for $\forall k \in \Phi_1$ and $\gamma^{(n)}_k > 0$. Then we can relate $\gamma^{(n)}_k$ to $\delta^{(n)}$ as follows. From (9) and (10), we derive

$$\gamma^{(n)}_k = \delta^{(n)} \prod_{p=1}^{M_1} v^{(n)}_{p}.$$ (15)

From (15), it holds that

$$\delta^{(n)} = \frac{\gamma^{(n)}_k}{(\omega^{(n)}_{M_1 + 1} - \gamma^{(n)}_1) \prod_{p=1}^{M} \omega^{(n)}_{p}}.$$ (16)

Hence, the condition $\delta^{(n)} > 0$ is equivalent to

$$0 < \gamma^{(n)}_k < \omega^{(n+1)}_{M_1 + 1}.$$ (17)

2.2. Positivity of the qd-type dhLV II Variables

We give a theorem concerning the positivity of the qd-type dhLV II variables $\omega^{(n)}_k$ and $\gamma^{(n)}_k$.

**Theorem 1.** Let us assume that $\omega^{(n)}_k > 0$, $\forall k \in \Phi_1$ and

$$0 < \gamma^{(n)}_k < \omega^{(n+1)}_{M_1 + 1},$$ then it holds that

$$\omega^{(n+1)}_k > 0, \quad \forall k \in \Phi_1,$$

$$\gamma^{(n)}_k > 0, \quad \forall k \in \Phi_2.$$ (18)

**Proof.** In the discussion for the positivity of the qd-type dhLV II variables, it is useful to introduce an auxiliary variable $d^{(n)}_k$ defined by

$$d^{(n)}_k = \omega^{(n)}_k - \gamma^{(n)}_k - \delta^{(n)}_{k-M}, \quad \forall k \in \Phi_1.$$ (19)

From (11) and (16), it follows that

$$\begin{align*}
d^{(n)}_1 &= \omega^{(n)}_1 - \gamma^{(n)}_1, \\
d^{(n)}_{M_1 + 1} &= \omega^{(n+1)}_{M_1 + 1} - \gamma^{(n+1)}_1.
\end{align*}$$ (20)

By combining (17) with the assumption, we have

$$d^{(n)}_k > 0, \quad \forall k \in \Phi_5 \cup \{M, M + 1\} \setminus \{0\}.$$ (21)
In terms of $d_k^{(n)}$, we may rewrite the 1st equation of (14) as
$$
\omega_k^{(n+1)} = \gamma_k^{(n)} + d_k^{(n)}, \quad \forall k \in \Phi_1.
$$
(18)

From (14) and (16), we also get the recursion formula for $d_k^{(n)}$ and $d_{k+1}^{(n)}$ as follows.
$$
d_k^{(n)} = \omega_k^{(n)} = \frac{\omega_{k+1}^{(n)} - \gamma_k^{(n)}}{\omega_k^{(n+1)}}
$$
(19)

From (14), (18) and (19), we get a differential form without subtraction as follows.
$$
\begin{cases}
\omega_k^{(n+1)} = \gamma_k^{(n)} + d_k^{(n)}, \\
\gamma_k^{(n)} = \frac{\omega_{k+1}^{(n)} + \gamma_k^{(n)}}{\omega_k^{(n+1)}}, \\
d_k^{(n)} = \frac{\omega_{k+1}^{(n)} + d_k^{(n)}}{\omega_k^{(n+1)}}.
\end{cases}
$$
(20)

By using (20) repeatedly, we can compute $\omega_k^{(n+1)}$, $\gamma_k^{(n)}$, and $d_k^{(n)}$ for $k = 1, 2, \ldots$. Since the initial values $\omega_k^{(n)}$ for $\forall k \in \Phi_1$, $\gamma_k^{(n)}$, and $d_k^{(n)}$ for $\forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$ are positive by assumption and there are no subtractions, we can conclude that all the computed variables are positive. 

\section{LR Transformations Associated with the dhLV$_II$}

In this section, we give a Lax representation of the dhLV$_II$ (4), and then present a sequence of LR transformations associated with the dhLV$_II$ (4). In addition, we briefly review [10] concerning the LR transformation associated with the dhToda (6).

A Lax representation of the qd-type dhLV$_II$ (13) is given by
$$
\hat{L}^{(n+1)} \hat{R}^{(n)} = \hat{R}^{(n)} \hat{L}^{(n)},
$$
(21)

where
$$
\hat{L}^{(n)} = \begin{pmatrix}
0 & \omega_1^{(n)} & & \\
\vdots & \ddots & \ddots & \\
0 & & \ddots & \omega_2^{(n)} \\
1 & & \ddots & \ddots
\end{pmatrix},
\hat{R}^{(n)} = \begin{pmatrix}
1 & 0 & \ldots & 0 & \gamma_1^{(n)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \gamma_{M-1}^{(n)} & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{pmatrix}.
$$

By focusing on the entries in the both sides of (21), we can get the qd-type dhLV$_II$ (13). This means that (21) is a Lax representation of the qd-type dhLV$_II$ (13). Of course, the Lax representation (21) with (8) and (9) is just equal to that of the dhLV$_II$ (4) in [6].

