AN ASPECT FOR SPECTRAL ANALYSIS OF NON-SELFADJOINT OPERATORS — SCHRÖDINGER AND WAVE EQUATIONS —

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Abstract. We give an overview for spectral analysis of non-self adjoint operators from a mathematical standpoints. Among other things, we concentrate our considerations on operators that appeared for Schrödinger and wave equations. To these examples, we describe some results of the relation between spectral structure of generator and asymptotic behavior of solutions (energy decay and scattering). Roughly speaking, continuous spectra which are on the real axis effects existence of scattering states (existence of wave and scattering operators) and eigenvalues with negative imaginary part provide total energy decay of solutions.

1. Introduction.

The scene is set in *Energy space* $\mathcal{E}$, which is a Hilbert space equipped with norm $||\cdot||_{\mathcal{E}}$ induced by scalar product $(\cdot,\cdot)_{\mathcal{E}}$. In $\mathcal{E}$, the governing equation is defined, which describes motion in system. It is assumed that it has the form of ordinary differential equations in $\mathcal{E}$ of Schrödinger type:

$$i\frac{du}{dt} = Hu, \qquad u(0) = f. \tag{1.1}$$

Here H is a linear operator in $\mathcal{E}$ and $f \in \mathcal{E}$ denote initial data. For the conservative system, the generator H is self adjoint operator and the spectrum is located only on a real line $\sigma(H) \subset \mathbb{R}$. In this case, since the spectral decomposition theorem holds, a comparatively clear result is obtained. On the other hand, under a more realistic situation, the generator does not become self adjoint. Therefore, the method for the analysis of the problem is limited to us. We occasionally face the unexpected situation for the spectrum structure. For instance, there is a possibility that point spectrum appears in complex lower half-plain, too.

As suggested in the linear algebra, the spectrum decides the behavior of solutions $u(t)$, where $u(t)$ is given at least formally in the form

$$u(t) = e^{-itH}f.$$

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In self adjoint case, spectral resolution theorem means that

$$u(t) = e^{-itH}f = \int_{\mathbb{R}} e^{-i\lambda t} dP_H(\lambda) f$$

in which $P_H(\lambda)$ denotes a spectral projection associated with H. Since energy conservation law holds in this case, the total energy of solutions at time t, $||u(t)||_{\mathcal{E}}$ is the same as it at initial time. This is consequence of the fact that spectrum lie only on the real line. On the other hand, non-self adjoint case, energy conservation law does not hold. For instance, let's think about the case where H is *maximal dissipative operator.* In this case, the spectrum has the possibility of appearing on the complex lower half-plane and on the real line $\sigma(H) \subset \mathbb{C}_- \cup \mathbb{R}$. If the point spectrum appear in $\mathbb{C}_-$;

$$Hf = zf \qquad z = z_{\Re} + iz_{\Im} \qquad z_{\Im} < 0,$$

then the total energy of solutions at time t decays as t goes to infinity:

$$u(t) = e^{-itH}f = e^{-itz}f = e^{z_{\Im}} \cdot e^{-iz_{\Re}t}f \to 0 \quad \text{as} \quad t \to \infty.$$

Mathematical picture is given as mentioned above. Here we shall point out the reason to analyze non-self adjoint problems, compared with self adjoint one:

1 : We want to handle a more realistic situation. The energy-preserving system does not exist in the actual world. In this situation, some frictional forces inevitably appear in the movement. This means the operator to consider becomes non-self adjoint.

2 : The tool of the analysis is insufficient. As is stated in the above, the spectral decomposition theorem does not hold. As for the rest, spectral structure remains in general incompletely understood. Under the circumstances it is difficult to foresee the motion in the future.

In the following, we shall explain some results obtained in the last few years.

Contents of a present paper are outlined as follows. In section 2, it will take a general view of some basic facts in functional analysis and spectral theory. In section 3, the Schrödinger equation with a dissipative perturbation term is treated. Since the perturbation term contains Dirac delta function we can clarify the spectrum structure by explicit calculations. It is especially understood that one eigenvalue appears in complex lower half-plain. It is shown that *wave* and *scattering operators* exist. This means the solution becomes asymptotically free. By the use of the wave operator *generalized Fourier transform* is composed, and, as a result, *generalized Parseval identity* is shown. Therefore, the classification of the asymptotic behavior of solutions by initial data becomes possible because of the relation to the spectrum. In section 4, we shall consider wave equations with some dissipations. Firstly, we consider that equation with more general rank one dissipation. For this, we cannot obtain a clear result for the spectrum structure. However, if we follow the method of section 2, we can reach the same kind of conclusions. For very special Coulomb type dissipation, we can explicitly solve the eigenvalue (stationary) problem to construct solutions of original time-dependent equations. Hole complex lower half-plains are covered with the eigenvalue, and the total energy decays exponentially. In section 5, we shall present the result concerning wave equation with dissipations in layered media. Though it does not make clear the spectrum structure, asymptotic behavior of solutions (decay and scattering) can be classified according to the condition of dissipation. In the final section 6, the reference literature is given and supplemented for not having been described up to now.

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2. Basic facts.

In this section, it describes some basic facts that will appear later without proofs. For readers who want to know in more details, please refer P. D. Hislop and I. M. Sigal [3] and Z. Yoshida [29] for examples. As for the spectral representation, see K. Mochizuki [20].

A *Hilbert space* $\mathcal{H}$ is a complete metric space with respect to a norm $||f||_{\mathcal{H}} = \sqrt{(f,f)_{\mathcal{H}}}$ induced by an inner product $(f,g)_{\mathcal{H}}$ for any f, $g \in \mathcal{H}$. Here, "complete" means that every Cauchy sequences in $\mathcal{H}$ converges to an element of $\mathcal{H}$. A *Banach space* is a complete metric space with respect to the norm $||f||$ in which inner product is not defined. For example, the *Lebesgue space* $L^2(X)$ is a Hilbert space with norm $||f||_{L^2(X)}$ and inner product

$$(f,g)_{L^2(X)} = \int_X f(x)\overline{g(x)}dx,$$

where $\overline{g}$ denotes the complex conjugate of g and $X \subseteq \mathbb{R}^N$ $(x = (x_1, \cdots, x_n) \in X)$. The *Sobolev space* $H^m(X)$ $(m = 0, 1, 2 \cdots)$ is also a Hilbert space with norm $||f||_{H^m(X)}$ and inner product

$$(f,g)_{H^m(X)} = \sum_{0 \le |\alpha| \le m} (D^\alpha f, D^\alpha g)_{L^2},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi index, $|\alpha| = \sum_{j=1}^n \alpha_j$, $D_j = \partial/\partial x_j$, $D^\alpha = D_1 \cdots D_n$ denotes a differential operator of order $|\alpha|$. Note that the following inclusion relation holds

$$\cdots \subset H^2(X) \subset H^1(X) \subset H^0(X) = L^2(X).$$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. A *linear operator* $A : \mathcal{H}_1 \to \mathcal{H}_2$ is consist of two objects: (i) a linear subspace $\mathcal{D}(A) \subset \mathcal{H}_1$ which is called the domain of A; (ii) a linear map $A : \mathcal{D}(A) \to \mathcal{H}_2$. For example, the *Laplacian* $A = -\Delta = -\sum_{j=1}^n \partial^2/\partial x_j^2$ with $\mathcal{D}(A) = H^2(\mathbb{R}^N)$ is a linear operator on $L^2(\mathbb{R}^n)$ (i.e., $-\Delta : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^n)$). The *kernel* of operator A is defined by $\ker A = \{f \in D(A) | Af = 0\}$. If $\ker A = \{0\}$, then the *inverse operator* $A^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ can be defined. An operator A is said to be *bounded* (or *continuous*), if there exists a constant $C > 0$ such that $||Af||_{\mathcal{H}_2} \le C||f||_{\mathcal{H}_1}$

holds for any $f \in \mathcal{D}(A)$. If each $f \in \mathcal{D}(A)$ is the limit of a sequence of elements of $\mathcal{H}_1$, $\mathcal{D}(A)$ is said to be *dense* in $\mathcal{H}_1$ and the operator A is called *densely defined.* An operator A is said to be *closed* if $f_n \in \mathcal{D}(A)$, $f_n \to 0$ in $\mathcal{H}_1$ and $Af_n \to g$ in $\mathcal{H}_2$ imply $f \in \mathcal{D}(A)$ and $Af = g$.

