Notes on Integral Geometry and Harmonic Analysis

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Introduction

There are two strands of integral geometry. One involves the study of measures on a set invariant under a group of transformations. This is exemplified by the Crofton formula expressing the length of a rectifiable plane curve γ as an integral on the set of lines intersecting γ, or Buffon's needle problem about the probability that a line segment of given length placed randomly on the plane would intersect a horizontal line of the form $y = 0, \pm 1, \ldots$.

The other strand, which is the object of the present set of notes, involves questions pertaining to integral transforms of functions or differential forms on manifolds X which are usually acted upon by a Lie group G. These functions are integrated over a collection of orbits of compatible closed subgroups of G. We wish to determine whether it is possible to recover a function f from its integrals over these orbits – that is to say, whether this integral transform is injective. We would also wish, if possible, to recover f by means of an *inversion formula*. There are other important questions we can ask about these integral transforms. For example, if the integrals of f vanish over all orbits not intersecting a given closed set B in X, is f supported in B? Likewise, are there ways by which we can characterize the integrals of various types of functions on X?

This strand of integral geometry goes back to the work of P. Funk [4] in 1916, who showed that a continuous even function on the two-sphere can be recovered from its integrals over great circles, as well as that of J. Radon in 1917, who obtained an explicit formula recovering a compactly supported C^∞ function on $\mathbb{R}^3$ from its plane integrals.

The study of Funk's transform depends crucially on the fact that it is invariant under rotations on S^2, and likewise, Radon's transform (now called the *classical Radon transform*) depends on its being invariant under the group $M(3)$ of rigid motions on $\mathbb{R}^3$. It is therefore natural to ask whether we can develop a group-theoretic framework covering both types of transforms above, and what possible conclusions we can extract from this framework. This framework was first introduced by S. Helgason in 1964 using the terminology *homogeneous spaces in duality* ([15]). It was then generalized and extended by others in non-group-theoretic settings under the setting of *double fibrations*. (See, for example [6] or

[14]).

In these notes, we'll see, through several examples, the power and beauty of Helgason's group-theoretic framework. The framework is too general to allow us to offer explicit formulas, but it does allow us to formulate our questions in such a way as to make efficient and fruitful study possible.

These are the notes accompanying the series of seven lectures the author gave at Kyushu University in December 2009 and January 2010. They are appropriate for any advanced undergraduate or beginning graduate student who is interested in the interaction of geometry, Lie group theory, and analysis. Because of the need for brevity, we will assume that the reader has some familiarity with real and functional analysis as well as some Lie group theory. In the last section, we will also assume some knowledge of highest weight representations. In any event, I will try to state the important extraneous results we will need, as well as provide the appropriate references.

For a more comprehensive (and assuredly better) treatment of the subject, I would highly recommend Helgason's upcoming book *Integral Geometry and Radon Transforms* [20] or its previous incarnation *The Radon Transform* [18].

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Chapter 1

Homogeneous Spaces in Duality

1.1 Double Fibrations and Integral Transforms

Let X and Ξ be coset spaces of a group G, with $X = G/K$ and $\Xi = G/H$ for subgroups K and H, respectively. Suppose that $x = gK$ and $\xi = \gamma H$ are elements of X and Ξ, respectively. We say that x and ξ are *incident* if they intersect as cosets in G. If we put $L = K \cap H$, then the incidence relation can be expressed in terms of the *double fibration*

$$\begin{tikzcd} & G/L \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ X = G/K & & \Xi = G/H \end{tikzcd} \tag{1.1}$$

where p and π are the natural projection maps.

Again assuming that $x = gK$, let $\check{x} = \{\xi \in \Xi \mid x \text{ and } \xi \text{ are incident}\}$. Then it is easy to see that $\check{x} = \pi(p^{-1}(x)) = \{gkH \mid k \in K\}$. Likewise, with $\xi = \gamma H$, the set $\widehat{\xi} = \{x \in X \mid x \text{ and } \xi \text{ are incident}\}$ equals $p(\pi^{-1}(\xi)) = \{\gamma hK \mid h \in H\}$.

It is also easy to see that the map $gL \mapsto (gK, gH)$ is a bijection of the set G/L onto the set $\{(x, \xi) \in X \times \Xi \mid x \text{ and } \xi \text{ are incident}\}$. Thus the incidence relation, which is given by the latter subset of $X \times \Xi$, may be identified with the coset space G/L.

The incidence relation is clearly invariant under the left action of G: x is incident to ξ if and only if $g \cdot x$ is incident to $g \cdot \xi$, so that $(g \cdot x)^\vee = g \cdot \check{x}$ and $(\gamma \cdot \xi)^\wedge = \gamma \cdot \widehat{\xi}$.

For convenience we let $o = \{K\}$ and $\xi_0 = \{H\}$ be the identity cosets in X and Ξ, respectively. Then each $\check{x}$ and each $\widehat{\xi}$ is an orbit of a conjugate of K and H, respectively:

$$(g \cdot o)^\vee = gK \cdot \xi_0 = (gKg^{-1}) \cdot (g \cdot \xi_0) = gKg^{-1}/gLg^{-1}$$

and

$$(\gamma \cdot \xi_0)^\wedge = \gamma H \cdot o = (\gamma H \gamma^{-1}) \cdot (\gamma \cdot o) = \gamma H \gamma^{-1}/\gamma L \gamma^{-1} \tag{1.2}$$

Lemma 1.1.1. *Let $H_K = \{h \in H \,|\, hK \cup h^{-1}K \subset KH\}$. Then the map $x \mapsto \check{x}$ is injective if and only if $H_K = H \cap K$.*

Proof. Assume that $H_K = K \cap H$. We show that $x \mapsto \check{x}$ is injective. Let $x_1 = g_1 \cdot o$ and $x_2 = g_2 \cdot o$ be elements of X such that $\check{x}_1 = \check{x}_2$. Letting $g = g_1^{-1} g_2$, we have $g \cdot \check{o} = \check{o}$, so that $\{gkH \,|\, k \in K\} = \{kH \,|\, k \in K\}$. Thus $gH = k_0 H$ for some $k_0 \in K$, so that $g = k_0 h_0$, and we obtain $h_0 \cdot \check{o} = \check{o}$; that is to say, $\{h_0 kH \,|\, k \in K\} = \{kH \,|\, k \in K\}$. This implies that $h_0 K \subset KH$. Since we also have $g^{-1} \cdot \check{o} = \check{o}$, we obtain $h_0^{-1} K \subset KH$. Thus $h_0 \in H_K \subset K$, and so $g \in K$, and thus $x_1 = x_2$.

Conversely, suppose that $x \mapsto \check{x}$ is injective. Let $h \in H_K$. Then $(h \cdot o)^\vee = \{hkH \,|\, k \in K\} \subset \check{o}$, since $hK \subset KH$. Likewise $h^{-1} \cdot \check{o} = (h^{-1} \cdot o)^\vee \subset \check{o}$, so that $(h \cdot o)^\vee \supset \check{o}$. By the injectivity, we obtain $h \cdot o = o$, so that $h \in K$. □

Exercise 1.1.2. (a) Show that $G = KHK$ if and only if for every pair of points x_1, $x_2 \in X$, there exists a $\xi \in \Xi$ such that x_1 and x_2 are incident to ξ.

(b) Show that $KH \cap HK = K \cup H$ if and only if any pair of points in X are incident to at most one $\xi \in \Xi$. (Or, equivalently, if and only if any pair of points in Ξ are incident to at most one $x \in X$.)

Let us now suppose that G is a Hausdorff topological group and that K and H are closed subgroups. We equip K and H with the relative topology, and X and Ξ with their respective quotient topologies. Let $p : G \to X$ be the quotient map. Since $p^{-1}(\widehat{\xi}_0) = HK$, we see that $\widehat{\xi}_0$ (and therefore every $\widehat{\xi}$) is closed in X if and only if HK is a closed subset of G. Thus we will assume that HK is closed. (This is automatic if H or K is compact.) Of course this will also imply that each $\check{x}$ is a closed subset of Ξ.

In order to define the integral transforms associated with the double fibration above, let us now assume that G is a locally compact group, so that K, H, and L likewise are, and that G, K, H, and L are all unimodular. We let dg, dk, dh, and dl denote their respective Haar measures. If any of these groups is compact, we assume that its Haar measure is normalized.

For any $F \in C_c(G)$, let F_p be the function on X defined by

$$F_p(gK) = \int_K F(gk)\, dk \tag{1.3}$$

Then F_p is well-defined since the support of F intersects each coset gK in a compact set. From the Dominated Convergence Theorem, it is clear that F_p is continuous, with support in $p(\mathrm{supp}(F))$. Thus $F_p \in C_c(X)$. In fact, the map $F \mapsto F_p$ maps $C_c(G)$ onto $C_c(X)$. (See Lemma 1.10, Chapter 1 of [17].)

The coset space $X = G/K$ has a unique natural left G-invariant measure dm such that

$$\int_G F(g)\, dg = \int_X F_p(x)\, dm(x) \tag{1.4}$$

for all $F \in C_c(G)$. When convenient, we will also denote the measure dm by dg_K, so that equation (1.4) reads

$$\int_G F(g)\, dg = \int_{G/K} \left(\int_K F(gk)\, dk \right) dg_K$$

Likewise we have unique left invariant measures $d\mu = dg_H$, dg_L, dh_L, and dk_L on $\Xi = G/H$, G/L, H/L, and K/L, respectively. We note that the following integral version of the "chain rule" holds:

$$\int_{G/L} \varphi(gL)\, dg_L = \int_{G/H} \left(\int_{H/L} \varphi(ghL)\, dh_L \right) dg_H \tag{1.5}$$

for all $\varphi \in C_c(G/L)$.

Since the orbit $\hat{\xi}_0 = H \cdot o$ may be identified with the homogeneous space H/L, the measure dh_L gives rise to an H-invariant measure m_{ξ_0} on $\hat{\xi}_0$. Similarly, by (1.2), if $\xi = \gamma H \in \Xi$, then $\hat{\xi}$ is an orbit of $H^\gamma = \gamma H \gamma^{-1}$, the isotropy subgroup of ξ in G. Thus $\hat{\xi}$ has a measure μ_ξ invariant under H^γ.

Since H is unimodular, the Haar measure on H can be made compatible with the Haar measure on H^γ in the sense that

$$\int_{H^\gamma} f(h^\gamma)\, dh^\gamma = \int_H f(\gamma h \gamma^{-1})\, dh,$$

for all $f \in C_c(H^\gamma)$, with the right hand side independent of the choice of γ in the coset γH. We can likewise make the Haar measure on L compatible with those of its conjugates L^γ.

In this manner, the measures m_ξ are left-invariant under G in that

$$m_\xi(A) = m_{g\cdot\xi}(g \cdot A) \tag{1.6}$$

for all Borel sets A in $\widehat{\xi}$.

For each $x = gK \in X$, we likewise have a measure μ_x on $\check{x} \subset \Xi$ invariant under the subgroup K^g of G fixing x. If we make the Haar measures on the K^g compatible, then the measures μ_x are compatible under left multiplication by G.

Suppose now that $f \in C_c(X)$. We define its *Radon transform* to be the function Rf on Ξ given by

$$Rf(\xi) = \int_{\widehat{\xi}} f(x)\, dm_\xi(x) \tag{1.7}$$

for any $\xi \in \Xi$. Group-theoretically, if $\xi = \gamma H$, we have

$$Rf(\gamma H) = \int_{H/L} f(\gamma hK)\, dh_L \tag{1.8}$$

Note that the right hand side above is independent of the choice of $\gamma \in G$ such that $\xi = \gamma H$. By taking local cross sections from Ξ to G, it is immediate from the Dominated Convergence Theorem that $Rf \in C(\Xi)$.

If $\varphi \in C_c(\Xi)$, its *dual transform* is the function on X given by

$$R^*\varphi(x) = \int_{\check{x}} \varphi(\xi)\, d\mu_x(\xi) \tag{1.9}$$

Its group-theoretic expression is given by

$$R^*\varphi(gK) = \int_K \varphi(gkH)\, dk_L \tag{1.10}$$

Note that if K is compact, $R^*\varphi$ may be defined for $\varphi \in C(\Xi)$.

Note that both R and R^* are linear maps.

Lemma 1.1.3. *The Radon transform R and its dual R^* are formal adjoints in the sense that*

$$\int_\Xi Rf(\xi)\, \varphi(\xi)\, d\mu(\xi) = \int_X f(x)\, R^*\varphi(x)\, dm(x) \tag{1.11}$$

for all $f \in C_c(X)$ and $\varphi \in C_c(\Xi)$.

Proof. Apply the integral chain rule and the Fubini theorem to the function $gL \mapsto (f \circ p)(gL)\, (\varphi \circ \pi)(gL)$ on G/L. □

Practically all group-invariant integral transforms fit into the double fibration framework given above. While the framework is too general to offer any specific conclusions, we can nonetheless consider the following general problems.

1. *Injectivity:* Is R injective; that is, is it possible to recover a function $f \in C_c(X)$ (or some other class of functions, such as $L^p(X)$) from its Radon transforms $Rf(\xi)$? If f is not injective, can one describe the kernel of R?

2. *Support:* Let B be a closed subset of X. If $Rf(\xi) = 0$ for all ξ such that $\widehat{\xi} \cap B = \varnothing$, does f have support in B? (Note that if $B = \varnothing$, this reduces to the injectivity question above.)

3. *Inversion:* If R is injective, is there a formula or procedure by which f can be recovered from Rf? In particular, what is the relation between any function $f \in C_c(X)$ and R^*Rf (if the latter exists)?

4. *Range characterization:* Describe the range, under R of certain spaces of functions in X, such as $C_c^\infty(X)$, or $L^p(X)$.

1.2 Continuity Properties of the Radon and Dual Transforms

In this section we examine the continuity properties of the Radon and dual transforms associated with homogeneous spaces in duality. In order to do so, we will need to consider some pertinent facts about topological vector spaces and their duals. These facts may be found in any standard text on topological vector spaces, such as or Rudin's book [38] or Treves' book [43].

Let V be a topological vector space, and let V' be its dual space, the vector space of continuous linear functionals on V. Since we are mainly concerned with function spaces of smooth complex-valued functions on manifolds, we will assume that V and V' are complex. The *strong topology* on V' is the locally convex topology defined by the seminorms

$$\|\lambda\|_B := \sup_{v \in B} |\lambda(v)| \qquad (\lambda \in V')$$

for all bounded sets B in V. Thus a sequence $\{\lambda_m\}$ in V' converges to λ in the strong topology if and only if $\{\lambda_m\}$ converges uniformly to λ on every bounded subset B of V.

The *weak topology* on V' is the locally convex topology defined by the seminorms $\lambda \mapsto |\lambda(v)|$, for each v in V. Thus the sequence $\{\lambda_m\}$ converges weakly to λ if and only if the sequence of scalars $\{\lambda_m(v)\}$ converges to $\lambda(v)$ for each v in V.