It is remarkable here that the Lax representation is not always uniquely given. In the following theorem, we present a new Lax representation for the qd-type dhLV$_II$ (13).

\textbf{Theorem 2.} As $\delta^{(n)} \to \infty$, a Lax representation of the qd-type dhLV$_II$ (13) becomes
$$
\hat{L}^{(n+1)} R_{j+1}^{(n)} = R_j^{(n)} \hat{L}_j^{(n)}, \quad j \in \Phi_5,
$$
(22)

where
$$
\hat{L}_j^{(n)} = \begin{pmatrix}
\omega_1^{(n)} & & & \\
& \ddots & \ddots & \\
& & \ddots & \omega_2^{(n)} \\
& & \ddots & \ddots
\end{pmatrix},
$$
(23)

$$
R_j^{(n)} = \begin{pmatrix}
1 & \gamma_1^{(n)} & & \\
\vdots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \gamma_{M-1}^{(n)} \\
0 & \ldots & \ldots & 1
\end{pmatrix}.
$$
(24)

\textbf{Proof.} In (20), let us assume that $\omega_k^{(n)} > 0$ for $\forall k \in \Phi_1$, $0 < c \leq \gamma_1^{(n)} \leq \omega_{M+1}^{(n)}$, where $c$ is some positive constant. We first show that the qd-type dhLV$_II$ variables satisfy the following inequality.
$$
\begin{cases}
\frac{d_k^{(n)}}{\omega_k^{(n+1)}} \leq \frac{d_k^{(n)}}{\omega_k^{(n+1)}}, & \forall k \in \Phi_1 \setminus \{i(M + 1)\}, \quad \forall i \in \Phi_3, \\
\frac{d_k^{(n)}}{\omega_k^{(n+1)}} \leq \frac{d_k^{(n)}}{\omega_k^{(n+1)}}, & \forall k \in \{i(M + 1)\}, \quad \forall i \in \Phi_3, \\
\frac{\gamma_k^{(n)}}{\gamma_k^{(n)}} \leq \frac{\gamma_k^{(n)}}{\gamma_k^{(n)}}, & \forall k \in \Phi_2, \\
\omega_k^{(n+1)} \leq \omega_k^{(n+1)} & \forall k \in \Phi_1,
\end{cases}
$$
(25)

where $d_k$, $\omega_k$, $\gamma_k$, $\overline{\gamma}_k$, and $\overline{\omega}_k$ are some positive constants that do not depend on $\gamma_1^{(n)}$.

We first consider the case where $\gamma_k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}$. From (17), we have
$$
d_k^{(n)} \leq \overline{d}_k^{(n)} \leq \overline{d}_k, \quad \forall k \in \Phi_5 \cup \{M\} \setminus \{0\},
$$
(26)
with \( d_k = \overline{d}_k = \omega_k^{(n)} \). Similarly, from (17), we get
\[
d^{(n)}_{M+1} \leq \overline{d}_{M+1},
\]
for \( d^{(n)}_{M+1} = \omega^{(n)}_{M+1} \). Obviously, the assumption leads to
\[
\gamma_1 \leq \gamma^{(n)}_1 \leq \overline{\gamma}_1,
\]
with \( \gamma_1 = c \) and \( \overline{\gamma}_1 = \omega^{(n)}_{M+1} \). By combining (26)–(28) with (20), we can prove the following inequalities by induction.
\[
\begin{align*}
\gamma_k & \leq \gamma^{(n)}_k \leq \overline{\gamma}_k, \quad \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0,1\}, \\
\omega_k & \leq \omega^{(n+1)}_k \leq \overline{\omega}_k, \quad \forall k \in \Phi_5 \cup \{M, M+1\} \setminus \{0\}.
\end{align*}
\]
In the case where \( \forall k \in \{i + M+1\}, \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0\} \), from the 3rd equation of (20), (26), (27) and the 2nd equation of (29), it holds that
\[
d_k \leq d^{(n)}_k \leq \overline{d}_k, \quad \forall k \in \{i + M+1\}, \forall i \in \Phi_5 \cup \{M\} \setminus \{0\};
\]
for \( d^{(n)}_{2(M+1)} \leq \overline{d}_{2(M+1)} \).

Moreover, from (29), it follows that
\[
\gamma_{M+2} \leq \gamma^{(n)}_{M+2} \leq \overline{\gamma}_{M+2}.
\]
Eqs. (30)–(32) lead to
\[
\begin{align*}
\gamma_k \leq \gamma^{(n)}_k \leq \overline{\gamma}_k, \\
\gamma_k \leq \gamma^{(n)}_k \leq \overline{\gamma}_k, \\
\omega_k \leq \omega^{(n+1)}_k \leq \overline{\omega}_k, \\
\gamma_{i+M+1}, \forall i \in \Phi_5 \cup \{M, M+1\} \setminus \{0,1\}, \\
\omega_k \leq \omega^{(n+1)}_k \leq \overline{\omega}_k,
\end{align*}
\]
and for \( \forall k \in \{i(M+1)\}, \forall i \in \Phi_3 \setminus \{1,2\} \), we have
\[
d_k \leq \overline{d}_k.
\]
We next consider the case where \( \forall k \in \{(m-1)(M+1) + i + 1\}, \forall i \in \Phi_5 \). By combining the 3rd equation of (20), (33) with (12), we have
\[
d_{k+1} \leq d^{(n)}_{k+1} \leq \overline{d}_{k+1}, \quad \forall i \in \Phi_5.
\]
From (34) and the 1st equation of (20), we have
\[
\omega_{k+1} \leq \omega^{(n+1)}_{k+1} \leq \overline{\omega}_{k+1}, \quad \forall i \in \Phi_5.
\]
To sum up, we obtain (25).