Let A be a linear operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. The *resolvent set* $\rho(A)$ of A is defined by

$$\rho(A) = \left\{ z \in \mathbb{C} \,\middle|\, \exists (A - zI)^{-1} : \text{bounded}, \ \mathcal{D}(A) \subset \mathcal{H} : \text{dense} \right\}.$$

If $z \in \rho(A)$, the operator function $R(z) = (A - zI)^{-1}$ can be defined and is called the *resolvent* of A. The *spectrum* of A denoted by $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$: $\sigma(A) = \mathbb{C} \setminus \rho(A)$. If A is a closed linear operator and $z \notin \sigma(A)$, then $\mathcal{D}(R(z)) = \mathcal{H}$ holds. Moreover, $\rho(A)$ is open set in $\mathbb{C}$, so $\sigma(A)$ is closed in $\mathbb{C}$. The spectrum $\sigma(A)$ is classified as follows:

$$\begin{aligned}
\sigma_p(A) &= \left\{ z \in \sigma(A) \,\middle|\, \ker(A - zI) \neq \{0\} \right\}, \\
\sigma_r(A) &= \left\{ z \in \sigma(A) \,\middle|\, \ker(A - zI) = \{0\}, \mathcal{D}(R(z)) \subset \mathcal{H} : \text{not dense} \right\}, \\
\sigma_c(A) &= \left\{ z \in \sigma(A) \,\middle|\, \ker(A - zI) = \{0\}, \mathcal{D}(R(z)) \subset \mathcal{H} : \text{dense}, R(z) : \text{unbounded} \right\}.
\end{aligned}$$

These are in turn called the point spectrum (set of eigenvalue), the residual spectrum and the continuous spectrum of A, respectively. By this definition, we have

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A), \quad \sigma_p(A) \cap \sigma_r(A) \cap \sigma_c(A) = \emptyset.$$

Another classification is given as follows:

$$\begin{aligned}
\sigma_d(A) &= \left\{ z \in \sigma_p(A) \,\middle|\, z \text{ is isolated points of finite multiplicity} \right\}, \\
\sigma_{ess}(A) &= \sigma(A) \setminus \sigma_d(A).
\end{aligned}$$

These are in sequence called the discrete spectrum and the essential spectrum of A, respectively. The set $\sigma_{ess}(A)$ consists of the continuous spectrum of A, the accumulation points of the point spectrum of A and eigenvalues of A of infinite multiplicities.

If A is a linear operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$ then the *adjoint operator* A^* is defined by the condition that $(Af, g)_{\mathcal{H}} = (f, A^*g)_{\mathcal{H}}$ for any $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(A^*)$, where

$$\mathcal{D}(A^*) = \left\{ g \in \mathcal{H} \,\middle|\, \exists h \in \mathcal{H} \text{ such that } (Af, g)_{\mathcal{H}} = (f, h)_{\mathcal{H}} \text{ for any } f \in \mathcal{D}(A) \right\}.$$

Then we find that such h is unique and we define $A^*g = h$. If A is a closed linear operator with dense domain then A^* is also a closed linear operator with dense domain. If $Af = A^*f$ for any $f \in \mathcal{D}(A) \subset \mathcal{D}(A^*)$, then A is said to be *symmetric.* If $Af = A^*f$ for any $f \in \mathcal{D}(A) = \mathcal{D}(A^*)$, then A is *self adjoint.* In this case, we have $\sigma(A) \subset \mathbb{R}$

and $\sigma_r(A) = \emptyset$. For example, Laplacian $-\Delta$ with $\mathcal{D}(-\Delta) = H^2(\mathbb{R}^n)$ is self adjoint and the spectral structure is given by $\sigma(-\Delta) = \sigma_c(-\Delta) = \sigma_{ess}(-\Delta) = [0,\infty)$, $\sigma_p(-\Delta) = \sigma_r(-\Delta) = \sigma_d(-\Delta) = \emptyset$ and $\rho(-\Delta) = \mathbb{C} \setminus [0,\infty)$.

Spectral decomposition theorem means that if A is a self adjoint operator with a domain $\mathcal{D}(A)$ in $\mathcal{H}$, then a uniquely determined projection operator family $P(\lambda)$ exists such that

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H} \,\middle|\, \int_{\sigma(A)} \lambda^2 d||P(\lambda)f||^2_{\mathcal{H}} < \infty \right\},$$

$$\forall f \in D(A),\ \forall g \in \mathcal{H},\ (Af,g)_{\mathcal{H}} = \int_{\sigma(A)} \lambda d(P(\lambda)f,g)_{\mathcal{H}}. \quad \left(A = \int_{\sigma(A)} \lambda dP(\lambda).\right)$$

The *Fourier transform* of $f^{(*_1)}$ denoted by $\hat{f}$ is the projection on $(2\pi)^{-n/2}e^{ix\cdot\xi}$:

$$\hat{f}(\xi) = \Big(f(x), (2\pi)^{-n/2}e^{ix\cdot\xi}\Big)^{(*_2)}_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx.$$

where $x,\ \xi \in \mathbb{R}^n$, $x\cdot\xi = \sum_{i=1}^n x_i\xi_i$. We also denote it by $\mathcal{F}_0 f^{(*_3)}$. The inverse Fourier transform has the similar form$^{(*_1)}$

$$g(x) = \left(\mathcal{F}_0{}^{-1}\hat{g}\right)(x) = \Big(\hat{g}(\xi), (2\pi)^{-n/2}e^{-ix\cdot\xi}\Big)^{(*_2)}_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g(\xi)}d\xi.$$

We shall describe the *spectral representation of* $-\Delta$ here. We define the bounded operator $\mathcal{F}_0(\sigma) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ by

$$\left[\mathcal{F}_0(\sigma)f\right](\omega_\xi) = \sigma^{(n-1)/2}\hat{f}(\sigma\omega_\xi),$$

where $\xi = \sigma\omega_\xi$ ($\sigma = |\xi|$, $\omega_\xi \in \mathbb{S}^{n-1}$) in polar coordinate. This is continuous with respect to σ. Its adjoint operator $\mathcal{F}_0^*(\sigma) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ is given by

$$\left[\mathcal{F}_0^*(\sigma)h\right](x) = \sigma^{(n-1)/2}(2\pi)^{-n/2}\int_{\mathbb{S}^{n-1}} e^{i\sigma x\cdot\omega_\xi}h(\omega_\xi)dS_{\omega_\xi}.$$

Using this, we can define

$$\left[\mathcal{F}_0 f\right](\sigma,\omega_\xi) = \left[\mathcal{F}_0(\sigma)f\right](\omega_\xi)$$

$^{(*_1)}$ In this case, we assume that the support of f or $\hat{g}$ is compact. By the *Parseval identity* which is stated later, we may assume $f,\ \hat{g} \in L^2$.