It is not difficult to show that if V' is endowed with the weak topology, then the evaluation map

$$v \mapsto (\lambda \mapsto \lambda(v)) \tag{1.12}$$

is a linear bijection from V onto the double dual $(V')'$. The weak topology on $(V')'$ gives rise to the *weak topology* on V.

V is said to be *semireflexive* if the map (1.12) is a bijection of V onto the dual space of V' when V' is equipped with the strong topology.

Let $T : V \to W$ be a continuous linear map of topological vector spaces. The dual map $T^* : W' \to V'$ is given by $T^*(\lambda) = \lambda \circ T$ for all $\lambda \in W'$. It is easy to see that T^* is continuous in the strong dual topologies as well as in the weak dual topologies.

A *Frechét space* is a locally convex metrizable topological vector space which is complete in its metric. This metric may be chosen to be translation-invariant.

The following result will be needed in Theorem 1.2.5 below, which will be important later in characterizing the range of the dual transform.

Theorem 1.2.1. 1. *([43], Proposition 35.2) Let V be a locally convex topological vector space. If C is any convex set, then the weak closure and the strong closure of C coincide.*

2. *([43], Theorem 37.2) Let $T : V \to W$ be a continuous map of Frechet spaces. Then T is onto if and only if the following conditions are satisfied:*
   (a) *The dual map $T^* : W' \to V'$ is one-to-one.*
   (b) *The image $T^*(W')$ is weakly closed in V'.*

The first assertion above is in fact an easy consequence of the Hahn-Banach theorem.

From now on it will be convenient to assume that G is a Lie group, so that X and Ξ are manifolds, and the orbits $\widehat{\xi}$ and $\check{x}$ closed submanifolds. Let $\mathcal{E}(X)$ denote the vector space of C^∞ functions on X. We equip $\mathcal{E}(X)$ with the topology of uniform convergence of all derivatives on compact sets. More precisely, for each linear differential operator D on X with C^∞ coefficients and each compact subset $C \subset X$, let us put

$$\|f\|_{D,C} = \sup_{x \in C} |Df(x)| \tag{1.13}$$

for each $f \in \mathcal{E}(X)$. The seminorms $\| \ \|_{D,C}$ give rise to a locally convex topology on $\mathcal{E}(X)$. In particular, a basis of neighborhoods of 0 consists of the sets

$$U(D, C, \epsilon) = \{f \in \mathcal{E}(X) \,|\, \|f\|_{D,C} < \epsilon\} \tag{1.14}$$

for all D and C. Since X is second countable, the topology on $\mathcal{E}(X)$ is generated by countably many seminorms of the form (1.13), and $\mathcal{E}(X)$ is in fact a Frechet space.

Next let $\mathcal{D}(X)$ be the subspace of $\mathcal{E}(X)$ consisting of all functions with compact support. We provide $\mathcal{D}(X)$ with the *inductive limit topology* as follows. For each compact set $C \subset X$, let $\mathcal{D}_C(X)$ be the subspace consisting of all functions in $\mathcal{E}(X)$ supported in C. Then $\mathcal{D}_C(X)$ is a closed subspace of $\mathcal{E}(X)$, and hence is a Frechet space with the relative topology. The inductive limit topology on $\mathcal{D}(X)$ is the strongest locally convex topology for which the inclusion maps $\mathcal{D}_C(X) \hookrightarrow \mathcal{D}(X)$ are continuous.

For details on the topologies of $\mathcal{D}$ and $\mathcal{E}$, we again refer the reader to Rudin's book [38], where X is replaced by an open subset of $\mathbb{R}^n$, but where the main topological results carry over.

Proposition 1.2.2. *The Radon transform $f \mapsto Rf$ is a continuous linear map from $\mathcal{D}(X)$ to $\mathcal{E}(\Xi)$.*

Proof. For each $g \in G$, we have

$$Rf(g \cdot \xi_0) = \int_{\widehat{\xi}_0} f(g \cdot x)\, dm_{\xi_0}(x)$$

Choose a local cross section of a coordinate neighborhood U of $g \cdot \xi_0$ into a slice of a coordinate system containing g in G, and let $y_1, \ldots, y_n$ be the local coordinates on this slice. Then we can write the above as

$$Rf(y_1, \ldots, y_n) = \int_{\widehat{\xi}_0} f((y_1, \ldots, y_n) \cdot x)\, dm_{\xi_0}(x)$$

Since the function $x \mapsto f((y_1, \ldots, y_n) \cdot x)$ is compactly supported on ξ_0, the integral above is well-defined and is C^∞ in $(y_1, \ldots, y_n)$. The continuity of the map $f \mapsto Rf$ then follows by expressing D in (1.14) in the coordinates $y_1, \ldots, y_n$ and differentiating under the integral sign. $\square$

The same proof, of course, shows that the dual transform $\varphi \mapsto R^*\varphi$ is a continuous map from $\mathcal{D}(\Xi)$ to $\mathcal{E}(X)$.

The duality (1.11) then allows us to define the Radon transform of a compactly supported distribution. If $T \in \mathcal{E}'(X)$, then RT is the distribution on Ξ defined by

$$RT(\varphi) = T(R^*\varphi) \tag{1.15}$$

for all $\varphi \in \mathcal{D}(\Xi)$. If we equip $\mathcal{E}'(X)$ and $\mathcal{D}'(\Xi)$ with the strong dual topologies, then the above shows that the map $T \mapsto RT$ is a continuous map from $\mathcal{E}'(X)$ to $\mathcal{D}'(\Xi)$.

Now suppose that K is compact. Then the coset map $p : G \to G/K$ is proper. By (1.10) this means that for all $\varphi \in \mathcal{E}(\Xi)$, the dual transform $R^*\varphi$ is well-defined. Moreover in this case R is now a map from $\mathcal{D}(X)$ to $\mathcal{D}(\Xi)$. For if $f \in \mathcal{D}(X)$ is supported on a compact set $C \subset X$, then Rf will have support in $\check{C} = \{\xi \in \Xi \mid \xi \in \check{x} \text{ for some } x \in C\} = \pi(p^{-1}(C))$, a compact set in Ξ.

The following result is thus an immediate consequence of Proposition 1.2.2.

Proposition 1.2.3. *If K is compact, then the Radon transform $f \mapsto Rf$ is a continuous map from $\mathcal{D}(X)$ to $\mathcal{D}(\Xi)$.*

As a corollary, from the duality, it follows that $R^* : \mathcal{D}'(\Xi) \to \mathcal{D}'(X)$ is continuous.

Proposition 1.2.4. *If K is compact, the dual transform $\varphi \mapsto R^*\varphi$ is a continuous map from $\mathcal{E}(\Xi)$ to $\mathcal{E}(X)$.*

Proof. Let C be any compact set in X. Then as was shown above, the set $\check{C}$ is a compact subset of Ξ. Now if $\varphi \in \mathcal{E}(\Xi)$, the restriction $(R^*\varphi)|_C$ is determined completely by the values of φ on the set $\check{C}$. Let U be any neighborhood of C in X with compact closure. Since the projection $\pi : G/L \to \Xi$ is an open map, the set $\check{U} = \pi(p^{-1}(U))$ is an open subset of Ξ with compact closure. Fix a function $\psi \in \mathcal{D}(\Xi)$ such that $\psi \equiv 1$ on $\check{U}$. If φ_n is a sequence in $\mathcal{E}(\Xi)$ converging to 0, then $\psi\,\varphi_n$ is a sequence in $\mathcal{D}(\Xi)$ which converges to 0. Hence by Proposition 1.2.2 for R^*, the sequence $R^*(\psi\,\varphi_n)$ converges to 0 in $\mathcal{E}(X)$. Let D be any differential operator on X. Then $D(R^*(\psi\,\varphi_n)) \to 0$ uniformly on U. But by our remark above, $D(R^*(\psi\,\varphi_n)) = D(R^*(\varphi_n))$ on C. Hence $\|R^*\varphi_n\|_{D,C} \to 0$. This shows that $R^*\varphi_n$ converges to 0 in $\mathcal{E}(X)$, and proves the proposition. ☐

Again, by duality, we conclude that the Radon transform $R : \mathcal{E}'(X) \to \mathcal{E}'(\Xi)$ is continuous.

The following theorem, due to Helgason ([19], Chapter 1, Theorem 3.7) provides an important general characterization of the range of the dual transform R^*.

Theorem 1.2.5. *Assume that K is compact. Suppose that the range $R\mathcal{E}'(X)$ is closed in $\mathcal{E}'(\Xi)$. Then $R^*\mathcal{E}(\Xi) = \mathcal{N}^\perp$, where*

$$\mathcal{N} = \{T \in \mathcal{E}'(X) \mid RT = 0\} \tag{1.16}$$

and

$$\mathcal{N}^\perp = \{f \in \mathcal{E}(X) \mid T(f) = 0 \textit{ for all } T \in \mathcal{N}\} \tag{1.17}$$

Proof. Note that $\mathcal{N}^\perp$ is a closed subspace of a Frechét space, and hence is a Frechét space. Moreover, by the hypothesis and Theorem 1.2.1, the range $R\mathcal{E}'(X)$ is weakly closed in $\mathcal{E}'(\Xi)$.

The duality (1.15) shows that $R^*\mathcal{E}(\Xi) \subset \mathcal{N}^\perp$. We put $T = R^*$, so that the dual T^* maps $(\mathcal{N}^\perp)'$ to $\mathcal{E}'(\xi)$. Now by the Hahn-Banach theorem, any $\alpha \in (\mathcal{N}^\perp)'$ extends to a continuous linear functional on $\mathcal{E}(X)$. Any two such extensions agree on $\mathcal{N}^\perp$, so differ by an element of $(\mathcal{N}^\perp)^\perp$. By the Hahn-Banach theorem, the latter set equals the weak closure of $\mathcal{N}$, which by Theorem 1.2.1, Part 1 coincides with the closure of $\mathcal{N}$. Since $\mathcal{N}$ is closed, we obtain $(\mathcal{N}^\perp)^\perp = \mathcal{N}$.

Hence the dual map T^* corresponds to a map of $\mathcal{E}'(X)/\mathcal{N}$ to $\mathcal{E}'(\Xi)$. By (1.15) we see that $T^*(\alpha + \mathcal{N}) = R\alpha$, for all $\alpha \in \mathcal{E}'(X)$. Now $T^* : (\mathcal{N}^\perp)' = \mathcal{E}'(X)/\mathcal{N} \to \mathcal{E}'(\Xi)$ is injective, and its range $T^*(\mathcal{E}'(X)) = R(\mathcal{E}'(X))$ is weakly closed in $\mathcal{E}'(\Xi)$. Thus by Theorem 1.2.1, the map $T = R^* : \mathcal{E}(\Xi) \to \mathcal{N}^\perp$ is onto. $\square$

1.3 Invariant Differential Operators

Since the Radon transform R is an integral operator, we can try to ask whether it can be inverted by a differential operator. Now both R and R^* are invariant under left translations by elements of G, so it also makes sense to ask whether R can be inverted by a differential operator (or even by a pseudodifferential operator) which is invariant under these left translations.

With this in mind, let us introduce some notation. If τ is a diffeomorphism of a manifold M onto a manifold N and f is any function on M, the *push-forward* of f with respect to τ is $f^\tau = f \circ \tau^{-1}$. Then of course f is C^∞ if and only if f^τ is. Note that $(f^\tau)^\sigma = f^{\sigma\tau}$ if σ is a diffeomeorphism from N to a manifold P.

If D is a differential operator on M, its push-forward with respect to τ is the differential operator D^τ on N given by $D^\tau(\varphi) = \left(D(\varphi^{\tau^{-1}})\right)^\tau$. Again we have $(D^\tau)^\sigma = D^{\sigma\tau}$. Moreover, if D and E are differential operators on M, it is easy to see that $(DE)^\tau = D^\tau E^\tau$. If τ is a diffeomorphism of M onto itself, we say that the differential operator D is *invariant* under τ if $D^\tau = D$.

Suppose that a Lie group G acts smoothly on a manifold M on the left. If $g \in G$, we let $\tau(g)$ denote the diffeomorphism $m \mapsto g \cdot m$. We say that a differential operator D on M is *left G-invariant* (or just *G-invariant*) if $D^{\tau(g)} = D$ for all $g \in G$. Note that the vector space $\mathbb{D}(M)$ of G-invariant differential operators on M is in fact a subalgebra of the (associative) algebra of all differential operators on M under compositions.

It will be useful for us later on to try to characterize the algebra $\mathbb{D}(M)$ for various G and M.

If $G = M$, with the left action by G given by left multiplication, then $\mathbb{D}(G)$ can be identified with the complexified *universal enveloping algebra* $U(\mathfrak{g}) \otimes \mathbb{C}$, where $\mathfrak{g}$ is the Lie algebra of G. The universal enveloping algebra $U(\mathfrak{g})$ is usually constructed in the following completely algebraic fashion. (For details, see, for example, Chapter III of Knapp's book [26].)

Let $T(\mathfrak{g})$ denote the (associative) tensor algebra of the vector space $\mathfrak{g}$: $T(\mathfrak{g}) = \sum_{k=0}^\infty \otimes^k \mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient algebra of $T(\mathfrak{g})$ by the two-sided ideal $\mathcal{I}$ of $T(\mathfrak{g})$ generated by the elements $X \otimes Y - Y \otimes X - [X, Y]$, for all X, $Y \in \mathfrak{g}$.

Then $U(\mathfrak{g})$ is an associative algebra, and its multiplication satisfies

$$XY = YX + [X, Y] \qquad (X, Y \in \mathfrak{g}) \tag{1.18}$$

where we have identified elements of $\mathfrak{g}$ with their images in $U(\mathfrak{g})$. This identification makes sense since it turns out that the quotient map from $T(\mathfrak{g})$ onto $U(\mathfrak{g})$ is injective on $\mathfrak{g}$. Now the theorem below provides a precise basis of $U(\mathfrak{g})$.

Theorem 1.3.1. *(The Poincare-Birkhoff-Witt Theorem.) Let $X_1, \ldots, X_n$ be an (ordered) basis of $\mathfrak{g}$. Then $U(\mathfrak{g})$ has basis given by the monomials*

$$X^I = X_1^{i_1} \cdots X_n^{i_n}$$

for all multiindices $I = (i_1, \ldots, i_n)$.

For a proof, see Chapter 17 of Humphreys' book [22] or Chapter III of Knapp's book [26].

From (1.18), the multiplication in $U(\mathfrak{g})$ is almost commutative, in the sense that $DE = ED +$ (lower order terms) for any D and E in $U(\mathfrak{g})$.