By using (25), we discuss the behavior of variables in (13) as \( \delta^{(n)} \to \infty \). We first consider the case of \( k = M_i + M, \forall i \in \Phi_3 \). By using (19), repeatedly, we derive
\[
d^{(n)}_{M_i+M} = \frac{\prod_{p=2}^{M_i+1} \omega^{(n)}_{M_p+M} \omega^{(n)}_M \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p \omega^{(n)}_M \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p}{\delta^{(n)} \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p \omega^{(n)}_M \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p + 1}.
\]
Eqs. (15) and (17) lead to
\[
d^{(n)}_{M_i+M} = \omega^{(n)}_{M_i+M} - \gamma^{(n)}_{M_i+M} = \omega^{(n)}_{M_i+M} - \frac{1}{\delta^{(n)} + \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p \omega^{(n)}_M \prod_{p=1}^{M_i+1} \omega^{(n+1)}_p}.
\]
As \( \delta^{(n)} \to \infty \), from (25), it follows that \( d^{(n)}_{M_i+M} \to 0 \) in (36).

From (35), we derive \( d^{(n)}_{M_i+M} \to 0 \) for \( \forall i \in \Phi_3 \setminus \{1\} \). By combining them with (16), we get
\[
\lim_{\delta^{(n)} \to \infty} \left( \omega^{(n)}_{M_i+M} - \gamma^{(n)}_{M_i+M} \right) = 0, \quad \forall i \in \Phi_3.
\]
Moreover, from (18) and \( d^{(n)}_{M_i+M} \to 0 \) for \( \forall i \in \Phi_3 \), we have
\[
\lim_{\delta^{(n)} \to \infty} \left( \omega^{(n+1)}_{M_i+M} - \gamma^{(n)}_{M_i+M} \right) = 0, \quad \forall i \in \Phi_3.
\]
Thus, as \( \delta^{(n)} \to \infty \), (13) becomes the trivial equalities \( \omega^{(n+1)}_{M_i+M} = \omega^{(n+1)}_{M_i+M} + \gamma^{(n+1)}_{M_i+M}, \forall i \in \Phi_3 \) and \( \omega^{(n+1)}_{M_i+M} + \gamma^{(n+1)}_{M_i+M} = \omega^{(n+1)}_{M_i+M} + \gamma^{(n+1)}_{M_i+M}, \forall i \in \Phi_3 \setminus \{m-1\} \).

Next, we consider the cases except for \( k = M_i + M, \forall i \in \Phi_3 \) in the 1st and 2nd equations of (13). We here focus on the product of \( L^{(n+1)}_j \) and \( R^{(n)}_{j+1} \). The \( (i, i) \) and \( (i, i+1) \) entries of \( L^{(n+1)}_j R^{(n)}_{j+1} \) are given as, respectively,
\[
\begin{align*}
\left(L^{(n+1)}_j R^{(n)}_{j+1}\right)_{j-i} &= \omega^{(n+1)}_{M_i+j} + \gamma^{(n+1)}_{M_i+j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \\
\left(L^{(n+1)}_j R^{(n)}_{j+1}\right)_{j-i+1} &= \omega^{(n+1)}_{M_i+j} + \gamma^{(n+1)}_{M_i+j}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5.
\end{align*}
\]
Similarly, it follows that
\[
\begin{align*}
\left(R^{(n)}_j L^{(n)}_{j+1}\right)_{j-i} &= \omega^{(n)}_{M_i+j} + \gamma^{(n)}_{M_i+j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5, \\
\left(R^{(n)}_j L^{(n)}_{j+1}\right)_{j-i+1} &= \omega^{(n)}_{M_i+j} + \gamma^{(n)}_{M_i+j}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5.
\end{align*}
\]
Moreover, we give a lemma concerning the relationship of the Lax matrices $\mathcal{R}_0^{(n+1)}$ and $\mathcal{R}_M^{(n)}$ as $\delta^{(n)} \to \infty$.