$^{(*_2)}$ Here, to give priority to comprehensible, we abuse the notation to write down variable x or ξ in inner-product.

$^{(*_3)}$ Subscript 0 means *free* . In later, we introduce *generalized Fourier transform.*

for any $f \in L^2(\mathbb{R}^n)$. Then $[\mathcal{F}_0 f](\sigma, \omega_\xi) \in L^2([0,\infty); L^2(\mathbb{S}^{n-1}))^{(*_4)}$ holds and we find that

$$[\mathcal{F}_0(-\Delta) f](\sigma\omega_\xi) = \sigma^2 [\mathcal{F}_0 f](\sigma, \omega_\xi),$$

which is the spectral representation of $-\Delta$. Therefor the *Parseval identity* is given by

$$(f,g)_{L^2(\mathbb{R}^n_x)} = (\mathcal{F}_0 f, \mathcal{F}_0 g)_{L^2(\mathbb{R}^n_\xi)} = \int_{\sigma(-\Delta)} \Big(\mathcal{F}_0(\sigma) f, \mathcal{F}_0(\sigma) g\Big)_{L^2(\mathbb{S}^{n-1})} d\sigma. \tag{2.1}$$

3. Schrödinger equations.

In this section, we consider the Schrödinger equation with dissipative perturbation of rank one [5]:

$$i\frac{du}{dt} = Hu, \quad u(0) = f \ \in \ \mathcal{E} = L^2(\mathbb{R}), \tag{3.1}$$

where

$$H = H_\alpha = H_0 + \alpha(\cdot, \delta)_{\mathcal{E}}^{(*_5)} \delta = -\frac{d^2}{dx^2} + \alpha(\cdot, \delta)_{\mathcal{E}} \delta.$$

Here δ is Dirac delta function and $\alpha = \alpha_\Re + i\alpha_\Im$ with $\alpha_\Re$, $\alpha_\Im \le 0$. The domain of the operator H_α is defined as follows;

$$\mathcal{D}(H_\alpha) = \Bigg\{ U = u + aH_0(H_0^2+1)^{-1}\delta \,\Bigg|\, u \in \mathcal{H}^2,\ a \in \mathbb{C},$$

$$(u,\delta)_{\mathcal{E}} = -a\Big(\alpha^{-1} + (\delta, H_0(H_0^2+1)^{-1}\delta)_{\mathcal{E}}\Big) \Bigg\},$$

where

$$\mathcal{H}^s = \left\{ f \,\middle|\, ||f||^2_{\mathcal{H}^s} = \int_{\mathbb{R}^1} (1+|\xi|^2)^s \big|[\mathcal{F}_0 f](\xi)\big|^2 d\xi < \infty \right\}$$

for $s \in \mathbb{R}$. Then the operator H_α with $\alpha_\Im < 0$ is maximal dissipative operator, i.e., $\Im(H_\alpha f, f)_{\mathcal{E}} \le 0$ for any $f \in \mathcal{D}(H_\alpha)$ and $\mathcal{R}(H_\alpha - i)^{(*_6)} = \mathcal{E}$ hold. Thus H_α with $\alpha_\Im < 0$ generates a contraction semi-group $\{e^{-itH_\alpha}\}_{t\ge0}$.

Now we shall start to state on spectral structure of H_α.

$^{(*_4)}$ $f(\sigma,\omega_\xi) \in L^2\big([0,\infty); L^2(\mathbb{S}^{n-1})\big) \Leftrightarrow ||f(\cdot,\cdot)||^2_{L^2([0,\infty);L^2(\mathbb{S}^{n-1}))} = \int_0^\infty ||f(\sigma,\cdot)||^2_{L^2(\mathbb{S}^{n-1})} d\sigma = \int_0^\infty \Big(\int_{\mathbb{S}^{n-1}} |f(\sigma,\omega_\xi)|^2 dS_{\omega_\xi}\Big) d\sigma < \infty$.

$^{(*_5)}$ Since $\delta \in H^{-1}(\mathbb{R})$; the dual space of $H^1(\mathbb{R})$, $(\cdot,\delta)_{\mathcal{E}}$ is the one to be interpreted as the dual couplings of H^1 and H^{-1}.

$^{(*_6)}$ For operator A, $\mathcal{R}(A)$ denotes the range of the operator.

Theorem 3.1. *The following assertions hold:*

(1) $$\sigma(H_\alpha) = \begin{cases} [0,\infty) \cup \{-\alpha^2/4\} & (\alpha_{\Re} < 0), \\ [0,\infty) \quad (\alpha_{\Re} = 0). \end{cases}$$

(2) $$\sigma_{ess}(H_\alpha) = \sigma_c(H_\alpha) = [0,\infty), \quad \sigma_r(H_\alpha) = \emptyset.$$

(3) $$\sigma_p(H_\alpha) = \begin{cases} \sigma_d(H_\alpha) = \{-\alpha^2/4\} & (\alpha_{\Re} < 0), \\ \emptyset \quad (\alpha_{\Re} = 0). \end{cases}$$

(4) *The projection with respect to the eigenvalue* $-\alpha^2/4$ $(\alpha_{\Re} \neq 0)$ *is given by*

$$P_{-\alpha^2/4} f = -\alpha/2 (f, e^{(\overline{\alpha}|\cdot|)/2})_{L^2(\mathbb{R})} e^{(\alpha|x|)/2}.$$

These are obtained from the explicit representation of the resolvent

$$R(z) = R_0(z)f + \int_{\mathbb{R}} K(x,y;z) f(y)dy,$$
$$K(x,y;z) = -\frac{\alpha}{2i\sqrt{z}(2i\sqrt{z}-\alpha)} e^{i\sqrt{z}(|x|+|y|)} \in L^2(\mathbb{R}_x \times \mathbb{R}_y)$$

($\Im z > 0$, $R(z) = (H_\alpha - z)^{-1}$, $R_0(z) = (H_0 - z)^{-1}$) and the residual theorem. The existence of wave operator is proved by essentially Enss methods (time-dependent method) (V. Enss [2], B. Simon [27], S.T. Kuroda [14] and P.A. Perry [24]. See also M. Kadowaki [4]):

Theorem 3.2. *Assume* $\alpha_{\Re} \leq 0$ *and* $\alpha_{\Im} < 0$. *Then the wave operator*

$$W(\alpha) = s\text{-}\lim_{t \to \infty} e^{itH_0} e^{-itH_\alpha} \,{}^{(*7)}$$

exists as non-trivial operator in $\mathcal{E}$.

By these two theorems, we find

Corollary 3.3. *Assume that* $\alpha_{\Re}$ *and* $\alpha_{\Im} < 0$. *Then*

$$\mathcal{R}(P_{-\alpha^2/4}) \subset \ker W(\alpha) = \Big\{ f \,\Big|\, \lim_{t\to+\infty} ||e^{-itH_\alpha} f||_{\mathcal{E}} = 0 \Big\}.$$

Now we shall state the *spectral representation of* H_α, where we need the *generalized Fourier transform* for it:

(*7) s-lim means the *strong limit* : $||e^{itH_0}e^{-itH}f - W(\alpha)f||_{\mathcal{E}} \to 0$ as $t \to \infty$.