Moreover, the construction of $U(\mathfrak{g})$ shows that it satisfies the following universal mapping property. If A is any associative algebra, then A can be given a Lie bracket by setting $[a, b] = ab - ba$ for all $a, b \in A$. Now suppose that Φ is a Lie algebra homomorphism from $\mathfrak{g}$ into A. Then Φ extends uniquely to a homomorphism from the associative algebra $U(\mathfrak{g})$ to the associative algebra A.

By definition, the Lie algebra $\mathfrak{g}$ of G is the tangent space of G at the identity e. If $X \in \mathfrak{g}$, then the left invariant vector field $\widetilde{X}$ coinciding with X at e is given by

$$\widetilde{X}f(g) = \frac{d}{dt} f(g \exp(tX))\Big|_{t=0} \qquad (f \in C^\infty(G))$$

Note that by the definition of the Lie bracket on $\mathfrak{g}$, $[X, Y]^\sim = [\widetilde{X}, \widetilde{Y}] = \widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X}$ for all X and Y in $\mathfrak{g}$. By the universal mapping property of $U(\mathfrak{g})$, the map $X \mapsto \widetilde{X}$ extends to a homomorphism $D \mapsto \widetilde{D}$ of $U(\mathfrak{g})$ into the algebra $\mathbb{D}(G)$ of all left invariant differential operators on G. Explicitly, if $Y_1, \ldots, Y_m \in \mathfrak{g}$, then

$$(Y_1 \cdots Y_m)^\sim f(g) = \widetilde{Y}_1 \cdots \widetilde{Y}_m f(g) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} f(g \exp(t_1 Y_1) \cdots \exp(t_k Y_k))\Big|_{(t_j = 0)}$$

for all $f \in C^\infty(G)$.

Theorem 1.3.2. *The map $D \mapsto \widetilde{D}$ is an isomorphism of $U(\mathfrak{g})$ onto $\mathbb{D}(G)$.*

Proof. Fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, and consider the map $\psi : (t_1, \ldots, t_n) \mapsto \exp(t_1 X_1) \cdots \exp(t_n X_n)$ from $\mathbb{R}^n$ into G. If ψ_* is the differential of ψ at the origin 0, then $\psi_*(\partial/\partial x_j) = X_j$, so ψ is a diffeomorphism near 0 and thus ψ^{-1} is a coordinate system on a neighborhood of the identity element e of G.

Suppose that $E \in \mathbb{D}(G)$. Then, using the local coordinates above, we see that at the identity, E has the form

$$Ef(e) = \sum_I a_I \frac{\partial^{|I|}(f \circ \psi)}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}(0) \tag{1.19}$$

for all $f \in C^\infty(G)$, where the a_I are constants and the sum ranges over a finite set of multiindices $I = (i_1, \ldots, i_n)$. This shows that

$$Ef(e) = \left(\sum_I a_I X_1^{i_1} \cdots X_n^{i_n}\right)^{\sim} f(e)$$

and by left-invariance,

$$Ef(g) = \left(\sum_I a_I X_1^{i_1} \cdots X_n^{i_n}\right)^{\sim} f(g). \tag{1.20}$$

Thus $E = (\sum_I a_I X_1^{i_1} \cdots X_n^{i_n})^{\sim}$.

To show that $D \mapsto \widetilde{D}$ is one-to-one, suppose that $D = \sum_I a_I X_1^{i_1} \cdots X_n^{i_n}$ is a nonzero element of $U(\mathfrak{g})$ such that $\widetilde{D} = 0$. Then at the identity, $\widetilde{D}$ is given by (1.19). Let I be any multiindex for which $a_I \neq 0$. If we define the function f near e by $f \circ \psi^{-1}(t_1, \ldots, t_n) = t_1^{i_1} \cdots t_n^{i_n}$, equation (1.19) shows that in fact $(i_1! \cdots i_n!) a_I = 0$, a contradiction. □

The map $D \mapsto \widetilde{D}$ can be thought of as the infinitesimal version of the *right regular representation* of G. For any $g \in G$, we let r_g denote the right translation $h \mapsto hg$. Then define the map $\pi(g)$ of $\mathcal{E}(G)$ by $\pi(g)\, f = f^{r_{g^{-1}}} = f \circ r_g$. It is easy to show that π is multiplicative, and it follows (from a straightforward argument using the fact that continuous functions on compact sets are uniformly continuous) that, for each $f \in \mathcal{E}(G)$, the map $g \mapsto \pi(g) f$ is continuous from G to $\mathcal{E}(G)$. Thus π is a representation of G on $\mathcal{E}(G)$.

By differentiation, the representation π gives rise to a representation $d\pi$ of $\mathfrak{g}$ on $\mathcal{E}(G)$, and we see immediately that for each $f \in \mathcal{E}(G)$ and each $g \in G$,

$$\begin{aligned} d\pi(X)\, f(g) &= \frac{d}{dt} f(g \exp tX)\bigg|_{t=0} \\ &= \widetilde{X} f(g). \end{aligned}$$

Thus $d\pi(X)$ coincides with the left invariant vector field $\widetilde{X}$; by extension to the universal enveloping algebra, we have $d\pi(D) = \widetilde{D}$, for all $D \in U(\mathfrak{g})$.

There is also a linear bijection of the symmetric algebra $S(\mathfrak{g})$ onto $\mathbb{D}(G)$, called the *symmetrization map*, and defined as follows. Again, let us first fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$; for a multiindex I, let X^I denote the monomial $X_1^{i_1} \cdots X_n^{i_n}$ in $S(\mathfrak{g})$. Now any element $P \in S(\mathfrak{g})$ has a unique expression $P = \sum_I a_I X^I$. For this P, let $\lambda(P)$ denote the operator on $C^\infty(G)$ given by

$$\lambda(P)\, f(g) = P(\partial/\partial t_1, \ldots, \partial/\partial t_n) f(g \exp(t_1 X_1 + \cdots + t_n X_n))|_{(t_j=0)} \qquad (1.21)$$

It is easy to see that $\lambda(P)$ is a differential operator on G which commutes with left translations, and that the map $P \mapsto \lambda(P)$ is linear. The proof that λ is one-to-one and onto is similar to that in the proof of Theorem 1.3.2, taking into account the fact that the map

$$\exp(t_1 X_1 + \cdots + t_n X_n) \mapsto (t_1, \ldots, t_n)$$

is a coordinate system on a neighborhood of e in G. For more details, see [17], Chapter 2.

The symmetrization map λ turns out to be independent of the choice of basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. To see this, let X be any element of $\mathfrak{g}$, so that $X = \sum_j a_j X_j$. Then for any $k \in \mathbb{Z}^+$, the definition (1.21) shows that

$$\begin{aligned}
\lambda(X^k)\, f(g) &= \lambda(a_1 X_1 + \cdots + a_n X_n)^k\, f(g) \\
&= \left(a_1 \frac{\partial}{\partial t_1} + \cdots + a_n \frac{\partial}{\partial t_n}\right)^k f(g \exp(t_1 X_1 + \cdots + t_n X_n))\Bigg|_{(t_j=0)} \\
&= \frac{d^k}{ds^k} f(g \exp(s(a_1 X_1 + \cdots + a_n X_n)))\Bigg|_{s=0} \\
&= \frac{d^k}{ds^k} f(g \exp sX)\Bigg|_{s=0} \\
&= \widetilde{X}^k\, f(g)
\end{aligned}$$

Thus,

$$\lambda(X^k) = \widetilde{X}^k \qquad (1.22)$$

for all k and all $X \in \mathfrak{g}$. Now let $Y_1, \ldots, Y_k$ be any elements of $\mathfrak{g}$. If we apply the relation (1.22) to $X = t_1 Y_1 + \cdots + t_k Y_k$ and equate the coefficients of $t_1 \cdots t_k$ on both sides, we see that

$$\lambda(Y_1 \cdots Y_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \widetilde{Y}_{\sigma(1)} \cdots \widetilde{Y}_{\sigma(k)}, \qquad (1.23)$$

where the right hand sum is taken over the symmetric group $\mathfrak{S}_k$. While λ is not multiplicative, equation (1.23) shows that $\lambda(Y_1 \cdots Y_k) = \widetilde{Y}_1 \cdots \widetilde{Y}_k +$ (lower order terms). Hence it follows by degree induction that for any P and Q in $S(\mathfrak{g})$, there exists an $R \in S(\mathfrak{g})$, with $\deg R < \deg P + \deg Q$, such that

$$\lambda(PQ) = \lambda(P)\lambda(Q) + \lambda(R)$$

for any P and Q in $S(\mathfrak{g})$.

Suppose that G is a Lie group acting smoothly on a manifold M on the left, with the action given by $(g,m) \mapsto g\cdot m$. For each $g \in G$, recall that $\tau(g)$ denotes the left translation $m \mapsto g \cdot m$. The *left regular representation* of G on $\mathcal{E}(M)$ is given by

$$\lambda(g)\, f(m) = f^{\tau(g)}(x) = f(g^{-1}\cdot x) \qquad\qquad (g \in G,\, x \in M) \tag{1.24}$$

λ is easily shown to be multiplicative and, using local coordinates on M, it is not too hard to verify that for each $f \in \mathcal{E}(M)$, $g \mapsto \lambda(g)f$ is continuous from G to $\mathcal{E}(M)$. Thus λ is a representation of G.

The left regular representation λ induces a representation $d\lambda$ of $\mathfrak{g}$ on $\mathcal{E}(M)$ given by

$$d\lambda(X)\, f(x) = \frac{d}{dt}\, f(\exp(-tX)\cdot x)\Big|_{t=0} \tag{1.25}$$

Each $d\lambda(X)$ is a smooth vector field on M, and it follows from general theory that

$$d\lambda[X,Y] = [d\lambda(X), d\lambda(Y)] \tag{1.26}$$

for all X, $Y \in \mathfrak{g}$. We can verify equation (1.26) directly as follows. Fix $x \in M$. For $f \in \mathcal{E}(M)$, define the function F on G by $F(g) = f(g^{-1}\cdot x)$. Then $d\pi(X)\, f(g^{-1}\cdot x) = \widetilde{X}F(g)$, and so

$$\begin{aligned}
&(d\lambda(X)\, d\lambda(Y) - d\lambda(Y)\, d\lambda(X))\; f(x)\\
&\quad = \frac{\partial^2}{\partial s\,\partial t}\,(f(\exp(-sY)\exp(-tX)\cdot x) - f(\exp(-tX)\exp(-sY)\cdot x))\Big|_{(s,t)=(0,0)}\\
&\quad = (\widetilde{X}\,\widetilde{Y}\,F)(e) - (\widetilde{Y}\,\widetilde{X}\,F)(e)\\
&\quad = [X,Y]^{\sim}\,F(e)\\
&\quad = d\lambda[X,Y]\,f(x)
\end{aligned}$$

Thus $d\lambda$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of smooth vector fields on M. By the universal mapping property, $d\lambda$ extends to a homomorphism, which we will also denote by $d\lambda$, from $U(\mathfrak{g})$ to the algebra of differential operators on M. We call $d\lambda$ the *infinitesimal left regular representation* of $U(\mathfrak{g})$ on $\mathcal{E}(M)$.

Explicitly, if $X_1,\ldots,X_m \in \mathfrak{g}$ and $f \in \mathcal{E}(M)$, then

$$\begin{aligned}
&d\lambda(X_1\cdots X_m)\, f(x) =\\
&\qquad = \frac{\partial^m}{\partial t_1\cdots\partial t_m}\, f(\exp(-t_mX_m)\cdots\exp(-t_1X_1)\cdot x)\Big|_{(t_j=0)}
\end{aligned} \tag{1.27}$$

Note that when $M = G$, then an argument similar to that of the proof of Theorem 1.3.2 (or by using the inversion map $f \mapsto \check{f}(g) = f(g^{-1})$) shows that the map $D \mapsto d\lambda(D)$ is an isomorphism of $U(\mathfrak{g})$ onto the algebra of all right-invariant differential operators on G.

Let us now return to the double fibration (1.1), with G a Lie group. From their group-theoretic expressions (1.8) and (1.10), we see that the Radon transform R and its dual R^* *intertwine* the left regular representations λ and ν of G on X and Ξ, respectively:

$$\begin{aligned} R(\lambda(g)\, f) &= \nu(g)\, Rf \\ R^*(\nu(g)\, \varphi) &= \lambda(g)\, R^*\varphi \end{aligned} \tag{1.28}$$

for all $f \in \mathcal{D}(X)$, $\varphi \in \mathcal{D}(\Xi)$, and $g \in G$.

Now for every $Z \in \mathfrak{g}$ and $f \in \mathcal{D}(X)$, the difference quotient

$$\frac{\lambda(\exp tZ) - \lambda(e)}{t}\, f$$

converges to $d\lambda(Z)\, f$ as $t \to 0$ in the topology of $\mathcal{D}(X)$. Hence by Proposition 1.2.2, we conclude that $R(d\lambda(Z)\, f) = d\nu(Z)\, Rf$, and by extension to the universal enveloping algebra, $R(d\lambda(U)\, f) = d\nu(U)\, Rf$, for all $U \in U(\mathfrak{g})$. Thus the following diagram commutes.

$$\begin{tikzcd} \mathcal{D}(X) \arrow[r, "R"] \arrow[d, "d\lambda(U)"'] & \mathcal{E}(\Xi) \arrow[d, "d\nu(U)"] \\ \mathcal{D}(X) \arrow[r, "R"] & \mathcal{E}(\Xi) \end{tikzcd} \tag{1.29}$$

Most of the transforms R we will study in these notes are injective. If this is the case, then $\ker(d\nu) \subset \ker(d\lambda)$. In fact, if $U \in \ker(d\nu)$, then $d\nu(U) = 0$, so for any $f \in \mathcal{D}(X)$, we have $0 = d\nu(U)\,(Rf) = R(d\lambda(U)\, f)$. But since R is injective, we obtain $d\lambda(U)\, f = 0$ for all $f \in \mathcal{D}(X)$, so $d\lambda(U) = 0$.

It is in general a difficult problem to determine $\ker(d\lambda)$ or $\ker(d\nu)$. If $\ker(d\nu) \subsetneqq \ker(d\lambda)$, then any function φ in the range of R satisfies the nontrivial differential equations

$$d\nu(U)\, \varphi = 0 \tag{1.30}$$

for all $U \in \ker(d\lambda) \backslash \ker(d\nu)$. These equations are therefore necessary conditions for any smooth function on Ξ to be in the range of R. As we will see in Chapter 4, these equations are often sufficient to characterize the range as well.

1.4 Regularization

In extending certain properties of classes of C^∞ functions in $\mathbb{R}^n$ to measures or distributions, it is often useful to regularize; that is to say, to take the convolution of a distribution with an approximate identity. The result is a C^∞ function which converges to the measure or distribution in question.

In this section we discuss the analogue to this procedure in the case of Lie groups and homogeneous spaces. Let us first consider some representation-thoeretic generalities.