**Lemma 1.** As $\delta^{(n)} \to \infty$, it holds that

$$\mathcal{R}_0^{(n+1)} = \mathcal{R}_M^{(n)}. \quad (43)$$

**Proof.** Obviously, from (37) and (38), $\omega_{M+1}^{(n+1)} \to \gamma_{M+1}^{(n)}$ and $\omega_{M+1}^{(n+1)} \to \gamma_{M+1}^{(n)}$ as $\delta^{(n)} \to \infty$. So, it holds that $\gamma_{M+1}^{(n)} \to \gamma_{M+1}^{(n)}$ as $\delta^{(n)} \to \infty$. This leads to (43). \qed

Let us introduce the matrix, given by the product of the Lax matrices $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_M$ in (23) and $\mathcal{R}_0^{(n)}$ in (24),

$$\mathcal{A}^{(n)} = \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)} \mathcal{R}_0^{(n)}. \quad (44)$$

Let us consider $\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1}$ as a similarity transformation of $\mathcal{A}^{(n)}$ by $\mathcal{R}_0^{(n)}$. Then, with the help of Theorem 2, we derive

$$\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} = \mathcal{A}^{(n+1)}. \quad (45)$$

By combining it with Lemma 1, we see that

$$\mathcal{R}_0^{(n)} \mathcal{A}^{(n)} (\mathcal{R}_0^{(n)})^{-1} = \mathcal{A}^{(n+1)}. \quad (46)$$

This means that the eigenvalues of $\mathcal{A}^{(n)}$ are invariant under the time evolution from $n$ to $n+1$. Eqs. (45) and (46) also lead to

$$\begin{cases} 
\mathcal{A}^{(n)} = (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)}) \mathcal{R}_0^{(n)}, \\
\mathcal{A}^{(n+1)} = \mathcal{R}_0^{(n)} (\mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} \cdots \mathcal{L}_M^{(n)}).
\end{cases} \quad (47)$$

Hence, we know that $\mathcal{A}^{(n+1)}$ is given through the LR transformation of $\mathcal{A}^{(n)}$. According to (45), the LR transformation in (47) coincides with $M$ times LR transformations in (22). Let us recall here that (22) is a Lax representation for the dhLV$_I$ (4) with $\delta^{(n)} \to \infty$. We therefore have the following theorem.

**Theorem 3.** The dhLV$_I$ (4) with $\delta^{(n)} \to \infty$ generates the LR transformation from $\mathcal{A}^{(n)}$ to $\mathcal{A}^{(n+1)}$ as in (47).

According to [8], the dhToda (6) satisfies the Lax representation,

$$L^{(n+M)} \mathcal{R}^{(n+1)} = \mathcal{R}^{(n)} L^{(n)}. \quad (48)$$

$$L^{(n)} = \begin{pmatrix} Q_1^{(n)} & 1 \\
& \ddots & \ddots \\
& 1 & Q_m^{(n)} \end{pmatrix}, \quad (49)$$

$$R^{(n)} = \begin{pmatrix} 1 & \cdots & 1 \\
& \ddots & \ddots \\
& & 1 \end{pmatrix}, \quad (50)$$

where $Q_i^{(n)} > 0$, $\forall i \in \Phi_4$ and $\mathcal{E}_i^{(n)} > 0$, $\forall i \in \Phi_4$. The Lax representation (48) may look different from that in [8]. Actually, we can easily get the same Lax representation as in [8] through matrix transposition on both sides of (48). Let $\mathcal{A}^{(n)}$ be the product of the Lax matrices $L^{(n)}$, $L^{(n+1)}$, …, $L^{(n+M-1)}$ in (49) and $R^{(n)}$ in (50), namely,

$$A^{(n)} = L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)} R^{(n)}. \quad (51)$$

Then, from (48), it follows that

$$R^{(n)} A^{(n)} (R^{(n)})^{-1} = R^{(n)} L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)} \quad = L^{(n+M)} L^{(n+1)} L^{(n+2)} \cdots L^{(n+M-1)} \quad \vdots \quad = L^{(n+M)} L^{(n+M+1)} L^{(n+2M-1)} R^{(n+M)} \quad = A^{(n+M)} \quad \text{(52)}$$

Obviously, from (52), the dhToda (6) gives the similarity transformation from $\mathcal{A}^{(n)}$ to $\mathcal{A}^{(n+M)}$. Eq. (52) is also rewritten as

$$\begin{cases}
\mathcal{A}^{(n+M)} = (L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)}) R^{(n)}, \\
\mathcal{A}^{(n+M)} = R^{(n)} (L^{(n)} L^{(n+1)} \cdots L^{(n+M-1)}). \quad (53)
\end{cases}$$

Thus, the dhToda (6) has a relationship with the LR transformation as follows.

**Theorem 4** ([10]). The dhToda (6) generates the LR transformation from $\mathcal{A}^{(n)}$ to $\mathcal{A}^{(n+M)}$ as in (53).

4. **Bäcklund Transformations Among the Discrete Hungry Systems**

In this section, by considering the relationship between the two LR transformations associated with the dhLV$_II$ (4) and the dhToda (6), we give a Bäcklund transformation between the dhLV$_II$ (4) and the dhToda (6). By referring to [10], we establish a Bäcklund transformation between the dhLV$_I$ (3) and the dhLV$_II$ (4). We also investigate the asymptotic behavior of the qd-type dhLV$_II$ (13) with the help of the obtained Bäcklund transformation.
4.1. The Bäcklund Transformation between the dhLV II and the dhToda

We first show the relationship of the matrices in two LR transformations associated with the dhLV II (4) and the dhToda (6).