Proposition 3.4. *Assume $\alpha_{\Re} \le 0$ and $\alpha_{\Im} < 0$ and define the generalized Fourier transform $\mathcal{F}_\alpha$ by*

$$\mathcal{F}_\alpha = \mathcal{F}_0 W(\alpha).$$

Its representation is given by

$$[\mathcal{F}_\alpha f](\xi) = \lim_{R\to+\infty} \int_{|x|<R} \overline{\varphi_\alpha(x,\xi)} f(x)dx \qquad in \quad \mathcal{E},$$

where

$$\overline{\varphi_\alpha(x,\xi)} = (2\pi)^{-1/2}\left(e^{-ix\xi} + \frac{\alpha}{(2i|\xi|-\alpha)} e^{i|x||\xi|}\right).$$

The spectral representation of H_α is given by this operator as follows:

$$[\mathcal{F}_\alpha H_\alpha f](\xi) = |\xi|^2 [\mathcal{F}_\alpha f](\xi) \qquad for \quad f \in \mathcal{D}(H_\alpha).$$

To show this, we may calculate $(W(\alpha)f, g)_{\mathcal{E}}$. The standard arguments in the stationary scattering theory as in S.T. Kuroda [15] and the property of Poisson integrals means Proposition 3.4. Now we describe the *generalized Parseval formula* (cf. B.S. Pavlov [23], Theorem 2.1).

Proposition 3.5. *Assume that $f, g \in \mathcal{E} \cap L^1(\mathbb{R})$.*

(1) *If $\alpha_{\Re}$, $\alpha_{\Im} < 0$, then we have*

$$(\mathcal{F}_\alpha f, \mathcal{F}_{\overline{\alpha}} g)_{\mathcal{E}} = (f,g)_{\mathcal{E}} + \frac{\alpha}{2}(f, e^{(\overline{\alpha}|\cdot|)/2})_{\mathcal{E}} (e^{(\alpha|\cdot|)/2}, g)_{\mathcal{E}}. \tag{3.2}$$

(2) *If $\alpha_{\Re} = 0$, $\alpha_{\Im} < 0$, then we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} (\mathcal{F}_{i\alpha_{\Im}} f, \chi_\varepsilon \mathcal{F}_{-i\alpha_{\Im}} g)_{\mathcal{E}} \\ &= (f,g)_{\mathcal{E}} + \frac{i\alpha_{\Im}}{4} \int_{\mathbb{R}} e^{(i\alpha_{\Im}/2)|x|} f(x)dx \int_{\mathbb{R}} e^{(i\alpha_{\Im}/2)|y|} \overline{g(y)} dy, \end{aligned} \tag{3.3}$$

where χ_ε is the characteristic function on $\left\{\xi \in \mathbb{R}_\xi \,\middle|\, a \le \left||\xi| + \alpha_{\Im}/2\right|\right\}$ for $a > 0$.

Remark 3.6. Using Theorem 3.1 (4), we find (3.2) is the same as

$$(f,g)_{\mathcal{E}} = (\mathcal{F}_\alpha f, \mathcal{F}_{\overline{\alpha}} g)_{\mathcal{E}} + (P_{-\alpha^2/4} f, g)_{\mathcal{E}}.$$

This is generalization of usual Parseval formula (2.1).

Now we shall state main theorem in this section.

Theorem 3.7.

(1) *If* $\alpha_{\Re}$, $\alpha_{\Im} < 0$, *then* $\ker W(\alpha) = \mathcal{R}(P_{-\alpha^2/4})$ *holds.*
(2) *If* $\alpha_{\Re} = 0$, $\alpha_{\Im} < 0$, *then* $\ker W(i\alpha_{\Im}) = \{0\}$ *holds.*

To explain the concrete meaning of this, we note

$$\ker P_{-\alpha^2/4} + \mathcal{R}(P_{-\alpha^2/4}) = \mathcal{E},$$
$$\ker P_{-\alpha^2/4} \cap \mathcal{R}(P_{-\alpha^2/4}) = \{0\}$$

(M. Reed - B. Simon [26], Theorem XII.5). Thus for each $f \in \mathcal{E}$, unique decomposition holds:

$$f = f_s + f_d^{(*8)}, \tag{3.4}$$

where

$$f_s \equiv f - P_{-\alpha^2/4} f \in \ker P_{-\alpha^2/4},$$
$$f_d \equiv P_{-\alpha^2/4} f \in \mathcal{R}(P_{-\alpha^2/4}).$$

Corollary 3.8 (The classification of asymptotics by the initial data).

(1) *If* $\alpha_{\Re}$, $\alpha_{\Im} < 0$, *then for each* $f \in \mathcal{E}$ *decomposed as in* (3.4), *we have*

(S) *The following two assertions are equivalent to each other:*

(i) $f_s \neq 0$,

(ii) $\lim_{t\to\infty} ||e^{-itH_\alpha} f - e^{-itH_0} W(\alpha) f||_{\mathcal{E}} = 0$ *with* $W(\alpha) f \neq 0$.

(D) *The following two assertions are equivalent to each other:*

(i) $f_s = 0$,

(ii) $\lim_{t\to\infty} ||e^{-itH_\alpha} f||_{\mathcal{E}} = 0$ $(e^{-itH_\alpha} f = e^{i(\alpha^2/4)t} f_d)$.

(2) *If* $\alpha_{\Re} = 0$, $\alpha_{\Im} < 0$, *then the following two assertions are equivalent to each other:*

(i) $f \in \mathcal{E}$ *and* $f \neq 0$,

(ii) $\lim_{t\to\infty} ||e^{-itH_{i\alpha_{\Im}}} f - e^{-itH_0} W(i\alpha_{\Im}) f||_{\mathcal{E}} = 0$ *with* $W(i\alpha_{\Im}) f \neq 0$.

(*8) Subscript "s" and "d" mean *scattering* and *decay*, respectively.

[*Proof of Theorem 3.7*]. (1) We have only to show the equivalence of $W(\alpha)f = 0$ and $f_s = 0$. Corollary 3.3 means

$$W(\alpha)f = W(\alpha)f_s. \tag{3.5}$$

Note that $f \in \ker P_{-\alpha^2/4}$ is equivalent to $(f, e^{(\overline{\alpha}|\cdot|)/2})_{\mathcal{E}} = 0$. This and (3.2) with a density argument give

$$\big(W(\alpha)f_s, W(\overline{\alpha})f_s\big)_{\mathcal{E}} = (\mathcal{F}_\alpha f_s, \mathcal{F}_{\overline{\alpha}} f_s)_{\mathcal{E}} = ||f_s||_{\mathcal{E}}^2.$$

This and (3.5) imply the desired conclusion above. □
(2) It suffices to show that $W(i\alpha_{\Im})f = 0$ imply $f = 0$. By the definition of $\mathcal{F}_\alpha$ in Proposition 3.4, we may assume $\mathcal{F}_{i\alpha_{\Im}} f = 0$. Then Proposition 3.5 (2) and the relation

$$(\mathcal{F}_{i\alpha_{\Im}} f, \mathcal{F}_{-i\alpha_{\Im}} g)_{\mathcal{E}} = (f, g)_{\mathcal{E}}$$

provide $(f, g)_{\mathcal{E}} = 0$ for any $g \in \mathcal{E} \cap L^1(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} |x||g(x)|dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} e^{-(i\alpha_{\Im}/2)|x|} g(x)dx = 0.$$

Since the space of whole such function g is dense in $\mathcal{E}$, we obtain $f = 0$. □