Let G be a unimodular Lie group, with Lie algebra $\mathfrak{g}$ and bi-invariant measure dg. Suppose that π is a representation of G on a locally convex topological vector space V. Then, by hypothesis, π is a group homomorphism from G to $\mathrm{GL}(V)$, the group of linear homeomorphisms of V, such that π is strongly continuous. This means that for each $v \in V$, the map $g \mapsto \pi(g)\, v$ is continuous from G to V.

For each $X \in \mathfrak{g}$, let $V(X)$ denote the subspace of V consisting of all vectors v for which the limit

$$d\pi(X)\, v := \lim_{t\to 0} \frac{\pi(\exp(tX)) - \pi(e)}{t}\, v$$

exists. Then let $V^{(1)}$ be the intersection of the subspaces $V(X)$, for all $X \in \mathfrak{g}$. Since $d\pi(X)\, \pi(g)\, v = \pi(g)\, d\pi(\mathrm{Ad}(g^{-1})X)\, v$, we see that $\pi(g)\, V^{(1)} = V^{(1)}$ for all $g \in G$. Next, for each $j \geqslant 1$ we define $V^{(j)}$ inductively by $V^{(j)} = (V^{(j-1)})^{(1)}$, and then put $V^\infty = \bigcap_{j\geqslant 1} V^{(j)}$.

If V is complete (i.e., Cauchy sequences in V converge), then we can define $\pi(f)\, v$ for each $f \in C_c(G)$, by

$$\pi(f)\, v = \int_G f(g)\, \pi(g)\, v\, dg$$

Note that $\pi(g)\, \pi(f)\, v = \pi(f^{L_g})\, v$, and that if $f \in \mathcal{D}(G)$, then

$$\pi(X)\, \pi(f)\, v = \pi(d\lambda(X) f)\, v$$

It follows that $\pi(f)\, v \in V^\infty$ whenever $f \in \mathcal{D}(G)$ and $v \in V$.

Now consider any countable basis $\{U_m\}$ of neighborhoods of $e \in G$ such that $\overline{U}_{m+1} \subset U_m$. For each m let f_m be a nonnegative function supported on U_m such that $\int_G f_m(g)\, dg = 1$. Then for any $v \in V$, the strong continuity of π implies that

$$\pi(f_m)\, v - v = \int_G f_m(g)\, (\pi(g)\, v - v)\, dg \to 0$$

in the topology of V, as $m \to \infty$. This shows that V^∞ is dense in V. We call $\{f_m\}$ an *approximate identity* in G.

The map $X \mapsto d\pi(X)$ is a representation of $\mathfrak{g}$ on V^∞, and by the universal mapping property of $U(\mathfrak{g})$, $d\pi$ extends to a representation of the associative algebra $U(\mathfrak{g})$ on V^∞.

Exercise 1.4.1. Let $V = C(G)$, the vector space of continuous functions on G, endowed with the topology of uniform convergence on compact sets. Show that $V^\infty = \mathcal{E}(G)$, and $d\pi$ the infinitesimal left regular representation of $U(\mathfrak{g})$ on $\mathcal{E}(\mathfrak{g})$.

Let V' be the dual space of V. For each $g \in G$, we define the map ${}^t\pi(g) : V' \to V'$ by ${}^t\pi(g)\,\lambda = \lambda \circ \pi(g)$. It is straightforward to verify that the map $g \mapsto {}^t\pi(g)$ is a group homomorphism from G to $GL(V')$, where V' is given either the strong or the weak topology.

If V is semireflexive, then ${}^t\pi$ turns out to be strongly continuous ([1]). Hence ${}^t\pi$ is a representation of G on V', called the representation *contragredient* to π.

V' is also easily verified to be complete (in both its weak and strong topologies), so that if V is semireflexive, then we can define the operator

$$ {}^t\pi(f) = \int_G f(g)\,{}^t\pi(g)\,dg \qquad (f \in C_c(G)) \tag{1.31}$$

on V'.

Now let $V = \mathcal{E}(G)$, and let π be the left regular representation $\pi(g)\varphi = \varphi^{L_g}$. Then $V = V^\infty$, and the algebra of right-invariant differential operators on G is $d\pi(U(\mathfrak{g}))$. If $f \in \mathcal{D}(G)$, then $\pi(f)$ is the convolution operator

$$\pi(f)\varphi(g) = f * \varphi(g) = \int_G f(u)\,\varphi(u^{-1}g)\,du = \int_G f(gy^{-1})\,\varphi(y)\,dy$$

Note that the support of $f * \varphi$ lies in $\mathrm{supp}(f)\,\mathrm{supp}(\varphi)$.

If $T \in \mathcal{D}'(G)$, then $\pi(f)\,T$ is the convolution $f * T$:

$$f * T(\varphi) = \int_G f(g)\,T^{L_g}(\varphi)\,dg \qquad (\varphi \in \mathcal{D}(G)) \tag{1.32}$$

If we put $\check{f}(g) = f(g^{-1})$, then we obtain

$$f * T(\varphi) = \int_G \int_G \check{f}(g)\,\varphi(g^{-1}u)\,dg\,dT(u) = T(\check{f} * \varphi) \tag{1.33}$$

Since G is unimodular, the above can also be written

$$\int_G \int_G f(gu^{-1})\,\varphi(g)\,du\,dT(u)$$

Now since the left regular representation is strongly continuous, the function $g \mapsto \int_G f(gu^{-1})\, dT(u)$ is continuous, and the above equals

$$\int_G \left(\int_G f(gu^{-1})\, dT(u) \right) \varphi(g)\, dg \tag{1.34}$$

(This is the result of a simple application of the Fubini theorem for distributions; see [43], §40.) The equation (1.34) therefore shows that $f * T$ is a continuous function on G.

For each $X \in \mathfrak{g}$, the function

$$\frac{\pi(\exp(tX)) - \pi(e)}{t}\, f$$

converges to $d\pi(X)\, f$ as $t \to 0$ in the topology of $\mathcal{D}(G)$. From this one sees that

$$\frac{d}{dt}\, f * T(\exp(tX)g)|_{t=0}$$

exists and equals $(d\pi(X)f) * T(g)$. Repeating this, we see that $f * T$ is a C^∞ function on G.

If $\{f_m\}$ is an approximate identity in G, then for each $\varphi \in \mathcal{D}(G)$, $f_m * \varphi \to \varphi$ uniformly on G as $m \to \infty$. Since

$$D(f_m * \varphi) = f_m * D\varphi$$

for every $D \in \mathbb{D}(G)$, we see that $f_m * \varphi \to \varphi$ in $D(G)$ as $m \to \infty$. Hence by (1.33), we see that $f_m * T \to T$ weakly in $\mathcal{D}'(G)$ for all $T \in \mathcal{D}'(G)$. Thus T may be approximated by C^∞ functions in $\mathcal{D}'(G)$.

Now suppose that H a closed subgroup of G and $X = G/H$ is the corresponding homogeneous space. Let us assume that H is unimodular, so that G/H has a left G-invariant measure dg_H, unique up to constant multiple.

Let $\pi : G \to G/H$ be the coset map, and let $\tau(g) : g_1H \mapsto gg_1H$ be left translation by $g \in G$. If $f \in \mathcal{E}(G/H)$, let $\widetilde{f} = f \circ \pi$ be its pullback to G.

If $F \in \mathcal{D}(G)$, then just as with (1.3), we put

$$F_\pi(gH) = \int_H F(gh)\, dh$$

Then F_π is compactly supported and is C^∞, so that $F_\pi \in \mathcal{D}(X)$. Using a a proof similar to that of Proposition 1.2.2, it can also be shown that the map $F \mapsto F_\pi$ is continuous from $\mathcal{D}(G)$ to $\mathcal{D}(X)$.

Exercise 1.4.2. Using partitions of unity and local cross sections of G/H into G, show that $F \mapsto F_\pi$ maps $\mathcal{D}(G)$ onto $\mathcal{D}(X)$.

For appropriate choices of the measures dg and dg_H, we have, just as with (1.4),

$$\int_G F(g)\,dg = \int_{G/H} F_\pi(gH)\,dg_H \qquad (F \in \mathcal{D}(G))$$

From this it follows that

$$\int_G \widetilde{f}(g)\,F(g)\,dg = \int_{G/H} f(gH)\,F_\pi(gH)\,dg_H \tag{1.35}$$

for all $f \in \mathcal{E}(X)$ and $F \in \mathcal{D}(G)$.

Suppose that $T \in \mathcal{D}'(X)$. We can use (1.35) to define the pullback $\widetilde{T}$:

$$\widetilde{T}(F) = T(F_\pi) \qquad (F \in \mathcal{D}(G)) \tag{1.36}$$

By the remarks preceding Exercise 1.4.2, we see that $\widetilde{T} \in \mathcal{D}'(G)$.

Let λ be the left regular representation of G on $\mathcal{D}(X)$. If $T \in \mathcal{D}'(X)$ and $f \in \mathcal{D}(G)$, then

$$\lambda(f)\,T = \int_G f(g)\,T^{\tau(g)}\,dg \tag{1.37}$$

We note that $(\lambda(f)\,T)^\sim = f * \widetilde{T}$, and since

$$(f * \widetilde{T})(gh) = f * \widetilde{T}^{R_{h^{-1}}} = f * \widetilde{T}(g)$$

for all $g \in G$ and $h \in H$, we see that $f * \widetilde{T}$ is the pullback of a smooth function on X. Thus $\lambda(f)\,T \in \mathcal{E}(X)$; we will also denote this function by $f * T$. Its support is $\operatorname{supp}(f) \cdot \operatorname{supp}(T)$.

Now suppose that $\{f_m\}$ is an approximate identity in G. If $F \in \mathcal{D}(G)$, then

$$(f_m * T)(F_\pi) = (f_m * \widetilde{T})\,(F) \to \widetilde{T}(F) = T(F_\pi)$$

By Exercise 1.4.2, we see that $f_m * T \to T$ weakly in $\mathcal{D}'(X)$. Thus T can be approximated weakly by smooth functions on X.

Chapter 2

The Classical Radon Transform

2.1 The Incidence Relation

The classical Radon transform integrates suitable functions on $\mathbb{R}^n$ over hyperplanes in $\mathbb{R}^n$. As we mentioned in the introduction, this transform dates back to Johann Radon's paper in 1917, and so the methods used to study it predate the double fibration framework. Nonetheless, it will be useful to keep this framework in mind while studying the transform.

Now it can be shown that any isometry, or rigid motion, of $\mathbb{R}^n$ is of the form $x \mapsto k{\cdot}x{+}v$, where $k \in \mathrm{O}(n)$ and $v \in \mathbb{R}^n$. We denote this isometry by $\tau(k, v)$, and we note that these isometries satisfy the composition rules $\tau(k, v) \circ \tau(k', v') = \tau(kk', v + k \cdot v')$. From this, we see that $\tau(k, v)^{-1} = \tau(k^{-1}, -k^{-1} \cdot v)$. If we let $\mathrm{M}(n)$ denote the group of all isometries of $\mathbb{R}^n$, then $\mathrm{M}(n)$ is the Cartesian product $\mathrm{O}(n) \times \mathbb{R}^n$ equipped with the group law

$$(k, v) \cdot (k', v') = (kk', v + k \cdot v') \tag{2.1}$$

We say that $\mathrm{M}(n)$ is the *semidirect product* of its subgroups $\mathrm{O}(n)$ and $\mathbb{R}^n$, and write $\mathrm{M}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. We also equip $\mathrm{M}(n)$ with the product manifold structure. From the above, we see that $\mathrm{M}(n)$ is a Lie group, and that $\mathrm{O}(n)$ and $\mathbb{R}^n$ are the subgroups consisting of all rotations and translations, respectively, on $\mathbb{R}^n$.

We also note that the subgroup of orientation-preserving isometries is $\mathrm{SO}(n) \ltimes \mathbb{R}^n$.

Now $\mathrm{M}(n)$ acts transitively on $\mathbb{R}^n$, and the subgroup fixing the origin 0 is $\mathrm{O}(n)$, so $\mathbb{R}^n = \mathrm{M}(n)/\mathrm{O}(n)$.

Let us now consider the space Ξ_n of unoriented $(n-1)$-dimensional planes in $\mathbb{R}^n$. Let $\xi \in \Xi_n$, and let ω be one of its two unit normal vectors. If p is the directed distance from the origin to ξ along ω, then

$$\xi = \{x \in \mathbb{R}^n \,|\, \langle \omega, x \rangle = p\}$$

We write $\xi = \xi(\omega, p)$ and note that $\xi(\omega, p) = \xi(-\omega, -p)$. In this way we obtain a two-to-one map of $S^{n-1} \times \mathbb{R}$ onto Ξ_n, and we give Ξ_n the quotient differentiable structure. In particular, we can identify C^∞ functions on Ξ_n with C^∞ functions φ on $S^{n-1} \times \mathbb{R}$ such that $\varphi(\omega, p) = \varphi(-\omega, -p)$ for all $\omega \in S^{n-1}$ and $p \in \mathbb{R}$.

Now $\mathrm{M}(n)$ acts smoothly and transitively on Ξ_n via

$$(k, v) \cdot \xi(\omega, p) = \xi(k \cdot \omega, p + \langle k \cdot \omega, v \rangle) \tag{2.2}$$

In particular, the translate $v + \xi(\omega, p)$ is the hyperplane $\xi(\omega, p + \langle \omega, v \rangle)$, and the rotated plane $k \cdot \xi(\omega, p)$ is just $\xi(k \cdot \omega, p)$.

Let ξ_0 be the $(n-1)$-plane $x_n = 0$. Then the isotropy subgroup H of $\mathrm{M}(n)$ at ξ_0 is $(\mathrm{O}(n-1) \times \mathbb{Z}_2) \ltimes \mathbb{R}^{n-1}$, where $\mathrm{O}(n-1) \times \mathbb{Z}_2$ is the group of matrices of the form

$$\begin{pmatrix} k & 0 \\ 0 & \pm 1 \end{pmatrix}$$

where $k \in \mathrm{O}(n-1)$. We can thus write $H = M(n-1) \times \mathbb{Z}_2$, where $M(n-1)$ is the motion group of ξ_0 and $\mathbb{Z}_2$ is the group generated by the reflection on ξ_0. Thus Ξ_n can be identified with the homogeneous space $\mathrm{M}(n)/H$.

$\mathbb{R}^n$ itself, of course, is the homogeneous space $\mathrm{M}(n)/\mathrm{O}(n)$, with $\mathrm{O}(n)$ being the subgroup of $\mathrm{M}(n)$ fixing the origin.