**Lemma 2.** For some fixed $n$, let $R^{(n)} = R^{(n)}_n$ and $L^{(n+1)} = L^{(n+1)}$, $\forall j \in \Phi_5$. Then, it holds that

$$L^{(n+1)}_{j+1} = L^{(n+M)+j}, \quad R^{(n+1)}_{j+1} = R^{(n)+j+1}, \quad \forall j \in \Phi_5.$$

**Proof.** The assumption leads to $R^{(n)}_0 L^{(n)}_1 = R^{(n)}_0 n^{(n)}_1$. Let us recall that $R^{(n)}_n L^{(n)}_1 = L^{(n+1)}_1 R^{(n)}_1$ in (22) and $R^{(n)}_n L^{(n)}_1 = L^{(n+M)} R^{(n+1)}_1$ in (48). So, it follows that

$$L^{(n+1)}_1 R^{(n)}_1 = L^{(n+M)}_1 R^{(n+1)}_1.$$

Recall that the upper bidiagonal matrices $R^{(n)}_1$ and $R^{(n+1)}_1$ have 1 in every diagonal entry. Hence, by taking account of the uniqueness of LR decomposition, we get

$$L^{(n+1)}_1 = L^{(n+M)}_1, \quad R^{(n)}_1 = R^{(n+1)}_1.$$ Similarly, it is easily proved by induction for $j = 1, 2, \ldots, M - 1$ that $L^{(n+1)}_j = L^{(n+M)+j}$ and $R^{(n)}_j = R^{(n+1)}_j$.

From (44) and (51), it is obvious that $A^{(n)} = A^{(n+M)}$ since $R^{(n+1)}_j = R^{(n)+j+1}, \forall j \in \Phi_5$ and $L^{(n+1)}_j = L^{(n+M)+j}, \forall j \in \Phi_5$. In other words, the evolution from $n$ to $n+1$ of the dhLV II (4) can generate the LR transformation given by the evolution from $n$ to $n+M$ of the dhToda (6).

Let us replace $n$ with $\ell M + j$ in the superscripts of the dhLV II and the dhToda variables. Hereinafter, we consider the evolution from $\ell$ to $\ell + 1$ by the dhLV II (4) and the dhToda (6). Let us assume that, for some fixed $\ell$, we have

$$L^{(\ell+1)}_j = L^{(\ell+M)+j}, \quad R^{(\ell)}_{j+1} = R^{(\ell+M)+j+1}, \quad \forall j \in \Phi_5. \tag{54}$$

Then, from Lemma 2, it follows that,

$$L^{(\ell+M)+j} = L^{(\ell+1)+M+1}, \quad R^{(\ell+M)+j+1} = R^{(\ell+1)+M+1}, \quad \forall j \in \Phi_5. \tag{55}$$

From the 2nd equation of (55) and Lemma 1, it holds that

$$R^{(\ell+1)}_0 = R^{(\ell+1)+M}.$$ By focusing on the entries of matrices in (54), we derive

$$\begin{align*}
E^{(\ell+M)}_i &= \gamma^{(\ell)}_i, \quad \forall i \in \Phi_3, \\
Q^{(\ell+M)+j}_i &= \omega^{(\ell)}_{M+j}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5,
\end{align*}$$

for $\ell = 0, 1, \ldots$. Moreover, from (55), we obtain

$$E^{(\ell+M)+j+1}_i = \gamma^{(\ell)}_{M+j+1}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5,$$ for $\ell = 0, 1, \ldots$. To sum up, we derive a theorem on the relationship of the variables, namely, the Bäcklund transformation between the qd-type dhLV II (13) and the dhToda (6).

**Theorem 5.** A Bäcklund transformation between the qd-type dhLV II (13) with $\delta^{(\ell)} \to \infty$ and the dhToda (6) is given by

$$\begin{align*}
E^{(\ell+M)+j+1}_i &= \gamma^{(\ell)}_{M+j+1}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5, \\
Q^{(\ell+M)+j+1}_i &= \omega^{(\ell)}_{M+j+1}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5,
\end{align*}$$

for $\ell = 0, 1, \ldots$.

It is observed that (8) and (9) are the Bäcklund transformation between the qd-type dhLV II (13) and the original dhLV II (4). So, by combining it with Theorem 5, we have a main theorem in this paper.

**Theorem 6.** A Bäcklund transformation between the dhLV II (4) with $\delta^{(\ell)} \to \infty$ and the dhToda (6) is given by

$$\begin{align*}
E^{(\ell+M)+j+1}_i &= \gamma^{(\ell)}_{M+j+1}, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5, \\
Q^{(\ell+M)+j+1}_i &= \omega^{(\ell)}_{M+j+1}, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5,
\end{align*}$$

for $\ell = 0, 1, \ldots$.