4. Wave equations.

The method similar to the preceding section is also applicable to the case of wave equation with some rank one dissipations, which is given by the following [8]:

$$\partial_t^2 w(x,t) - \partial_x^2 w(x,t) + (\partial_t w(\cdot,t), \psi(\cdot))_{L^2(\mathbb{R})}\psi(x) = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \tag{4.1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\psi \in L_s^2(\mathbb{R})$ with $s > 1/2$. Here, $L_s^2(\mathbb{R})$ is weighted L^2-space defined as

$$L_s^2(\mathbb{R}) = \Big\{ f(x) \,\Big|\, ||f||_s < \infty \Big\}, \quad ||f||_s^2 = \int_{\mathbb{R}} \big(1 + |x|^2\big)^s |f(x)|^2 dx$$

for $s \in \mathbb{R}$. We deal with (4.1) as a perturbed system of

$$\partial_t^2 w(x,t) - \partial_x^2 w(x,t) = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty). \tag{4.2}$$

These equations (4.1) and (4.2) are able to reduced to the ordinary differential equation (1.1) in the energy space $\mathcal{E}$ as follows. Here, the energy space $\mathcal{E} = \mathcal{E}(\mathbb{R})$ is a Hilbert space associated with energy conservation law, its inner product is given by

$$\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{E}} = \int_{\mathbb{R}} \Big\{ \partial_x f_1(x) \overline{\partial_x g_1(x)} + f_2(x) \overline{g_2(x)} \Big\} dx.$$

The norm derived from this is denoted by $||\cdot||_{\mathcal{E}}$. For equation (4.1), perturbed operator H is defined by

$$H = i\begin{pmatrix} 0 & 1 \\ \partial_x^2 & -(\cdot,\psi)_{L^2(\mathbb{R})}\psi \end{pmatrix}$$

with domain

$$\mathcal{D}(H) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E} \,\middle|\, \partial_x^2 f_1 \in L^2(\mathbb{R}),\, f_2 \in H^1(\mathbb{R}) \right\}.$$

Similarly, for equation (4.2), unperturbed operator H_0 is defined by

$$H_0 = i\begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}$$

with the same domain $\mathcal{D}(H_0) = \mathcal{D}(H)$. It is well-known that $\sigma(H_0) = \sigma_c(H_0) = \sigma_{ess}(H_0) = \mathbb{R}$ and $\sigma_p(H_0) = \sigma_r(H_0) = \emptyset$ hold. Since H is maximal dissipative and H_0 is self adjoint in $\mathcal{E}$, we find by M. Reed and B, Simon [25] Theorem X-50 that H and H_0 generate a contraction semi-group $\{e^{-itH}\}$ and unitary group $\{e^{-itH_0}\}$, respectively. Therefore, the next theorem holds good by a reason like Theorem 3.2.

Theorem 4.1. *The following statements hold.*

(1) $\sigma_p(H) \cap \mathbb{R} = \emptyset$.

(2) *The wave operator W exists as a non-trivial operator in $\mathcal{E}$:*

$$W = s\text{-}\lim_{t\to\infty} e^{itH_0}e^{-itH}.$$

Since the function ψ does not clear, more detailed spectrum structure of H less becomes clear. From the above theorem, the problem similar to the former section arises. To answer this, we assume that the function ψ satisfies

$$\begin{aligned} &(A1) \qquad \psi \in L^2_{s+1}(\mathbb{R}) \quad \text{with} \quad s > 1/2, \\ &(A2) \qquad \Psi(\alpha) \le \Psi(\beta) \quad \text{if} \quad 0 \le \alpha \le \beta, \end{aligned}$$

where

$$\Psi(\alpha) = |\hat{\psi}(\alpha)|^2 + |\hat{\psi}(-\alpha)|^2.$$

For example, $\psi(x) = e^{-|x|^2/2}$ satisfies $(A1)$ and $(A2)$ since $\hat{\psi}(\alpha) = e^{-|\alpha|^2/2}$. These assumptions are the condition to guarantee that the singularity of the resolvent of H is simple. Then we find out that the spectral structure depends on the size of ψ as follows.

Theorem 4.2. *Under the assumption on* ψ *as above,*

$$\sigma(H) \cap \mathbb{C}_- = \begin{cases} \emptyset & \left(\left|\int_{\mathbb{R}} \psi(x)dx\right| \le \sqrt{2}\right), \\ \{i\kappa_0\} & \left(\left|\int_{\mathbb{R}} \psi(x)dx\right| > \sqrt{2}\right) \end{cases}$$

for some $\kappa_0 < 0$. *Moreover,* $i\kappa_0$ *is an eigenvalue and its multiplicity is one.*

In the above example, the function $\psi(x) = \varepsilon e^{-|x|^2/2}$ satisfies $|\int_{\mathbb{R}} \psi(x)dx| = \varepsilon\sqrt{2\pi}$. Thus, only one eigenvalue appears in case of $\varepsilon > 1/\sqrt{\pi}$, but also it does not appear at all in case of $\varepsilon \le 1/\sqrt{\pi}$.

If the eigenvalue exists, we can define a projection $P_{i\kappa_0}$ with respect to this eigenvalue $i\kappa_0$ as follows:

$$(P_{i\kappa_0} f, g)_{\mathcal{E}} = -\frac{1}{2\pi i}\int_{\Gamma} (R(z)f, g)_{\mathcal{E}} dz \quad \text{for any} \quad f, g \in \mathcal{E},$$

where $R(z) = (H - z)^{-1}$ is the resolvent of H and $\Gamma(\subset \mathbb{C}_-)$ is a closed curve enclosed $i\kappa_0$. The claim corresponding to Corollary 3.3 is given as follows:

Corollary 4.3. $\mathcal{R}(P) \subset \ker W$ *holds.*

Now we shall state the construction of spectral representation for the free (unperturbed) operator H_0.

Proposition 4.4. *For* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ *and* $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{E}$, *we define operators* $\mathcal{F}_0$ *by*

$$(\mathcal{F}_0 f)(\lambda) = \begin{cases} \begin{pmatrix} \dfrac{\lambda \hat{f}_1(\lambda) + i\hat{f}_2(\lambda)}{\sqrt{2}} \\ \dfrac{\lambda \hat{f}_1(-\lambda) + i\hat{f}_2(-\lambda)}{\sqrt{2}} \end{pmatrix} & (\lambda > 0), \\ \begin{pmatrix} \dfrac{-\lambda \hat{f}_1(-\lambda) - i\hat{f}_2(-\lambda)}{\sqrt{2}} \\ \dfrac{-\lambda \hat{f}_1(\lambda) - i\hat{f}_2(\lambda)}{\sqrt{2}} \end{pmatrix} & (\lambda < 0). \end{cases}$$

Then

(1) $\mathcal{F}_0$ *is extended*[(*9)] *to a unitary operator from* $\mathcal{E}$ *onto* $L^2(\mathbb{R}; \mathbb{C}^2)$.

(2) *For any* $f \in \mathcal{D}(H_0)$ *and* $g \in \mathcal{E}$,

$$(H_0 f, g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \lambda \Big((\mathcal{F}_0 f)(\lambda), (\mathcal{F}_0 g)(\lambda) \Big)_{\mathbb{C}^2} d\lambda$$

holds, where $(\cdot,\cdot)_{\mathbb{C}^2}$ *denotes usual inner-product in* $\mathbb{C}^2$.

[(*9)] $\mathcal{F}_0$ is originally defined from a slightly narrower space than $\mathcal{E}$ to a slightly larger space than this (these are some kinds of weighted energy space associated with $\mathcal{E}$). The possibility of this extension follows from the boundedness of $\mathcal{F}_0$ in $\mathcal{E}$.