Next let us verify that the incidence relation between the homogeneous spaces $\mathbb{R}^n = \mathrm{M}(n)/\mathrm{O}(n)$ and $\Xi_n = \mathrm{M}(n)/H$ coincides with the usual incidence relation between points and planes: for g and γ in $\mathrm{M}(n)$, $x = g \cdot 0$ and $\xi = \gamma \cdot \xi_0$ are incident if and only if the point x lies in ξ. Suppose that x lies in ξ. Since $\xi_0 = H \cdot 0$, we must have $g \cdot 0 = \gamma h \cdot 0$ for some $h \in H$, so $\gamma h = gk$ for some $k \in \mathrm{O}(n)$. Conversely, if x and ξ are incident, then $gk = \gamma h$ for some $k \in \mathrm{O}(n)$ and $h \in H$, so $x = g \cdot 0 = \gamma h \cdot 0 \in \gamma \cdot \xi_0 = \xi$.

Thus, under this incidence relation, the set $\widehat{\xi}$ can be identified with ξ itself. On the other hand, if 0 is the origin, the orbit $\check{0} = \mathrm{O}(n) \cdot \xi_0$ coincides with the set of all $(n-1)$-planes through 0, and by translation, we see that for each $x \in \mathbb{R}^n$, we have $\check{x} = \{x + k \cdot \xi_0 \,|\, k \in \mathrm{O}(n)\}$.

From (2.1) it is easy to see that the motion group $\mathrm{M}(n)$ is unimodular, with Haar measure given by $f \mapsto \int f(k, v)\, dk\, dv$, where dk is the normalized Haar measure on $\mathrm{O}(n)$ and dv the Lebesgue measure on $\mathbb{R}^n$. It follows that $H = M(n-1) \times \mathbb{Z}_2$

is also unimodular, and of course so is the compact group $L = \mathrm{O}(n) \cap H = \mathrm{O}(n-1) \times \mathbb{Z}_2$.

The Lebesgue measure on $\widehat{\xi}_0 = \widehat{\xi}$ is invariant under the subgroup H fixing ξ_0, so we will take this to be the measure dm_{ξ_0}. By translation by an appropriate $g \in \mathrm{M}(n)$, we can likewise take, for $\xi \in \Xi_n$, dm_ξ to be the Lebesgue measure on ξ. For simplicity, we will denote dm_ξ by dm. For $f \in C_c(\mathbb{R}^n)$, it follows that the Radon transform R is given by

$$Rf(\xi) = \int_\xi f(x)\, dm(x) \tag{2.3}$$

It is possible to define Radon transform (2.3) for $f \in L^1(\mathbb{R}^n)$, since by Fubini's Theorem, the integrals are convergent for almost all $\xi \in \Xi_n$.

The dual transform is given by

$$R^*\varphi(x) = \int_{\mathrm{O}(n)} \varphi(x + k \cdot \xi)\, dk \tag{2.4}$$

Using our earlier notation, we also have $\check{x} = \{\xi(\omega, \langle \omega, x\rangle) \,|\, \omega \in S^{n-1}\}$, so $\check{x}$ is parametrized by $\omega \in S^{n-1}$. The map $k \mapsto k_x = (k, x - k \cdot x)$ is a Lie group isomorphism of $\mathrm{O}(n)$ onto the subgroup of $M(n)$ fixing x. Since $k_x \cdot \xi(\omega, \langle \omega, x\rangle) = \xi(k \cdot \omega, \langle k \cdot \omega, x\rangle)$, the normalized measure on $\check{x}$ invariant under the group of all rotations about x is just a multiple of the area measure $d\omega$ on S^{n-1}. The dual transform can thus be written as

$$R^*\varphi(x) = \frac{1}{\Omega_n} \int_{S^{n-1}} \varphi(\omega, \langle \omega, x\rangle)\, d\omega, \tag{2.5}$$

where Ω_n is the area of S^{n-1}: $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

From (2.2) one sees that the measure

$$\varphi \in C_c(\Xi_n) \mapsto \int_{S^{n-1} \times \mathbb{R}} \varphi(\omega, p)\, d\omega\, dp$$

is invariant under the action of $M(n)$. The duality (1.11) is then given by

$$\int_{S^{n-1} \times \mathbb{R}} Rf(\omega, p)\, \varphi(\omega, p)\, d\omega\, dp = \Omega_n \int_{\mathbb{R}^n} f(x)\, R^*\varphi(x)\, dx, \tag{2.6}$$

for $f \in C_c(\mathbb{R}^n)$ and $\varphi \in C(\Xi_n)$. In fact,

$$\begin{aligned}
\int_{S^{n-1}\times\mathbb{R}} Rf(\omega,p)\,\varphi(\omega,p)\,d\omega\,dp &= \int_{S^{n-1}} \int_{\mathbb{R}} \left(\int_{\langle x,\omega\rangle = p} f(x)\,dm(x) \right) \varphi(\omega,p)\,dp\,d\omega \\
&= \int_{S^{n-1}} \int_{\mathbb{R}^n} f(x)\,\varphi(\omega,\langle \omega, x\rangle)\,dx\,d\omega \\
&= \int_{\mathbb{R}^n} \int_{S^{n-1}} f(x)\,\varphi(\omega,\langle \omega, x\rangle)\,d\omega\,dx \\
&= \Omega_n \int_{\mathbb{R}^n} f(x)\,R^*\varphi(x)\,dx
\end{aligned}$$

In light of this, we define the Radon transform of a compactly supported distribution T on $\mathbb{R}^n$ by

$$RT(\varphi) = \Omega_n\,T(R^*\varphi) \tag{2.7}$$

for any $\varphi \in \mathcal{E}(\Xi_n)$.

2.2 The Projection-Slice Theorem and the Inversion Formula

Let us recall that the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widetilde{f}(y) = \int_{\mathbb{R}^n} f(x)\,e^{-i\,\langle x,y\rangle}\,dx \qquad (y \in \mathbb{R}^n) \tag{2.8}$$

For fixed y, we observe that the exponential $e^{-i\langle x,y\rangle}$ is constant on hyperplanes orthogonal to y. Thus we can relate the Fourier transform to the Radon transform by integrating (2.8) along such hyperplanes. Explicitly, let us write $y = s\omega$, for $s \in \mathbb{R}$ and $\omega \in S^{n-1}$. Then

$$\begin{aligned}
\widetilde{f}(s\omega) &= \int_{\mathbb{R}} \int_{\langle x,\omega\rangle = p} f(x)\,e^{-is\,\langle x,\omega\rangle}\,dm(x)\,dp \\
&= \int_{\mathbb{R}} \left(\int_{\langle x,\omega\rangle = p} f(x)\,dm(x) \right) e^{-ips}\,dp \\
&= \int_{\mathbb{R}} Rf(\omega,p)\,e^{-ips}\,dp
\end{aligned} \tag{2.9}$$

Equation (2.9) is known as the *Projection-Slice Theorem.* It expresses the intimate relation between the Fourier and the Radon transform; essentially it says that the Fourier transform is a one-dimensional Fourier transform of the Radon transform. It immediately implies that R is injective on $L^1(\mathbb{R}^n)$. It also enables us to invert the Radon transform on, say $\mathcal{S}(\mathbb{R}^n)$, by employing the well-known inversion formula for the Fourier transform.

Another consequence of (2.9) is that Radon transforms preserve convolutions. More precisely, suppose that f and g belong to $L^1(\mathbb{R}^n)$. Then it is easy to show that $f * g \in L^1(\mathbb{R}^n)$, and (2.9) shows that

$$R(f * g)(\omega, p) = \int_{\mathbb{R}} Rf(\omega, p - t)\, Rg(\omega, t)\, dt \tag{2.10}$$

This equation can also be verified directly by a simple application of Fubini's Theorem.

To facilitate our derivation of the inversion formula, let us now recall the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$. By definition, $\mathcal{S}(\mathbb{R}^n)$ consists of all $f \in \mathcal{E}(\mathbb{R}^n)$ satisfying the estimates

$$\|f\|_{k,N} := \sup_{x \in \mathbb{R}^n, |I| \leqslant k} (1 + \|x\|^N) \left|\partial^I f(x)\right| < \infty \tag{2.11}$$

for all k, $N \in \mathbb{Z}^+$. In the inequality above, I represents a multiindex $(i_1, \ldots, i_n)$, $|I|$ is the sum $i_1 + \cdots + i_n$ and ∂^I is the differential operator $\partial^{|I|}/(\partial x_1^{i_1} \cdots \partial x_n^{i_n})$.

Since the Fourier transform interchanges partial differential operators and multiplication by polynomials, it is not hard to show that it maps $\mathcal{S}$ into itself (it is in fact a bijection by the formula below). Moreover, for any $f \in \mathcal{S}(\mathbb{R}^n)$, we have the *Fourier Inversion Formula*

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widetilde{f}(y)\, e^{i\langle x, y\rangle}\, dy \tag{2.12}$$

There are several different formulas which invert the classical Radon transform R. For even dimensions, one of them involves the Hilbert transform, which we now define as follows.

Consider the linear mapping p.v.$[1/t]$ from $\mathcal{S}(\mathbb{R})$ to $\mathbb{C}$ given by

$$\text{p.v.}\left[\frac{1}{t}\right] h = \lim_{\epsilon \to 0^+} \int_{|t| > \epsilon} \frac{h(t)}{t}\, dt \tag{2.13}$$

(The p.v. stands for "principal value.") The right hand limit above exists even though $1/t$ is not locally integrable; in fact,

$$\text{p.v.}\left[\frac{1}{t}\right] h = \int_0^\infty \frac{h(t) - h(-t)}{t}\, dt$$

It is not difficult to show that p.v.$[1/t]$ is an element of $\mathcal{S}'(\mathbb{R})$; that is to say, is a tempered distribution on $\mathbb{R}$.

Exercise 2.2.1. Show that p.v.$[1/t]$ is the distribution derivative of the tempered distribution $\log|t|$ (which we note is locally integrable and vanishes at infinity), given by

$$(\log|t|)\,(h) = \int_{\mathbb{R}} h(t)\, \log|t|\, dt$$

Since p.v.$[1/t]$ is a tempered distribution, it makes sense to take its Fourier transform on $\mathbb{R}$.

Lemma 2.2.2. *Let $sgn(t)$ be the signum function on $\mathbb{R}$ given by*

$$sgn(t) = \begin{cases} 1 & \text{if } t \geqslant 0 \\ -1 & \text{if } t < 0 \end{cases}$$

Then the Fourier transform of p.v.$(1/t)$ is $-\pi\, i\, sgn(t)$.

Proof. Let F denote the (distributional) Fourier transform of p.v.$[1/t]$. Now the product $t\,(\text{p.v.}[1/t])$ equals the constant function 1, and this corresponds under the Fourier transform to the distribution derivative $i\,F'$. On the other hand, the Fourier transform of 1 is the Dirac distribution $2\pi\,\delta_0$, so we obtain $F' = -2\pi\, i\,\delta_0$. This implies that $F = -2\pi\, i\, H(t) + C$, where $H(t)$ is the Heaviside function and C is some constant. But since p.v.$[1/t]$ is an odd distribution in $\mathbb{R}$, so is F. This means that $C = \pi i$; that is to say $F(t) = -\pi i\,\text{sgn}(t)$. $\square$

The *Hilbert Transform* of a function $g \in \mathcal{S}(\mathbb{R})$ is the convolution $\mathcal{H}g = g *$ p.v.$[1/t]$. Thus

$$\mathcal{H}g(p) = \lim_{\epsilon\to 0} \int_{|t|>\epsilon} \frac{g(p-t)}{t}\, dt \tag{2.14}$$

By Lemma 2.2.2, the Fourier transform of $\mathcal{H}g$ is the function $(\mathcal{H}g)^{\sim}(y) = -\pi i\, \widetilde{g}(y)\,\text{sgn}(y)$.

Finally, let us introduce the space of Schwartz-class functions on Ξ_n. By definition, a function $\varphi \in C^\infty(S^{n-1}\times\mathbb{R})$ belongs to $\mathcal{S}(S^{n-1}\times\mathbb{R})$ if for every $k, N \in \mathbb{Z}^+$ and every C^∞ linear differential operator D on S^{n-1}, we have

$$\sup_{(\omega,p)} (1+|p|)^N \left| \frac{\partial^k}{\partial p^k} D_\omega \varphi(\omega,p)\right| < \infty$$

The space $\mathcal{S}(\Xi_n)$ then consists of the even functions in $\mathcal{S}(S^{n-1}\times\mathbb{R})$. The *partial Fourier transform* of φ is its Fourier transform in the p variable

$$\varphi \mapsto \widetilde{\varphi}(\omega,s) = \int_{-\infty}^{\infty} \varphi(\omega,p)\, e^{-ips}\, dp$$

The partial Fourier transform maps $\mathcal{S}(S^{n-1}\times\mathbb{R})$ into itself, and $\mathcal{S}(\Xi_n)$ into itself. Moreover, the projection-slice theorem says that the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is the partial Fourier transform of its Radon transform.

Let $f \in \mathcal{S}(\mathbb{R}^n)$. If we change to polar coordinates $(\omega, s) \mapsto s\omega$ in $\mathbb{R}^n$, a straightforward application of the chain rule shows that its Fourier $(\omega, s) \mapsto \widetilde{f}(s\omega)$ belongs to $\mathcal{S}(S^{n-1} \times \mathbb{R})$. By Projection-Slice and our observation above, we now see that $Rf \in \mathcal{S}(\Xi_n)$.

With these preliminaries out of the way, let us now invert the Radon transform R. Put $y = s\omega$ in (2.12) to obtain

$$\begin{aligned}(2\pi)^n f(x) &= \int_{S^{n-1}} \int_0^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, s^{n-1}\, ds\, d\omega \\ &= \frac{1}{2} \int_{S^{n-1}} \int_0^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, s^{n-1}\, ds\, d\omega \\ &\qquad + \frac{1}{2} \int_{S^{n-1}} \int_0^\infty \widetilde{f}(s(-\omega))\, e^{is\langle x,-\omega\rangle}\, s^{n-1}\, ds\, d\omega \\ &= \frac{1}{2} \int_{S^{n-1}} \int_0^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, s^{n-1}\, ds\, d\omega \\ &\qquad + \frac{1}{2} \int_{S^{n-1}} \int_{-\infty}^0 \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, (-s)^{n-1}\, ds\, d\omega \\ &= \frac{1}{2} \int_{S^{n-1}} \int_{-\infty}^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, |s|^{n-1}\, ds\, d\omega \qquad (2.15)\end{aligned}$$

Now if n is odd, then $|s|^{n-1} = s^{n-1}$, and so the right hand side above becomes

$$\begin{aligned}&\frac{1}{2} \int_{S^{n-1}} \int_{-\infty}^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, s^{n-1}\, ds\, d\omega \\ &\qquad = \frac{1}{2(i)^{n-1}} \int_{S^{n-1}} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty \frac{\partial^{n-1}}{\partial p^{n-1}} Rf(\omega, p)\, e^{-ips}\, dp \right)\, e^{is\langle x,\omega\rangle}\, ds\, d\omega,\end{aligned}$$

by the Projection-Slice Theorem. Then by the Fourier inversion formula for one variable, we obtain

$$(2\pi)^n f(x) = (-i)^{n-1}\, \pi \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial p^{n-1}} Rf(\omega, \langle \omega, x\rangle)\, d\omega$$

Note that $(\partial^{n-1}/\partial p^{n-1}) Rf(\omega, p)$ is an even function in $\mathcal{S}(S^{n-1} \times \mathbb{R})$, and so belongs to $\mathcal{S}(\Xi_n)$.