4.2. The Bäcklund Transformation between the dhLV I and the dhLV II

Let us introduce the new variables

$$\begin{align*}
U^{(n)}_k &= u^{(n)}_k \prod_{p=1}^{M} \left( 1 + \delta^{(n)} u^{(n)}_{k-p} \right), \quad \forall k \in \Phi_2 \cup \{ M_m \}, \\
V^{(n)}_k &= \frac{1}{\delta^{(n)}} \prod_{p=0}^{M} \left( 1 + \delta^{(n)} v^{(n)}_{k-p} \right), \quad \forall k \in \Phi_1 \cup \{ M_m + M \},
\end{align*}$$

in terms of the dhLV I variable $u^{(n)}_k$. Then the dhLV I (3) can be rewritten as

$$\begin{align*}
U^{(n)}_{k+1} + V^{(n)}_{M+k+1} &= u^{(n)}_{k+1} + v^{(n)}_{M+k}, \quad \forall k \in \Phi_2, \\
U^{(n)}_{k+1} V^{(n)}_{k} &= U^{(n)}_{k} V^{(n)}_{k}, \quad \forall k \in \Phi_2 \cup \{ M_m \}, \\
U^{(n)}_{M_m + j+1} &= 0, \quad V^{(n)}_{M_m + j+1} := \frac{1}{\delta^{(n)}}, \quad \forall j \in \Phi_5.
\end{align*}$$

Eq. (57) is named the qd-type dhLV I in [13]. Eq. (56) is a Bäcklund transformation between the original dhLV I (3) and the qd-type dhLV I (57). Some of the authors, in [10], give a Bäcklund transformation between the qd-type
From (59) and (60), we have that
\[ \gamma \]
Note here that
\[ U_{\ell}^{(\ell)} \]
\[ Q_{i}^{(\ell+1)} \]
\[ Q_{i}^{(\ell+1)M+1} \] \( Q_{i}^{(\ell+1)M+2} \cdot \ldots \cdot Q_{i}^{(\ell+1)M+1} \)
\[ \times \left( \gamma_{M_{i}+j-1}^{(\ell)} \omega_{M_{i}+M+j}^{(\ell)} \right) \]
\[ \gamma_{M_{i}+j}^{(\ell)} \rightarrow \gamma_{M_{i}+M}^{(\ell)} \text{ as } \delta^{(n)} \rightarrow \infty. \] So, it follows that
\[ U_{M_{i}+j}^{(\ell)} = \prod_{p=0}^{M-1} \omega_{M_{i}+p+1}^{(\ell)} \gamma_{M_{i}+j}^{(\ell)}, \quad \forall i \in \Phi_{3}, \quad \forall j \in \Phi_{5}. \] (60)

From (59) and (60), we have
\[ U_{k}^{(n)} = \prod_{p=0}^{M-1} \omega_{k+p}^{(n)} \gamma_{k}^{(n)}, \quad \forall k \in \Phi_{2} \cup \{ M_{m} \}. \] (61)

By combining (61) with (8) and (56), it follows that
\[ u_{k}^{(n)} \prod_{p=0}^{M-1} \left( 1 + \delta^{(n)} u_{k-p}^{(n)} \right) \]
\[ = \prod_{p=0}^{M-1} v_{k+p}^{(n)} \left( 1 + \delta^{(n)} \prod_{r=1}^{M-1} v_{k+r-p}^{(n)} \right) \]
\[ = \prod_{p=0}^{M-1} v_{k+p}^{(n)} \prod_{r=0}^{M-1} \left( 1 + \delta^{(n)} \prod_{p=0}^{M-1} v_{k+r-p}^{(n)} \right) \] (62)

From the boundary condition of \( u_{k}^{(n)} \) and \( v_{k}^{(n)} \), the case where \( k = 1 \) in (62) leads to
\[ u_{1}^{(n)} = \prod_{p=0}^{M-1} v_{1+p}^{(n)}. \]

Similarly, by considering the cases where \( k = 2, 3, \ldots, M_{m} \), we have \( u_{k}^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)} \).

The above discussion leads to the following theorem.

**Theorem 7.** As \( \delta^{(n)} \rightarrow \infty \), a Bäcklund transformation between the dhLV (3) and the dhLVII (4) is given by
\[ u_{k}^{(n)} = \prod_{p=0}^{M-1} v_{k+p}^{(n)} \]
\[ \forall k \in \Phi_{2} \cup \{ M_{m} \}, \quad \forall n = 0, 1, \ldots. \]

4.3. The asymptotic behavior of the qd-type dhLVII variables

We next clarify the asymptotic behavior of the qd-type dhLVII variables by combining Theorem 5 with the asymptotic behavior of the dhToda variables given in [9]. Let us again replace \( n \) with \( \ell M + j \), \( \forall j \in \Phi_{5} \) in the superscript of the dhToda variable. Of course, the limit of \( \ell \rightarrow \infty \) is equivalent to that of \( n \rightarrow \infty \). A minor change of the limit in [9] brings to the following theorem with respect to the convergence of the dhToda variables.