We call $\mathcal{F}_0$ the spectral representation for H_0. As for the spectral representation of perturbed operator H, we have the following proposition.

Proposition 4.5. *Define two operators $\mathcal{F}$ and $\mathcal{G}$ by*

$$(\mathcal{F}f)(\lambda) = (\mathcal{F}_0 f)(\lambda) + \frac{i\big(f, v(\lambda - i0)\big)_{\mathcal{E}}}{\Gamma(\lambda + i0)} \left(\mathcal{F}_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right)(\lambda),$$

$$(\mathcal{G}f)(\lambda) = (\mathcal{F}_0 f)(\lambda) - \frac{i\big(f, v(\lambda - i0)\big)_{\mathcal{E}}}{\overline{\Gamma(\lambda - i0)}} \left(\mathcal{F}_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right)(\lambda),$$

where $\Gamma(z)$ is the term appeared in the representation of perturbed resolvent $R(z)$ of H as follows:

$$R(z)f = R_0(z)f + \frac{i\big(f, v(\overline{z})\big)_{\mathcal{E}}}{\Gamma(z)} v(z),$$

where

$$R_0(z)f = \begin{pmatrix} r_0(z)(zf_1 + if_2) \\ i\partial_x r_0(z)\partial_x f_1 + zr_0(z)f_2 \end{pmatrix}$$

is free resolvent of H_0,

$$v(z) = \begin{pmatrix} ir_0(z)\psi \\ zr_0(z)\psi \end{pmatrix}, \quad r_0(z) = \big(-\partial_x^2 - z\big)^{-1} \text{ is resolvent of operator } -\partial_x^2$$

$$\Gamma(z) = 1 - iz\big(r_0(z)\psi, \psi\big)_{L^2(\mathbb{R})}, \quad \Gamma(\lambda \pm i0)^{(*_{10})} = 1 - i\lambda\big(r_0(\lambda \pm i0)\psi, \psi\big)_{L^2(\mathbb{R})}.$$

Then $\mathcal{F}$ is extended$^{(*_{11})}$ *to a bounded operator from $\mathcal{E}$ to $L^2(\mathbb{R};\mathbb{C}^2)$ and satisfies $\mathcal{F} = \mathcal{F}_0 W$. Moreover we have*

$$\int_{-\infty}^{\infty} \Big((\mathcal{F}Hf)(\lambda), \tilde{g}(\lambda)\Big)_{\mathbb{C}^2} d\lambda = \int_{-\infty}^{\infty} \lambda \Big((\mathcal{F}f)(\lambda), \tilde{g}(\lambda)\Big)_{\mathbb{C}^2} d\lambda \tag{4.3}$$

for any $f \in \mathcal{D}(H)$ and $\tilde{g} \in L^2(\mathbb{R};\mathbb{C}^2)$.

Therefore, we call the operator $\mathcal{F}$ the spectral representation for H. The operator $\mathcal{G}$ is the formal spectral representation for the adjoint operator H^*. Now we shall state the generalized Parseval formula.

$^{(*_{10})}$ This is justified from the *principle of limiting absorption* for the operator $-\partial_x^2$ i.e., the existence of the limit $\lim_{\varepsilon \downarrow 0} r_0(\lambda \pm i\varepsilon)$ in $L_s^2(\mathbb{R}) \to L_{-s}^2(\mathbb{R})$ $(s > 1/2)$.

$^{(*_{11})}$ This depends on a reason totally same as what we stated a while ago in footnote $(*_9)$.

Proposition 4.6. *Assume (A1) and (A2). Then*

(1) *If* $\left|\int_{\mathbb{R}} \psi(x)dx\right| \neq \sqrt{2}$, *then*

$$(f,g)_{\mathcal{E}} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\mathbb{C}^2} d\lambda, & \left(\left|\displaystyle\int_{\mathbb{R}} \psi(x)dx\right| < \sqrt{2}\right) \\ \displaystyle\int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\mathbb{C}^2} d\lambda + (Pf,g)_{\mathcal{E}}\,, & \left(\left|\displaystyle\int_{\mathbb{R}} \psi(x)dx\right| > \sqrt{2}\right) \end{cases}$$

for any $f, g \in \mathcal{E}$.

(2) *If* $\left|\int_{\mathbb{R}} \psi(x)dx\right| = \sqrt{2}$, *then*

$$(f,g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\mathbb{C}^2} d\lambda$$

for any $f \in \mathcal{H}, g \in \tilde{\mathcal{E}}$, *where* $\tilde{\mathcal{E}} \subset \mathcal{E}$ *is defined by*

$$\tilde{\mathcal{E}} = \left\{ g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})^{(*12)} \,\middle|\, (v(-i0), g)_{\mathcal{E}} = 0 \right\}.$$

Remark 4.7. We may consider the above Proposition 4.6 as the spectral decomposition theorem for the dissipative operator H. For instance, in the case $\left|\int_{\mathbb{R}} \psi(x)dx\right| > \sqrt{2}$, it holds by (4.3) that

$$\big(Af,g\big)_{\mathcal{E}} = \int_{-\infty}^{\infty} \lambda\big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\mathbb{C}^2} d\lambda + i\kappa_0\big(Pf,g\big)_{\mathcal{E}}$$

for any $f \in \mathcal{D}(H)$ and $g \in \mathcal{E}$.

Now we shall state counterpart to Theorem 3.7.

Theorem 4.8. *Assume (A1) and (A2).*

(1) *If* $\left|\int_{\mathbb{R}} \psi(x)dx\right| \leq \sqrt{2}$, *then* $\ker W = \{0\}$ *holds.*

(2) *If* $\left|\int_{\mathbb{R}} \psi(x)dx\right| > \sqrt{2}$, *then* $\ker W = \mathcal{R}(P)$ *holds.*

(*12) In general, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the *Schwartz space* on $\mathbb{R}^n$, which consists of *rapidly decreasing functions.*

[*Proof*]. (1) Consider the case $\left|\int_{\mathbb{R}} \psi(x)dx\right| < \sqrt{2}$. By Corollary 4.3, we may show $Wf = 0$ implies $f = 0$. Since $\mathcal{F}_0$ is unitary in $\mathcal{E}$, we have $\mathcal{F}f = 0$ by $\mathcal{F} = \mathcal{F}_0 W$ in Proposition 4.5. By Proposition 4.6 (1), we have $(f, g)_{\mathcal{E}} = 0$ for any $g \in \mathcal{E}$ to obtain the desired result. In case $\left|\int_{\mathbb{R}} \psi(x)dx\right| = \sqrt{2}$, similar arguments with this works well to find that $Wf = 0$ implies $(f, g)_{\mathcal{E}} = 0$ by Proposition 4.6 (2) since the space $\tilde{\mathcal{E}}$ is dense in $\mathcal{E}$. This shows $f = 0$. □
(2) In the same way, if $\left|\int_{\mathbb{R}} \psi(x)dx\right| > \sqrt{2}$, we find $Wf = 0$ implies $\mathcal{F}f = 0$ to conclude that $(f, g)_{\mathcal{E}} = (Pf, g)_{\mathcal{E}}$ by Proposition 4.6 (1), from which the desired result follows. □

Although we obtain the result corresponding to Corollary 3.8 by this theorem, we omit it.