Taking (2.4) into account, in the odd case (2.15) now becomes

$$f(x) = c_n\, R^* \left(\frac{\partial^{n-1}}{\partial p^{n-1}} Rf \right)(x) \qquad (2.16)$$

where

$$c_n = \Omega_n / 2(2\pi i)^{n-1}. \qquad (2.17)$$

When n is even, (2.15) equals

$$\frac{1}{2} \int_{S^{n-1}} \int_{-\infty}^\infty \widetilde{f}(s\omega)\, e^{is\langle x,\omega\rangle}\, s^{n-1}\, \mathrm{sgn}(s)\, ds\, d\omega$$

But for each ω, the function $s \mapsto \widetilde{f}(s\omega)\, s^{n-1}\, \mathrm{sgn}(s)\, e^{is\langle x,\omega\rangle}$ belongs to $L^1(\mathbb{R})$, and so by the Fourier inversion formula and Lemma 2.2.2 the inner integral above equals

$$2\pi\, \mathcal{H}\left(\frac{\partial^{n-1}}{\partial p^{n-1}} Rf\right)(\omega, \langle x,\omega\rangle)$$

Thus, again by (2.4), we obtain

$$f(x) = c_n\, R^*\left(\mathcal{H}\frac{\partial^{n-1}}{\partial p^{n-1}} Rf\right)(x) \tag{2.18}$$

where c_n is also given by (2.17). Note that when n is even and $\varphi \in C^\infty(\Xi_n)$, then $(\partial^{n-1}\varphi/\partial p^{n-1})$ is an odd function in $\mathcal{S}(S^{n-1} \times \mathbb{R})$, but the convolution $\mathcal{H}(\partial^{n-1}\varphi/\partial p^{n-1})$ is even, and thus can be seen to represent a smooth function on Ξ_n.

The following theorem summarizes our calculations above.

Theorem 2.2.3. *For any $\varphi \in \mathcal{S}(\Xi_n)$, let*

$$\Lambda\varphi(\omega, p) = \begin{cases} \dfrac{\partial^{n-1}}{\partial p^{n-1}}\varphi(\omega,p) & \text{if } n \text{ is odd} \\ \mathcal{H}\left(\dfrac{\partial^{n-1}}{\partial p^{n-1}}\varphi\right)(\omega,p) & \text{if } n \text{ is even} \end{cases} \tag{2.19}$$

If $f \in \mathcal{S}(\mathbb{R}^n)$, the Radon transform Rf is inverted by

$$f(x) = c_n\, R^*(\Lambda Rf)(x) \tag{2.20}$$

where c_n is given by (2.17).

For n odd, we can also express the Radon inversion formula (2.20) by means of invariant differential operators.

Exercise 2.2.4. Let D be any linear differential operator on $\mathbb{R}^n$, with C^∞ coefficients, invariant under the action of $\mathrm{M}(n)$. Use the following steps to show that D is a polynomial in the Laplacian L:

$$D = a_l L^l + \cdots + a_1 L + a_0.$$

1. Since D invariant under translations, it has constant coefficients.
2. For each $v \in \mathbb{R}^n$, let D_v denote the directional derivative $D_v f(x) = (d/dt) f(x+tv)|_{t=0}$. The map $v \mapsto D_v$ extends to an isomorphism of the symmetric algebra $S(\mathbb{R}^n)$ onto the algebra of constant coefficient differential operators on $\mathbb{R}^n$.
3. The dot product in $\mathbb{R}^n$ identifies the dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$, via $v \in \mathbb{R}^n \mapsto v^* = \langle v, \cdot\, \rangle$. This extends to a map $D \mapsto D^*$ identifying the symmetric algebra $S(\mathbb{R}^n)$ can be identified with $S((\mathbb{R}^n)^*)$, the algebra of all polynomial functions on $\mathbb{R}^n$. We have $(\partial/\partial x_j)^* = e_j^* = x_j$.

4. Under this identification, we have $(k \cdot v)^* = (v^*)^{\tau(k)}$ for all $V \in \mathbb{R}^n$ and $k \in \mathrm{O}(n)$. Thus the algebra $\mathbb{D}(\mathbb{R}^n)$ of $\mathrm{M}(n)$-invariant differential operators on $\mathbb{R}^n$ may be identified with the algebra $I(\mathbb{R}^n)$ of polynomial functions on $\mathbb{R}^n$ invariant under the action of $\mathrm{O}(n)$.

5. Let $p \in I(\mathbb{R}^n)$. Then p is constant on each sphere centered at the origin 0. p is therefore determined by its restriction to the x_1-axis, where it is an even polynomial function of x_1. Thus $p(x_1, 0, \ldots, 0) = a_k x_1^{2l} + \cdots + a_1 x_1^2 + a_0$.

6. The polynomial $a_l \|x\|^{2l} + \cdots + a_1 \|x\|^2 + a_0$ is also constant on spheres centered at the origin – hence is $\mathrm{O}(n)$-invariant – and coincides with p on the x_1-axis. Hence it equals p.

7. Let $D \in \mathbb{D}(\mathbb{R}^n) = S(\mathbb{R}^n)$. Then $D^* = \sum_{j=0}^{l} a_j \|x\|^{2j}$, and hence $D = \sum_{j=0}^{l} a_j\, L^j$.

Since functions on Ξ_n correspond to even functions on $S^{n-1} \times \mathbb{R}$, there is a well-defined differential operator $\Box$ on Ξ_n given by

$$\Box\varphi(\omega, p) = \frac{\partial^2}{\partial p^2}\varphi(\omega, p) \qquad\qquad (\varphi \in \mathcal{E}(\Xi_n)) \tag{2.21}$$

From (2.2) we see that $\Box$ is invariant under $\mathrm{M}(n)$. It can be shown that any $\mathrm{M}(n)$-invariant differential operator on Ξ_n is a polynomial in $\Box$. (See [9] or [10].)

Let us denote the left regular representations of $M(n)$ on $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{E}(\Xi_n)$ by λ and ν, respectively. Then since the Radon transform R commutes with the left action by $M(n)$, we have

$$\begin{aligned} R(f^{\tau(v)})(\omega, p) &= (Rf)^{\tau(v)}(\omega, p) \\ &= Rf(\omega, p - \langle \omega, v\rangle) \end{aligned}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$.

If we replace v by $-tv$ above and take the derivative with respect to t at $t = 0$, we obtain

$$R(D_v\, f)(\omega, p) = \langle \omega, v\rangle \frac{\partial}{\partial p} Rf(\omega, p).$$

where $D_v\, f$ is the directional derivative of f in the direction of v. When $v = e_i$, this gives

$$R\left(\frac{\partial f}{\partial x_i}\right)(\omega, p) = \omega_i \frac{\partial}{\partial p} Rf(\omega, p)$$

Differentiating both sides above again with respect to x_i and summing, we see that

$$R(Lf)(\omega, p) = \Box(Rf)(\omega, p)$$

Conversely, if $\varphi \in \mathcal{E}(\Xi_n)$, then we can differentiate both sides of the formula for $R^*\varphi$ in (2.5) with respect to x_i to obtain

$$\frac{\partial}{\partial x_i} R^*\varphi(x) = \frac{1}{\Omega_n} \int_{S^{n-1}} \omega_i \, \frac{\partial}{\partial p}\varphi(\omega, \langle \omega, x\rangle)\, d\omega$$

Taking $\partial/\partial x_i$ again and summing, we conclude that

$$L(R^*\varphi)(x) = R^*(\Box\varphi)(x)$$

From this, we see that the Radon inversion formula (2.16) for n odd can be written as

$$\begin{aligned} f(x) &= c_n\, R^*(\Box^{\frac{n-1}{2}} Rf)(x) \\ &= c_n\, L^{\frac{n-1}{2}}\,(R^*Rf)(x) \end{aligned} \tag{2.22}$$

This formula says that when n is odd, the value of $f(x)$ is determined by the values of R^*Rf on an arbitrarily small neighborhood of x. (R^*R is said to be *locally invertible.*) This is not at all the case when n is even.

Application: The Wave Equation. Let us apply the inversion formula (2.20) to solve the initial value problem for the wave equation in $\mathbb{R}^n$. Given functions f, $g \in \mathcal{S}(\mathbb{R}^n)$, we want to find solutions $u(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ to the wave equation

$$L_x u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) \tag{2.23}$$

satisfying the initial conditions $u(x,0) = f_0(x)$ and $u_t(x,0) = f_1(x)$.

General theory says that the solution of this initial value problem is unique. In order to write the Radon transform solution, we first make the following observation, which can be checked by a simple computation. Let h be any smooth function on $\mathbb{R}$. Then for any $\omega \in S^{n-1}$, the function

$$(x,t) \mapsto h(\langle \omega, x\rangle + t)$$

satisfies the wave equation (2.23).

Now given f_0 and f_1 as above, put

$$Sf(\omega,p) = \begin{cases} \dfrac{\partial^{n-1}}{\partial p^{n-1}} Rf_0(\omega,p) + \dfrac{\partial^{n-2}}{\partial p^{n-2}} Rf_1(\omega,p) & \text{if } n \text{ is odd} \\ \mathcal{H}\left(\dfrac{\partial^{n-1}}{\partial p^{n-1}} Rf_0\right)(\omega,p) + \mathcal{H}\left(\dfrac{\partial^{n-2}}{\partial p^{n-2}} Rf_1\right)(\omega,p) & \text{if } n \text{ is even} \end{cases}$$

We now claim that the function

$$u(x,t) = c \int_{S^{n-1}} Sf(\omega, \langle \omega, x\rangle + t)\, d\omega \tag{2.24}$$

where $c = 1/1(2\pi i)^{n-1}$ solves the problem above.

From our observation, we know that the right hand side of (2.24) satisfies the wave equation. Therefore we just need to check that the initial conditions are satisfied. For simplicity, let us assume that n is odd, as the argument for n even is similar.

According to (2.24), we have

$$u(x,0) = c\int_{S^{n-1}} \frac{\partial^{n-1}}{\partial p^{n-1}} Rf_0(\omega, \langle \omega, x\rangle)\, d\omega + c\int_{S^{n-1}} \frac{\partial^{n-2}}{\partial p^{n-2}} Rf_1(\omega, \langle \omega, x\rangle)\, d\omega$$

The first integral equals $f_0(x)$ by the Radon inversion formula (2.16), and since n is odd, the integrand on the right is an odd function of ω, so the integral vanishes.

On the other hand,

$$u_t(x,0) = c\int_{S^{n-1}} \frac{\partial^{n}}{\partial p^{n-1}} Rf_0(\omega, \langle \omega, x\rangle)\, d\omega + c\int_{S^{n-1}} \frac{\partial^{n-1}}{\partial p^{n-2}} Rf_1(\omega, \langle \omega, x\rangle)\, d\omega$$

and in this case the first integral on the right vanishes and the second integral equals $f_1(x)$, again by (2.16).

General theory also says that solutions to the wave equation (2.23) propagate at unit speed. In particular, if the initial data f_0 and f_1 have support in the ball $B_\epsilon(0)$, then $u(x,t)$ will have support in the cone $\|x\| \leqslant \epsilon + |t|$.

When n is odd, the solution (2.24) is in fact supported in the shell $|t| - \epsilon \leqslant \|x\| \leqslant |t| + \epsilon$. (This is known as *Huygens' principle.*) To see this, suppose that $\|x\| < |t| - \epsilon$. Then $|\langle \omega, x\rangle + t| > |t| - |\langle \omega, x\rangle| \geqslant |t| - \|x\| > \epsilon$. Thus the plane $\xi(\omega,\ \langle \omega, x\rangle + t)$ does not intersect $B_\epsilon(0)$, and hence $Rf_j(\omega,\ \langle \omega, x\rangle + t) = 0$. Since Sf consists of derivatives of the initial data f_0 and f_1, it follows that $Sf(\omega,\ \langle \omega, x\rangle + t) = 0$ and thus $u(x,t) = 0$.

2.3 Filtered Backprojection

By far the most important application of Radon transforms lies in imaging, and more specifically, medical imaging. *Tomography* is the study of useful and efficient algorithms which are applied to recover functions (such as the mass density function) from its various integral transforms.

The field of medical tomography was founded by Alan Cormack, who established its theoretical basis in two papers published in the Journal of Applied Physics in 1963 and 1964 ([2]). For this work, he was awarded the Nobel Prize in Medicine in 1979 together with Godfrey Hounsfield, whose team developed the first CT (*computerized tomography*) scanner in 1971.

Let us consider the following simple model used in CT scanning. Suppose that $f(x)$ represents the mass density at the point x in a planar cross section of a human head, as shown in the figure below. Let ℓ be a line representing an X-ray beam. If $I(x)$ denotes the beam intensity at the point x, then I is attenuated along a short segment Δx along ℓ according to the relation

$$-\frac{\Delta I}{I} = f(x)\,\Delta x$$

If I_0 represents the initial intensity of the beam and I_1 the intensity recorded at the detector, it follows from the above that

$$\log\frac{I_0}{I_1} = \int_\ell f(x)\,dx$$

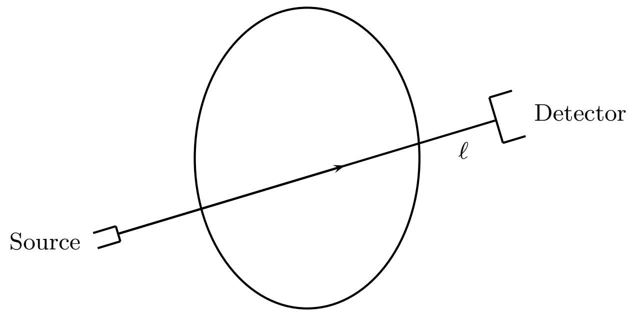

Figure 2.1: X-ray beam ℓ going through a human head

Thus the problem is to reconstruct the density $f(x)$ from the measured data I_0/I_1 along all lines ℓ, or more realistically, along a large but finite set of lines ℓ. This is, of course, just the inversion problem for the Radon transform on $\mathbb{R}^2$.

Rather than using (a discrete version of) the Radon inversion formula (2.20), the most common method for reconstructing X-ray images is the method of *filtered backprojection*. Its main advantage is its ability to cancel out high frequency noise. The key result is the convolution formula in Proposition 2.3.1 below.

If $\varphi,\ \psi \in \mathcal{S}(\Xi_n)$, their convolution $\varphi * \psi$ is the well-defined function on Ξ_n given by

$$\varphi * \psi(\omega, p) = \int_{\mathbb{R}} \varphi(\omega, p-t)\,\psi(\omega, t)\,dt$$

(See (2.10).)