**Theorem 8** (cf.[9]). Let \( Q_{i}^{(\ell)} > 0, Q_{i}^{(1)} > 0, \ldots, Q_{i}^{(M-1)} > 0, \forall i \in \Phi_{4} \) and \( E_{i}^{(\ell)} > 0, \forall i \in \Phi_{3} \). As \( \ell \rightarrow \infty \), the limits of \( Q_{i}^{(\ell M+j)} \) and \( E_{i}^{(\ell M+j)} \) are given by
\[ \lim_{\ell \rightarrow \infty} \prod_{p=0}^{M-1} Q_{i}^{(\ell M+p)} = C_{i}, \quad \forall i \in \Phi_{4}, \]
\[ \lim_{\ell \rightarrow \infty} E_{i}^{(\ell M+j)} = 0, \quad \forall i \in \Phi_{3}, \quad \forall j \in \Phi_{5}, \] (63)

where \( C_{i} \) is some constant and \( C_{1} \geq C_{2} \geq \cdots \geq C_{m} > 0 \). In [9], it is also shown that \( \{ Q_{i}^{(\ell M+j)} \}_{\ell=0,1,\ldots} \) is a Cauchy sequence. This implies that \( Q_{i}^{(\ell M+j)} \) converges to some constant \( C_{i,j} > 0 \) as \( \ell \rightarrow \infty \), namely,
\[ \lim_{\ell \rightarrow \infty} Q_{i}^{(\ell M+j)} = C_{i,j}, \quad \forall i \in \Phi_{4}, \quad \forall j \in \Phi_{5}. \] (64)
Since it is shown in Theorem 5 that $Q_i^{(M+j)} = \omega^{(t)}_{M+j}$, by using it in (64), we get
\[
\lim_{t \to \infty} \omega^{(t)}_{M+j} = C_i, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5.
\] (65)

By taking account that $E_i^{(M+j)} = \gamma^{(t)}_{M+j}$ shown in Theorem 5, from (63), we derive
\[
\lim_{t \to \infty} \gamma^{(t)}_{M+j} = 0, \quad \forall i \in \Phi_3, \quad \forall j \in \Phi_5.
\] (66)

It is remarkable that (65) and (66) show the convergence of $\omega^{(n)}_k$ and $\gamma^{(n)}_k$ except for $k = M_1 + M$, $\forall i \in \Phi_3$ as $t \to \infty$. We next study the convergence of $\omega^{(t)}_{M_1+M}$ and $\gamma^{(t)}_{M_1+M}$. Eqs. (37) and (66) with $j = 0$ leads to
\[
\lim_{t \to \infty} \omega^{(t)}_{M_1+M} = 0, \quad \forall i \in \Phi_3.
\] (67)

From (38) and (67), it follows that
\[
\lim_{t \to \infty} \gamma^{(t)}_{M_1+M} = 0, \quad \forall i \in \Phi_3.
\]

We summarize the asymptotic behavior of the qd-type dhLV variables as follows.

**Theorem 9.** Let us assume that $\omega^{(0)}_k > 0$, $\forall k \in \Phi_1$. Then, the limits of the qd-type dhLV variables as $\delta^{(n)} \to \infty$ are
\[
\lim_{n \to \infty} \omega^{(n)}_{M_1+j} = C_i, \quad \forall i \in \Phi_4, \quad \forall j \in \Phi_5,
\]
\[
\lim_{n \to \infty} \omega^{(n)}_{M_1+M} = 0, \quad \forall i \in \Phi_3,
\]
\[
\lim_{n \to \infty} \gamma^{(n)}_k = 0, \quad \forall k \in \Phi_2.
\]

5. Numerical Examples

In this section, we numerically observe some properties of the qd-type dhLV (13) shown in the previous sections. We easily realize numerical properties of the original dhLV (4) through those of the qd-type dhLV (13).

We first demonstrate the asymptotic behavior of the qd-type dhLV (13) shown in Theorem 9. Numerically let $\omega^{(0)}_0 = \omega^{(0)}_2 = 5$, $\omega^{(0)}_3 = \omega^{(0)}_4 = 2$, $\omega^{(0)}_6 = \omega^{(0)}_8 = 1 = M = 2$, $m = 3$, $\delta^{(n)} = 10^{12}$, respectively, in the qd-type dhLV (13). It is emphasized here that the qd-type dhLV variables, except for $\omega^{(0)}_k$, depend on the value of $\delta^{(n)}$ through $\omega^{(n)}_k$, as shown in (15). Figures 1 and 2 show the behavior of $\omega^{(n)}_k$, $k = 1, 2, 3, 4, 5, 7, 8$ and $\omega^{(n)}_k$, $k = 3, 6$, $\gamma^{(n)}_k$, $k = 1, 2, 3, 4, 5, 6$, for $n = 0, 1, \ldots, 19$, respectively.

We see from Figures 1 and 2 that, as $n$ becomes larger, $\omega^{(n)}_k$, $k = 1, 2, 4, 5, 7, 8$ and $\omega^{(n)}_k$, $k = 3, 6$ approach some positive constants and zero, respectively. This numerical result agrees with Theorem 9.

We next give a numerical example in order to confirm the Bäcklund transformation, shown in Theorem 5, between the dhLV (4) and the dhToda (6), as $\delta^{(n)} \to \infty$. Let $Q_i^{(0)} = 5$, $i = 1, 2, \ldots, 12$ and $E_i^{(0)} = 2$, $i = 1, 2, 3$ with $M = 3$ and $m = 4$ in the dhToda (6). Moreover, let
\[
Q_i^{(0)} = 5, \quad i = 1, 2, 3, 4, \quad j = 0, 1, 2, 3, 4, 5, 6 \quad (y-axis), \quad \times : \omega^{(n)}_1, \quad \circ : \omega^{(n)}_6, \quad \square : \gamma^{(n)}_1, \quad \circledast : \gamma^{(n)}_6, \quad \circledast : \gamma^{(n)}_6.
\]

Figure 1: A graph of the iteration number $n$ (x-axis) and the values of $\omega^{(n)}_1, \omega^{(n)}_2, \omega^{(n)}_3, \omega^{(n)}_4, \omega^{(n)}_5, \omega^{(n)}_6$ and $\omega^{(n)}_8$ (y-axis).