In the method of up to now, it is essential that the singular point of resolvent is of order one i.e., the perturbed operator is close to a self adjoint operator in some sense [9]. In the rest of this section, we shall state a little irregular method [6]. Consider the radially symmetric solution of wave equation with very special dissipations of Coulomb type:

$$\partial_t^2 w(x,t) - \Delta w(x,t) + b(x)\partial_t w(x,t) = 0 \tag{4.4}$$

in $(x, t) \in \mathbb{R}^n \times (0, \infty)$, where the function $b(x)$ is defined by

$$(B) \qquad b(x) = b(|x|) = (|n-2|+1)|x|^{-1}$$

and initial data are given by

$$w(x, 0) = w_0(x), \quad \partial_t w(x, 0) = w_1(x).$$

Theorem 4.9. *Assume that the initial data are given by*

$$w_0(x) = \begin{cases} |x|f(|x|), & (n=1) \\ f(|x|), & (n \geq 2) \end{cases}, \quad w_1(x) = \partial_{|x|}\{w_0(x)\}, \tag{4.5}$$

where $f(|x|) = e^{z_{\Im}|x|} g(|x|)$ *with* $z_{\Im} < 0$ *and* $g \in \mathcal{S}'^{(*_{13})}$. *Then the explicit radial solution of* (4.4) *with* (B) *and* (4.5) *is given by*

$$w(x,t) = \begin{cases} |x|f(|x|+t), & (n=1) \\ f(|x|+t). & (n \geq 2) \end{cases}$$

Therefore if $f \in H^1$, *then the total energy* $||w(\cdot,t)||_{\mathcal{E}}$ *decays exponentially as* t *goes to infinity, where*

$$||w(\cdot,t)||_{\mathcal{E}}^2 = \frac{1}{2}\Big(||\partial_t w(\cdot,t)||_{L^2(\mathbb{R}^n)}^2 + ||\nabla w(\cdot,t)||_{L^2(\mathbb{R}^n)}^2\Big)$$

$(*_{13})$ $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ denotes the *tempered distribution,* which is the dual space of $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ (see the footnote $(*_{12})$)

with $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)^{(*_{14})}$.

Remark 4.10. The solution obtained in the above is an example of *disappearing solution* studied by A. Majda [16], which means that the solution disappears at some t_0: there exists some $t_0 > 0$ such that $w(x,t) = 0$ holds for any $t \geq t_0$.

As for the spectral structure of generator H_b for (4.4), we have

Theorem 4.11. *Define the operator* H_b *by*

$$H_b = i \begin{pmatrix} 0 & 1 \\ \Delta & -b(x) \end{pmatrix}$$

with domain

$$\mathcal{D}(H_b) = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{E}(\mathbb{R}^n) \,\middle|\, H_b f \in \mathcal{E}(\mathbb{R}^n) \right\}.$$

Assume $n \geq 3$ *and* (B)*. Then*

$$\sigma_p(H_b) = \mathbb{C}_-, \quad \sigma_r(H_b) = \emptyset, \quad \sigma_c(H_b) = \mathbb{R}, \quad \rho(H_b) = \mathbb{C}_+.$$

Result of supplementing the above is

Theorem 4.12. *Assume* $n \geq 3$ *and* $|b(x)| \leq b_1|x|^{-1}$ *with* $0 < b_1 < n-2$*. Then the following inclusion relations holds:*

$$\sigma_p(H_b) \subset \left\{ z = z_{\Re} + iz_{\Im} \in \mathbb{C} \,\middle|\, z_{\Im}^2 \leq \frac{b_1^2}{(n-2)^2 - b_1^2} z_{\Re}^2 \right\}.$$

[*Proof of Theorem 4.9*]. Consider the stationary problem associated with (4.4):

$$(-\Delta - izb(x) - z^2)u(x) = 0 \tag{4.6}$$

with $z = z_{\Re} + iz_{\Im} \in \mathbb{C}$. If we define $u(x)$ by $u(x) = e^{p(|x|)}$ with

$$p(|x|) = -iz|x| - \frac{(n-1)}{2} \log|x| + \frac{1}{2} \int_1^{|x|} b(s)ds$$

(Kato [11], Kawashita-Nakazawa-Soga [13]), then we have the Riccati type equation on $b = b(|x|)$:

$$2\frac{db}{d|x|} + b^2 - \frac{(n-1)(n-3)}{|x|^2} = 0.$$

$^{(*_{14})}$ See the description on $\mathcal{E}$ just behind (4.2)

Putting $h = h(|x|) = |x|b(|x|)$, we have

$$2|x|\frac{dh}{d|x|} + (h - n + 1)(h + n - 3) = 0.$$

Since this equation is of variable separation type, we can easily solve to obtain the function $b(|x|)$ given by (B). Then direct computations give the solution for (4.6)

$$u(|x|) = \begin{cases} |x|e^{-iz|x|}, & (n = 1) \\ e^{-iz|x|}. & (n \geq 2) \end{cases} \tag{4.7}$$

Now we assume that $n \geq 2$ (the proof for the case $n = 1$ is done by the same way). By (4.7), we find that the function $u(|x|; z_{\Re})$ defined by $u(|x|; z_{\Re}) = e^{-i(z_{\Re}+iz_{\Im})|x|}$ solves (4.6). Thus the solution $w(|x|; z_{\Re})$ of (4.4) with initial data

$$w_0(|x|) = u(|x|; z_{\Re}), \quad w_1(|x|) = \frac{du}{d|x|}(|x|; z_{\Re})$$

is given by $w(|x|; t; z_{\Re}) = u(|x| + t; z_{\Re})$. Since

$$g(|x|) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iz_{\Re}|x|} \mathcal{F}_0{}^{-1}\tilde{g}(z_{\Re}) dz_{\Re},$$

where we extend g to the following $\tilde{g}$

$$\tilde{g}(s) = \begin{cases} g(s), & (s \geq 0) \\ 0, & (s < 0) \end{cases}$$

we find that

$$(2\pi)^{-1/2} \int_{\mathbb{R}} \mathcal{F}^{-1}\tilde{g}(z_{\Re}) w(|x|, t; z_{\Re}) dz_{\Re} = e^{z_{\Im}(|x|+t)}\tilde{g}(|x| + t) = f(|x| + t)$$

is the desired solution (4.5). $\Box$

[*Proof of Theorem 4.11*]. (i) *On point spectrum.* Consider the eigenvalue problem

$$H_b v = zv \qquad \text{for} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{E}.$$

Then both components of v satisfy stationary equation (4.4). Thus, by Theorem 4.9, we find that

$$\mathbb{C}_- \subset \sigma_p(H_b).$$

If $z \in \mathbb{C}_+$ (i.e., $z_{\Im} > 0$), then

$$z_{\Im}||v||_{\mathcal{E}} \leq ||(H_b - z)v||_{\mathcal{E}}.$$

From this, we obtain

$$\sigma_p(H_b) \cap \mathbb{C}_+ = \emptyset.$$

If $z \in \mathbb{R}$, integration by parts of $\overline{v_j}$-times of (4.4) ($j = 1$ or 2) gives $||\sqrt{b}v_j||_{L^2(\mathbb{R}^n)} = 0$ to conclude

$$\sigma_p(H_b) \cap \mathbb{R} = \emptyset.$$

These arguments show

$$\sigma_p(H_b) = \mathbb{C}_-.$$

(ii) On residual spectrum. Since $H_b^* = H_{-b}$, we have $\sigma_p(H_b^*) = \mathbb{C}_+$. If we note the relation

$$z \in \sigma_r(H_b) \quad \Leftrightarrow \quad \overline{z} \in \sigma_p(H_b^*) \quad \& \quad z \notin \sigma_p(H_b),$$

we have

$$\sigma_r(H_b) = \emptyset.$$

(iii) On resolvent set. Since the operator H_b is maximal dissipative, we have $\mathbb{C}_+ \subset \rho(H_b) \subset \mathbb{R} \cap \mathbb{C}_+$. Using the fact that the resolvent set is open in $\mathbb{C}$, we conclude

$$\rho(H_b) = \mathbb{C}_+.$$

(iv) On continuous spectrum. We obtain $\sigma_c(H_b) = \mathbb{R}$ from the arguments $(i) - (iii)$. □