Proposition 2.3.1. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\Xi_n)$. Then*

$$R^*(\varphi * Rf) \;=\; (R^*\varphi) * f \tag{2.25}$$

Proof. For any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
R^*(\varphi * Rf)(x) &= \frac{1}{\Omega_n}\int_{S^{n-1}}\left(\int_{-\infty}^{\infty}\varphi(\omega,\langle\omega,x\rangle - p)\,Rf(\omega,p)\,dp\right)d\omega \\
&= \frac{1}{\Omega_n}\int_{S^{n-1}}\left(\int_{-\infty}^{\infty}\varphi(\omega,\langle\omega,x\rangle - p)\left(\int_{\langle\omega,y\rangle=p} f(y)\,dm(y)\right)dp\right)d\omega \\
&= \frac{1}{\Omega_n}\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}\varphi(\omega,\langle\omega,x-y\rangle)\,f(y)\,dy\right)d\omega \\
&= \int_{\mathbb{R}^n}\left(\frac{1}{\Omega_n}\int_{S^{n-1}}\varphi(\omega,\langle\omega,x-y\rangle)\,d\omega\right)f(y)\,dy \\
&= \int_{\mathbb{R}^n} R^*\varphi(x-y)\,f(y)\,dy \\
&= \;(R^*\varphi) * f(x)
\end{aligned}$$

□

Let $\varphi \in \mathcal{S}(\Xi_n)$. There is a constant C such that $|\varphi(\xi)| < C$ for all $\xi \in \Xi_n$, and so by (2.5), $|\,R^*\varphi(x)| < C$ for all x.

Exercise 2.3.2. Prove that $R^*\varphi(x) \;=\; O(\|x\|^{-1})$ and that this is the best possible estimate.

Thus $R^*\varphi$ may be viewed as a tempered distribution. The following lemma gives the relation between the Fourier transform of $R^*\varphi$ and the partial Fourier transform of φ.

Lemma 2.3.3. *Let $\varphi \in \mathcal{S}(\Xi_n)$. Then the Fourier transform of $R^*\varphi$ is given by*

$$(\,R^*\varphi)^{\sim}\,(x) = \frac{2\,(2\pi)^{n-1}}{\Omega_n}\,\|x\|^{1-n}\,\widetilde{\varphi}\left(\frac{x}{\|x\|},\|x\|\right) \tag{2.26}$$

for all $x \neq 0$ in $\mathbb{R}^n$.

Proof. We begin by considering the following variant of the projection-slice theorem (2.9). If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned}
f(s\omega) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\widetilde{f}(y)\,e^{is\langle y,\omega\rangle}\,dy \\
&= (2\pi)^{-n}\int_{-\infty}^{\infty} R(\widetilde{f})(\omega,p)\,e^{ips}\,dp
\end{aligned}$$

By the Fourier inversion formula in $\mathbb{R}$, we then obtain

$$R(\widetilde{f})(\omega, p) = (2\pi)^{n-1} \int_{-\infty}^{\infty} f(s\omega)\, e^{-ips}\, ds$$

Thus by (2.7),

$$\begin{aligned}
(R^*\varphi)^{\sim}(f) &= R^*\varphi\,(\widetilde{f}) \\
&= \frac{1}{\Omega_n}\,\varphi\,(R\widetilde{f}) \\
&= \frac{1}{\Omega_n} \int_{S^{n-1}} \int_{-\infty}^{\infty} \varphi(\omega, p)\, R(\widetilde{f})(\omega, p)\, dp\, d\omega \\
&= \frac{(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{-\infty}^{\infty} \varphi(\omega, p) \left(\int_{-\infty}^{\infty} f(s\omega)\, e^{-ips}\, ds \right) dp\, d\omega \\
&= \frac{(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{-\infty}^{\infty} f(s\omega) \left(\int_{-\infty}^{\infty} \varphi(\omega, p)\, e^{-ips}\, dp \right) ds\, d\omega \\
&= \frac{(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{-\infty}^{\infty} f(s\omega)\, \widetilde{\varphi}(\omega, s)\, ds\, d\omega \\
&= \frac{(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{0}^{\infty} \left[f(s\omega)\, \widetilde{\varphi}(\omega, s) + f(-s\omega)\, \widetilde{\varphi}(\omega, -s) \right] ds\, d\omega \\
&= \frac{2(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{0}^{\infty} f(s\omega)\, \widetilde{\varphi}(\omega, s)\, ds\, d\omega \qquad (2.27)
\end{aligned}$$

For the last equality we used the fact that $\widetilde{\varphi}(\omega, -s) = \widetilde{\varphi}(-\omega, s)$. The right hand side of (2.27) thus equals

$$\begin{aligned}
&\frac{2(2\pi)^{n-1}}{\Omega_n} \int_{S^{n-1}} \int_{0}^{\infty} f(s\omega)\, (s^{1-n}\widetilde{\varphi}(\omega, s))\, s^{n-1}\, ds\, d\omega \\
&\qquad = \frac{2(2\pi)^{n-1}}{\Omega_n} \int_{\mathbb{R}^n} f(x) \left(\|x\|^{1-n}\, \widetilde{\varphi}\left(\frac{x}{\|x\|}, \|x\| \right) \right) dx,
\end{aligned}$$

proving the lemma. □

Lemma 2.3.3 above holds, with essentially the same proof, for $\varphi \in L^1(\Xi_n)$, since the partial Fourier transform $\widetilde{\varphi}(\omega, s)$ exists for almost all $(\omega, s) \in S^{n-1} \times \mathbb{R}$.

In applying (2.25) to recover f, we choose φ so that $R^*\varphi$ is an *approximate identity* and is *band-limited* (i.e., its Fourier transform is compactly supported). Thus we want to put $\varphi = \varphi_t$, a function parametrized by t, such that $(R^*\varphi_t)^{\sim}$ is compactly supported and

$$R^*\varphi_t \to \delta_0$$

in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$.

One choice for φ_t would be the function on $\mathbb{R}^n$ such that

$$(R^*\varphi_t)^{\sim}(y) = \chi_{B_t(0)}(y), \qquad (2.28)$$

where the right hand side is the characteristic function of the open ball $B_t(0)$. Then clearly $(R^*\varphi_t)^\sim \to 1$, and hence $R^*\varphi_t \to \delta_0$ as $t \to \infty$. In this case, let us determine φ_t explicitly.

Lemma 2.3.4. *Let* $F(x) = \chi_{B_1(0)}$*. Then*

$$\widetilde{F}(r\omega) = (2\pi)^{\frac{n}{2}}\, r^{-\frac{n}{2}}\, J_{\frac{n}{2}}(r) \tag{2.29}$$

for all $r > 0$ *and* $\omega \in S^{n-1}$*.*

Here J_p is the Bessel function

$$J_p(x) = \sum_{k=0}^{\infty} (-1)^k \,\frac{(x/2)^{2k+p}}{k!(k+p)!}$$

with $(k+p)!$ given by $\Gamma(k+p+1)$ when p is not an integer.

Proof. Since F is radial, so is $\widetilde{F}$. Thus we need only calculate $\widetilde{F}(-re_n)$, which equals

$$\int_{B_1(0)} e^{ir\,x_n}\,dx = \frac{\Omega_{n-1}}{n-1}\int_{-1}^{1} e^{irt}\,\left(1-t^2\right)^{\frac{n-1}{2}}\,dt$$

Now according to [12], equation 3.387(2), the right hand integral equals

$$\sqrt{\pi}\,\left(\frac{2}{r}\right)^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)\,J_{\frac{n}{2}}(r),$$

and from this equation (2.29) follows. □

If $F_t = \chi_{B_t(0)}$, then $F_t(x) = \chi_{B_1(0)}(x/t)$, so it follows from (2.29) that

$$(F_t)^\sim(r\omega) = (2\pi)^{\frac{n}{2}}\, t^{\frac{n}{2}}\, r^{-\frac{n}{2}}\, J_{\frac{n}{2}}(rt)$$

Thus if φ_t is to satisfy (2.28), we must have

$$R^*\varphi_t(x) = (2\pi)^{-\frac{n}{2}}\, t^n\, \frac{J_{\frac{n}{2}}(t\|x\|)}{(t\|x\|)^{\frac{n}{2}}}$$

To calculate φ_t itself, we make use of the relation (2.26). This gives

$$\widetilde{\varphi}_t(\omega, s) = \frac{\Omega_n}{2(2\pi)^{n-1}}\,|s|^{n-1}\,\chi_{(-t,t)}(s)$$

Thus

$$\begin{aligned}\varphi_t(\omega, p) &= \frac{\Omega_n}{2(2\pi)^n}\int_{-t}^{t}|s|^{n-1}\,e^{ips}\,ds \\ &= \frac{\Omega_n}{(2\pi)^n}\int_0^t s^{n-1}\,\cos(ps)\,ds\end{aligned}$$

When $n = 2$, this equals

$$\frac{t^2}{2\pi}\left(\frac{\sin(pt)}{pt} - \frac{1-\cos(pt)}{(pt)^2}\right)$$

In the method of filtered backprojection, the density $f(x)$ is approximated using the left hand side of (2.25). First, an appropriate "filter" φ is chosen to convolve with the data Rf, and then the dual transform R^* is applied (this is the "backprojection.")

2.4 The Support Theorem

In this section we consider for the classical Radon transform the following fundamental problem regarding supports of functions. Let us retain the notation of Chapter 1. Let B be a closed subset of X, and suppose that f is a function on X integrable over all orbits $\widehat{\xi}$ such that $Rf(\xi) = 0$ for all $\xi \in \Xi$ such that $\widehat{\xi}$ is disjoint from B. Is f supported in B?

The problem of whether $f \mapsto Rf$ is injective is a special case of the support problem above, if we put $B = \emptyset$.

In the case of the classical Radon transform, with $X = \mathbb{R}^n$ and $\Xi = \Xi_n$, it is easy to see that we need to require that B be convex. If we assume that f decreases reasonably rapidly, and that B is a closed ball, then we obtain the following support result due to Helgason.

Theorem 2.4.1. *Let f be a continuous function on $\mathbb{R}^n$ such that for all $k \in \mathbb{Z}^+$,*

$$\sup_{x\in\mathbb{R}^n} \|x\|^k\,|f(x)| < \infty \tag{2.30}$$

If $Rf(\xi) = 0$ for all hyperplanes ξ such that $d(0,\xi) > R$, then f has support on the closed ball $\overline{B_R(0)}$.

The case $n = 1$ being trivial, we can assume that $n > 1$.

Fix any $\epsilon > 0$ and let ϕ_ϵ be an approximate identity supported on $B_\epsilon(0)$. Then the convolution $f * \phi_\epsilon$ belongs to $\mathcal{E}(\mathbb{R}^n)$ and satisfies the estimate (2.30). In addition, by (2.10), we see that $R(f * \phi_\epsilon)(\omega, p) = 0$ whenever $|p| > R + \epsilon$. If we can prove that $f * \phi_\epsilon$ has support in $\overline{B_{R+\epsilon}(0)}$, then letting $\epsilon \to 0$ we see that f has support in $\overline{B_R(0)}$.

Thus it suffices to assume that $f \in \mathcal{E}(\mathbb{R}^n)$.

We first deal with the simplest case possible, in which f is a *radial* function. Thus there is an even C^∞ function F on $\mathbb{R}$ such that $f(x) = F(\|x\|)$. Then the

Radon transform Rf is also radial: this means that there exists an even C^∞ function G on $\mathbb{R}$ such that $Rf(\omega, p) = G(p)$. Integrating f in polar coordinates, it follows that F and G are related by the equation

$$\begin{aligned} G(p) &= \Omega_{n-1} \int_0^\infty F((p^2+s^2)^{\frac{1}{2}})\, s^{n-2}\, ds \\ &= \Omega_{n-1} \int_p^\infty F(t)\,(t^2-p^2)^{\frac{n-3}{2}}\, t\, dt \end{aligned} \tag{2.31}$$

This is an integral equation of Abel type for which there are standard methods of solution [44], which we give below.

In case n is odd, the integral equation (2.31) can be solved using a differential operator. In fact, if we apply the differential operator $d/d(p^2) = (1/2p)\, d/dp$ to both sides $(n-3)/2$ times, we obtain

$$\left(\frac{1}{2p}\frac{d}{dp}\right)^{\frac{n-3}{2}} G(p) = (-1)^{\frac{n-3}{2}}\, \Omega_{n-1} \left(\frac{n-3}{2}\right)! \int_p^\infty F(t)\, t\, dt$$

We note that differentiating inside the integral sign is permissible because of the decay assumption (2.30) on F.

Applying $(1/p)\, d/dp$ one more time to the last equation above allows us to solve for F in terms of G:

$$F(p) = \left(-\frac{1}{2\pi p}\frac{d}{dp}\right)^{\frac{n-1}{2}} G(p) \tag{2.32}$$

Since, by hypothesis $G(p) = 0$ whenever $p > R$, it follows that $F(p) = 0$ whenever $p > R$.

For the general solution of (2.31), we put $H(p) = G(p)/\Omega_{n-1}$ and let $m = (n-3)/2$. Thus we want to solve the integral equation

$$H(p) = \int_p^\infty F(t)\,(t^2-p^2)^m\, t\, dt \tag{2.33}$$

for F. Fix $s > 0$, then multiply both sides above by $(p^2-s^2)^m\, p$ and integrate with respect to p from s to ∞:

$$\begin{aligned} \int_s^\infty H(p)\,(p^2-s^2)^m\, p\, dp &= \int_s^\infty \left(\int_p^\infty F(t)\,(t^2-p^2)^m\, t\, dt\right)(p^2-s^2)^m\, p\, dp \\ &= \int_s^\infty F(t)\, t \left(\int_s^t [(t^2-p^2)(p^2-s^2)]^m\, p\, dp\right) dt \end{aligned}$$

The change of order of integration is justified due to the decay assumption on F and since the inner integral converges and is clearly bounded by a power of t. To evaluate it, we use the substitution $(t^2-s^2)\, v = t^2+s^2-2p^2$, so that

$$(t^2-p^2)(p^2-s^2) = \frac{1}{4}(t^2-s^2)^2\,(1-v^2)$$

and therefore

$$\int_s^t [(t^2 - p^2)(p^2 - s^2)]^m \, p \, dp = \frac{(t^2 - s^2)^{2m+1}}{4^{m+1}} \int_{-1}^1 (1 - v^2)^m \, dv$$
$$= \frac{\Gamma(m+1)\,\Gamma\left(\frac{1}{2}\right)}{4^{m+1}\,\Gamma\left(m + \frac{3}{2}\right)} (t^2 - s^2)^{2m+1}$$

Thus

$$\int_s^\infty H(p)\,(p^2 - s^2)^m \, p \, dp = \frac{\Gamma(m+1)\,\Gamma\left(\frac{1}{2}\right)}{4^{m+1}\,\Gamma\left(m + \frac{3}{2}\right)} \int_s^\infty F(t)\,(t^2 - s^2)^{2m+1} \, t \, dt$$

Since $2m + 1 = n - 2 \in \mathbb{Z}^+$, we obtain

$$F(s) = \frac{2\,\Gamma\left(m + \frac{3}{2}\right)}{\Gamma(m+1)\,\Gamma\left(\frac{1}{2}\right)\,(2m+1)!} \left(-\frac{1}{s}\frac{d}{ds}\right)^{2m+2} \int_s^\infty H(p)\,(p^2 - s^2)^m \, p \, dp$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\,(n-2)!} \left(-\frac{1}{s}\frac{d}{ds}\right)^{n-1} \int_s^\infty G(p)\,(p^2 - s^2)^{\frac{n-3}{2}} \, p \, dp \tag{2.34}$$

By hypothesis, $G(p) = 0$ for $p > R$, so we see from the above that $F(s) = 0$ for $s > R$. This proves the support theorem in the case when f is radial.