Figure 2: A graph of the iteration number $n$ (x-axis) and the values of $\omega^{(n)}_1, \omega^{(n)}_2, \omega^{(n)}_3, \omega^{(n)}_4, \omega^{(n)}_5, \omega^{(n)}_6$ and $\omega^{(n)}_8$ (y-axis). 

\[
\omega^{(0)}_{M_1+j} = 5, \quad i = 1, 2, 3, 4, \quad j = 0, 1, 2, 3, 4, 5, 6 \quad (y-axis), \quad \times : \omega^{(n)}_1, \quad \circ : \omega^{(n)}_6, \quad \square : \gamma^{(n)}_1, \quad \circledast : \gamma^{(n)}_6, \quad \circledast : \gamma^{(n)}_6.
\]

Figure 1: A graph of the iteration number $n$ (x-axis) and the values of $\omega^{(n)}_1, \omega^{(n)}_2, \omega^{(n)}_3, \omega^{(n)}_4, \omega^{(n)}_5, \omega^{(n)}_6$ and $\omega^{(n)}_8$ (y-axis).

\[
\omega^{(0)}_{M_1+M} = 2, \quad i = 1, 2, 3 \quad (x-axis) \quad \text{and} \quad \delta^{(n)} = 10^{12} \quad (y-axis).
\]

\[
\omega^{(0)}_{M_1+M} = 2, \quad i = 1, 2, 3, 4, 5, 6 \quad (y-axis).
\]
Table 1: Values of $Q_1^{(M)}, \omega_1^{(M)}$ and $E_1^{(M)}, \gamma_1^{(M)}$ in the case where $\delta^{(\ell)} = 0.5$ for $\ell = 0, 1, \ldots, 5$ and 50.

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<table>
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Table 2: Values of $Q_1^{(M)}, \omega_1^{(M)}$ and $E_1^{(M)}, \gamma_1^{(M)}$ in the case where $\delta^{(\ell)} = 10^{12}$ for $\ell = 0, 1, \ldots, 5$ and 50.

<table>
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<th>$\ell$</th>
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<th>$\omega_1^{(M)}$</th>
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<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$E_1^{(M)}$</th>
<th>$\gamma_1^{(M)}$</th>
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between $Q_1^{(M)}, E_1^{(M)}$ and $\omega_1^{(M)}, \gamma_1^{(M)}$, respectively. Tables 1 and 2 show that Theorem 5 approximately holds for a sufficiently large $\delta^{(\ell)}$.

In [9], some of the authors have proposed an algorithm for computing eigenvalues of totally nonnegative matrices, for which all the minors are nonnegative. So, from the Bäcklund transformation among the dhLV$_I$ (4), the qd-type dhLV$_I$ (13) and the dhToda (6) shown in Section 4, it is easily expected that the dhLV$_I$ (4) and the qd-type dhLV$_I$ (13) are applicable for computing the eigenvalues of a totally nonnegative matrix. In particular, $\lim_{n \to \infty} \prod_{p=0}^{M-1} \omega_{M+p}^{(n)} \gamma_{M+p} \in \Phi_1$ give the eigenvalues of the totally nonnegative matrix $A^{(n)}$.

6. Conclusion

In this paper, we first introduce the qd-type dhLV$_I$ and show the positivity of its variables. We also give a new Lax representation for the dhLV$_I$. As $\delta^{(n)} \to \infty$, it is observed that the Lax representation for the dhLV$_I$ is related to the LR transformation for a band matrix. In other words, the time evolution of the dhLV$_I$ with $\delta^{(n)} \to \infty$ corresponds to the LR transformation. We next explain how to associate the dhToda with the LR transformation. Remarkably, the dhLV$_I$ with $\delta^{(n)} \to \infty$ is associated with the same form of LR transformation associated with the dhToda. By identifying two these LR transformations, we finally obtain a Bäcklund transformation between the dhLV$_I$ and the dhToda. Additionally, through considering a Bäcklund transformation between the dhLV$_I$ and the dhToda in [10], we establish a Bäcklund transformation between the dhLV$_I$ and the dhLV$_I$ for the case of $\delta^{(n)} \to \infty$. We therefore have Bäcklund transformations among the dhLV$_I$, the dhLV$_I$ and the dhToda. The asymptotic convergence of the qd-type dhLV$_I$ variables as $n \to \infty$ through that of the dhToda. Finally, we give some numerical examples which demonstrate our theoretical results.

We give a comment that the dhLV$_I$ and the qd-type dhLV$_I$ are applicable for computing eigenvalues of a totally nonnegative matrix. A future work is to derive the Bäcklund transformations among the dhLV$_I$, the dhLV$_I$ and the dhToda in the case where $\delta^{(n)}$ is finite.

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