5. Decay and scattering for wave equations with dissipation in layered media.

In this section we shall describe a result on wave equations with dissipations in some layered media [10]. Consider the following initial-boundary value problems for $w = w(x, y, t)$:

$$\begin{cases} \partial_t^2 w - \Delta w + b(x,y)\partial_t w = 0, & (x,y,t) \in \Omega \times (0,\infty), \\ w(x,y,0) = w_1(x,y), \quad \partial_t(x,y,0) = w_2(x,y), & (x,y) \in \Omega, \\ w(x,0,t) = w(x,\pi,t) = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \end{cases}$$

where the space Ω is given by

$$\Omega = \mathbb{R}^n \times (0,\pi) = \left\{ (x,y) \,\middle|\, x \in \mathbb{R}^n, 0 < y < \pi \right\}$$

for $n = 1, 2, 3, \cdots$ and $b(x, y)$ is a measurable function decaying as $|x| \to \infty$. Under these settings, we shall study the behavior of solutions, i.e., total energy decay and existence of scattering states. To explain our results, we prepare some notations. The equations can be written as (1.1) under the following settings; the energy space $\mathcal{E}$ is the Hilbert space with the inner product:

$$\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{E}} = \int_\Omega (\nabla f_1(x,y) \cdot \overline{\nabla g_1(x,y)} + f_2(x,y)\overline{g_2(x,y)}) dxdy.$$

The norm of $\mathcal{E}$ is denoted by $\|\cdot\|_{\mathcal{E}}$. The operator $H = H_0 + V$ is defined by

$$H_0 = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad V = i\begin{pmatrix} 0 & 0 \\ 0 & -b(x,y) \end{pmatrix}$$

with domain

$$\mathcal{D}(H) = \mathcal{D}(H_0) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E} \;\middle|\; \Delta f_1 \in L^2(\Omega), f_2 \in H_0^1(\Omega) \right\}.$$

As for the function $b(x,y)$ we consider the following two conditions:

(1) (some kinds of long-range conditions)

$$(L) \quad : \quad b_1 \left(\prod_{k=0}^{m} \log^{[m]}(e_m + r) \right)^{-1} \le b(x,y) \le b_2$$

for some $b_1, b_2 > 0$.

(2) (some kinds of short-range conditions)

$$(S) \quad : \quad 0 \le b(x,y) \le b_3 \left(\prod_{k=0}^{m} \log^{[k]}(e_m + r) \right)^{-1} \left(\log^{[m]}(e_m + r) \right)^{-\delta}$$

for some $0 < \delta \le 1$, $b_3 > 0$.

Here, m is non-negative integer, $r = |x|$ and

$$e_0 = 1, \quad e_m = e^{e_{m-1}}, \quad \log^{[0]} s = s, \quad \log^{[m]} s = \log \log^{[m-1]} s \quad (m \ge 1).$$

Under these assumptions, operators H and H_0 is maximal dissipative and self adjoint in $\mathcal{E}$, respectively, therefore, H and H_0 generates a contraction semi-group $\{e^{-itH}\}_{t \ge 0}$ and unitary group $\{e^{-itH_0}\}_{t \in \mathbb{R}}$, respectively.

The results are as follows:

Theorem 5.1. *Under the assumption (L), we have for any $f \in \mathcal{E}$,*

$$\lim_{t \to \infty} \|e^{-itH} f\|_{\mathcal{E}} = 0.$$

Theorem 5.2. *Under the assumption (S), we have*

(1) *H has no real eigenvalue-values.*

(2) *The wave operator exists:*

$$W = s\text{-}\lim_{t \to \infty} e^{itH_0} e^{-itH}.$$

Moreover W is not zero as an operator in $\mathcal{E}$.

Corollary 5.3. *Under the assumption* (S)*, there exist non-trivial initial data* $f \in \mathcal{D}(H)$ *and* $f_+ \in \mathcal{D}(H_0)$ *such that*

$$\lim_{t\to\infty} \|e^{-itH} f - e^{-itH_0} f_+\|_{\mathcal{E}} = 0.$$

We omit proofs of the above. Only some comment are given here. As for Theorem 5.1 on the total energy decay, we prove the following

Proposition 5.4. *Assume* (L) *for fixed* m *and the initial data* $w_0 = (w_1, w_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$(*15)*. Let* ε *be positive number satisfying* $0 < \varepsilon \leqq \min\{1, b_1/2\}$*. Then*

$$\|e^{-itH} w_0\|_{\mathcal{E}} \leq C_1 \left\{\log^{[m]}(e_m + t)\right\}^{-\varepsilon/2}$$

holds for some positive constant $C_1 = C_1(w_1, w_2, b_1, b_2, \varepsilon) > 0$*.*

For the proof of this, we follow the same arguments as in Mochizuki and Nakazawa [21]. On the other hand, for the proof of Theorem 5.2, we need careful manipulation to control singular points (thresholds) in the spectrum. Although using Kato's smooth perturbation theory (Kato [12], Mochizuki [19], Kadowaki [4]), it seems difficult to apply the same methods with these since the operator $\sqrt{V}$ is not H_0-smooth near threshold $\pm k$, $k = 1, 2, 3 \cdots$. To relax these singularities,we chose $\sqrt{V}(H_0-i)^{-2}(H_0 \pm k)$ as smooth operator. Then we need density argument by using the operator $\prod_{k=1}^{n}(H-i)^{-4}(H^2-k^2)$ instead of approximate operator $(H-i)^{-2}H$ by Simon [27].

6. Final remarks.

For basic facts in functional analysis and spectral theory, we refer P.D. Hislop and I. M. Segal [3] and Z. Yoshida [29]. Especially, [29] has the angle of application of nonself adjoint problem in fluid mechanics. Z. Yosida [28] is helpful as the book which dealt with a similar problem more deeply, in which we can find the example that the point spectrum covers hole complex plain (Theorem 4.5 (2)). For spectral representation we consult K. Mochizuki [20]. In that book, spectral and scattering theory for Schrödinger and wave equations are argued. Energy decay-non decay problem for dissipative wave equations and existence, blow-up and scattering for non-linear wave equations are also treated.

For rank one perturbation, we quote S. Albeverio and P. Kurasov [1]. Although non-self adjoint perturbation of Schrödinger equations were studied by many authors, for example, B.S. Pavlov [23], K. Mochizuki [17], [18] e.t.c. (for detailed informations, please see the references in [7]), spectral structure for it is not yet made clear.

The situation does not also change to wave equations with dissipations. Smallness of the function $b(x)$ in (4.4) means the conformance of spectral structure with the free

(*15) The space $C_0^\infty(\Omega)$ denotes the vector spaces consisting of all functions which are continuous on Ω and have compact support in Ω

case and the validity of the principle of limiting absorption ([22]). But we have no answer to spectral structure without such a smallness condition except for Theorem 4.11. At present, it is when the order of the singularity of resolvent is one for the moment, that classification of asymptotic behavior of solutions by initial data in the relation to spectral structure is possible ([9]).

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