Let us now prove the main assertion in Theorem 2.4.1. For this, we will need to make use of the *mean value operator* in $\mathbb{R}^n$. For any $r \geqslant 0$, we let M^r denote the operator on $C(\mathbb{R}^n)$ given by

$$M^r f(x) = \frac{1}{\Omega_n} \int_{S^{n-1}} f(x + r\omega)\, d\omega \qquad (x \in \mathbb{R}^n) \tag{2.35}$$

for all $f \in C(\mathbb{R}^n)$. Thus $M^r f$ is the average value of f on the sphere $S_r(x)$ with center x and radius r. Now, up to constant multiple, $d\omega$ is the unique measure on S^{n-1} invariant under the left action of $\mathrm{O}(n)$ (or $\mathrm{SO}(n)$). Hence we can also write the above as

$$M^r f(x) = \int_{\mathrm{O}(n)} f(x + k \cdot y)\, dk \tag{2.36}$$

for any y such that $\|y\| = r$.

Suppose that $f \in \mathcal{E}(\mathbb{R}^n)$ and satisfies the estimate (2.30). Fix $x_0 \in \mathbb{R}^n$ and define the function G on $\mathbb{R}^n$ by

$$G(x) = M^{\|x\|} f(x_0) \tag{2.37}$$

Then G is radial and satisfies the same estimate since for any $m \in \mathbb{Z}^+$,

$$\|x\|^m \, |M^{\|x\|} f(x_0)| \leqslant \int_{\mathrm{O}(n)} |f(x_0 + k \cdot x)| \, (\|x_0 + k \cdot x\| + \|x_0\|)^m \, dk$$

so $\sup_x \|x\|^m |G(x)| < \infty$.

Next we observe that $RG(\xi) = 0$ whenever $d(0, \xi) > R + \|x_0\|$. In fact, for such ξ we have

$$\begin{aligned} RG(\xi) &= \int_{x \in \xi} \int_{\mathrm{O}(n)} f(x_0 + k \cdot x)\, dk\, dx \\ &= \int_{\mathrm{O}(n)} Rf(x_0 + k \cdot \xi)\, dk \end{aligned} \tag{2.38}$$

But then

$$d(0, x_0 + k \cdot \xi) \geqslant d(0, \xi) - \|x_0\| > R$$

and so, by the hypothesis on Rf, the integrand in the right hand side of (2.38) vanishes. Thus $RG(\xi) = 0$.

By the support theorem for radial functions, we conclude that $G(x) = 0$ whenver $\|x\| > R + \|x_0\|$. Thus we have proved that

$$\int_{\mathrm{O}(n)} f(x_0 + k \cdot x)\, dk = 0 \tag{2.39}$$

for all x such that $\|x\| > R + \|x_0\|$. Now any sphere enclosing the closed ball $\overline{B}_R(0)$ is of the form $S_r(x_0)$, where $r > R + \|x_0\|$. Thus (2.39) shows that the integral of f over any sphere enclosing $\overline{B}_R(0)$ vanishes.

Lemma 2.4.2. *Suppose that $f \in \mathcal{E}(\mathbb{R}^n)$ satisfies the decay property* (2.30)*. If*

$$\int_S f(x)\, dm(x) = 0$$

for any sphere S enclosing $\overline{B}_R(0)$, then $f(x) = 0$ for all x outside $\overline{B}_R(0)$.

To prove the lemma, we note that the hypothesis on f implies that

$$\int_B f(x)\, dx = \int_{\mathbb{R}^n} f(x)\, dx$$

for any ball B containing $\overline{B}_R(0)$ in its interior. Thus $\int_B f(x)\, dx$ is constant for all balls containing $\overline{B}_R(0)$.

Let $B = B_r(x)$ be any such ball. A slight perturbation of B leaves the value of the $\int_B f(x)\, dx$ unchanged, and hence

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_j} \int_{B_r(0)} f(x + y)\, dy \\ &= \int_{B_r(0)} \partial_j f(x + y)\, dy \\ &= \int_{B_r(x)} \partial_j f(y)\, dy \end{aligned}$$

for each j, where $\partial_j f$ denotes the partial derivative of f with respect to its jth argument. Let e_j denote the jth standard basis vector and let $f e_j$ be the vector field $(0, \ldots, f, \ldots, 0)$ with f in the jth place. Then by the divergence theorem the last integral above equals

$$\int_{S_r(x)} (f\, e_j \cdot n)(y)\, dm(y)$$

where n is the outward unit normal to the sphere $S_r(x)$ at y. This shows that

$$\int_{S^{n-1}} f(x + r\omega)\, \omega_j\, d\omega = 0$$

But by hypothesis,

$$\int_{S^{n-1}} f(x + r\omega)\, d\omega = 0$$

Thus

$$\int_{S^{n-1}} f(x + r\omega)\, (x_j + r\omega_j)\, d\omega = 0$$

and so

$$\int_S f(y)\, y_j\, dm(y) = 0$$

for all spheres S enclosing $B_R(0)$. Now $f(y)\, y_j$ satisfies the estimates (2.30) so applying our argument inductively we conclude that

$$\int_S f(y)\, p(y)\, dm(y) = 0$$

for any such sphere S, and for any polynomial $p(y)$ on $\mathbb{R}^n$. Since the set of restrictions of polynomials $p(y)$ to $S_r(x)$ is dense in $L^2(S_r(x))$, this shows that $f = 0$ on S and proves our lemma, as well as Theorem 2.4.1.

Note that we can replace the ball $B_R(0)$ in Theorem 2.4.1 by any compact convex set B, since any such set is the intersection of the closed balls containing it.

Corollary 2.4.3. *Suppose that f and g are continuous functions on $\mathbb{R}^n$ satisfying the estimates* (2.30). *If $Rf(\xi) = Rg(\xi)$ for all ξ outside a compact convex set B, then $f = g$ outside B.*

The Support Theorem requires that f satisfy decrease properties such as (2.30), as there are counterexamples when f decreases less rapidly at infinity. For example, consider the function on $\mathbb{R}^2$ given by

$$f(x, y) = (x + iy)^{-5}$$

when (x, y) is outside the unit disk, with f smooth on all of $\mathbb{R}^2$. (Such an f is possible.) Then $f \in L^1(\mathbb{R}^2)$ and by Cauchy's theorem, $Rf(\xi) = 0$ for all lines ξ outside the unit disk.

Convolution with an approximate identity in $\mathrm{M}(n)$ allows us to formulate a support theorem for $\mathcal{E}'(\mathbb{R}^n)$.

Corollary 2.4.4. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ such that RT has support in $\{\xi \in \Xi_n \mid d(0,\xi) \leqslant R\}$. Then T has support in the closed ball $\overline{B}_R(0)$. In particular, R is injective on $\mathcal{E}'(\mathbb{R}^n)$.*

Proof. Let $\{f_m\}$ be an approximate identity in $\mathrm{M}(n)$. For any $\epsilon > 0$, $f_m * RT = R(f_m * T)$ has support in $\{\xi \in \Xi_n \mid d(0,\xi) \leqslant R + \epsilon\}$ when m is sufficiently large. Since $f_m * T \in \mathcal{D}(\mathbb{R}^n)$, the support theorem shows that it has support in $\overline{B}_{R+\epsilon}(0)$. Since $f_m * T$ converges weakly to T, we see that T is supported in $\overline{B}_{R+\epsilon}(0)$. This proves the corollary, since ϵ is arbitrary. ☐

If $f \in C(\mathbb{R}^n)$ satisfies (2.30), Corollary 2.4.3 shows us that there is a unique solution to the following *exterior problem*: determine $f(x)$ for all x outside a compact convex set B from its integrals $Rf(\xi)$ for all ξ disjoint from B. While the Radon inversion formula (2.20) takes into account the integrals $Rf(\xi)$ for all hyperplanes ξ, it is often desirable to recover $f(x)$ outside B only from Radon data outside B. For example, B may contain a beating heart, a pacemaker, or some metallic object which completely absorbs X rays.

In practice, one recovers f from the exterior data $Rf(\xi)$ through reconstruction algorithms using singular value decompositions and other methods, which we will not discuss in these notes. See, for example, [34] for one such algorithm.

The Support Theorem also shows that a function f can be recovered from *limited angle data*. More precisely, a compactly supported function f on $\mathbb{R}^n$ can be recovered from its Radon transforms $Rf(\omega, p)$, where the normal vectors ω are allowed to vary only inside a spherical cap in S^{n-1}.

Theorem 2.4.5. *For any $\omega_0 \in S^{n-1}$ and any number α such that $0 < \alpha < 1$, let $C_\alpha(\omega_0)$ be the spherical cap $\langle \omega, \omega_0 \rangle > \alpha$. Suppose that $f \in C_c(\mathbb{R}^n)$ satisfies $Rf(\omega, p) = 0$ for all $p > 0$ and all $\omega \in C_\alpha(\omega_0)$. Then $f(x) = 0$ for all x in the half space $\langle x, \omega_0 \rangle > 0$.*

Proof. We can assume that ω_0 is the north pole e_n. Let $B_R(0)$ be an open ball containing the support of f. Fix $\epsilon > 0$, and for any $s > 0$ consider the ball $B = B_{s+\epsilon}(-s\, e_n)$. We can choose s so large that for all $\omega \notin C_\alpha(e_n)$ and all $p > 0$, any hyperplane $\xi(\omega, p)$ disjoint from B is also disjoint from $B_R(0)$. (See Figure 2.2 below.)

We now claim that $Rf(\xi) = 0$ for all hyperplanes ξ disjoint from B. Since B contains $B_\epsilon(0)$, any such hyperplane is of the form $\xi = \xi(\omega, p)$ with $\omega \in S^{n-1}$ and $p > \epsilon$. If $\omega \in C_\alpha(e_n)$, our claim is true by hypothesis. If $\omega \notin C_\alpha(e_n)$, the our claim is also true by the observation in the previous paragraph.

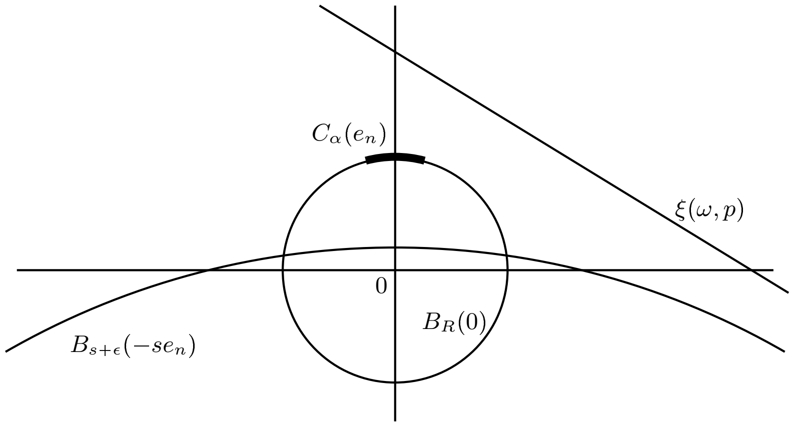

Figure 2.2: $\omega \notin C_\alpha(e_n)$ and $\xi(\omega,p)$ disjoint from B

Thus, by the Support Theorem, we conclude that $f(x) = 0$ for all x outside B. Lettin $s \to \infty$, it follows that $f(x) = 0$ for all x in the half plane $\langle x, e_n\rangle > \epsilon$. Then letting $\epsilon \to 0$, we obtain our desired conclusion. $\square$

As a corollary, we see that if $f \in C_c(\mathbb{R}^n)$ such that $Rf(\omega,p) = 0$ for all p and all $\omega \in C_\alpha(\omega_0)$, then $f(x) = 0$ for all x.

In *limited angle tomography*, algorithms for recovering a function f are analyzed given tomographic data $Rf(\omega,p)$, where ω is limited to some open subset of S^{n-1}. See [3], [13], or [33] for methods and analysis behind aspects of this important problem.

2.5 Moment Conditions and Range Characterization

The Projection-Slice Theorem shows that the classical Radon transform R is injective on $L^1(\mathbb{R}^n)$, and hence on $\mathcal{S}(\mathbb{R}^n)$. Our objective in this section is to characterize the range $R\mathcal{S}(\mathbb{R}^n)$.

So let $f \in \mathcal{S}(\mathbb{R}^n)$. For any $\omega \in S^{n-1}$, the function $x \mapsto \langle x,\omega\rangle$ is constant on each hyperplane orthogonal to ω. This fact allows us to obtain necessary conditions on the range of R.

For any $k \in \mathbb{Z}^+$, we have

$$\int_{-\infty}^{\infty} Rf(\omega, p)\, p^k\, dp = \int_{-\infty}^{\infty} p^k \left(\int_{\langle x, \omega \rangle = p} f(x)\, dm(x) \right) dp$$
$$= \int_{\mathbb{R}^n} f(x)\, \langle x, \omega \rangle^k\, dx$$

The right hand side above is a homogeneous polynomial $P_k(\omega)$ of degree k in ω.

A function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is said to satisfy the *Helgason moment conditions* if, for every $k \in \mathbb{Z}^+$, there exists a homogeneous polynomial function P_k of degree k on $\mathbb{R}^n$ such that

$$\int_{-\infty}^{\infty} \varphi(\omega, p)\, p^k\, dp = P_k(\omega) \tag{2.40}$$

Let $\mathcal{S}_H(\Xi_n)$ denote the vector space of all $\varphi \in \mathcal{S}(\Xi_n)$ satisfying the Helgason moment conditions. We have shown that $R\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}_H(\Xi_n)$.

Theorem 2.5.1. $R\mathcal{S}(\mathbb{R}^n) = \mathcal{S}_H(\Xi_n)$.

Proof. Suppose that $\varphi \in \mathcal{S}_H(\Xi_n)$. We want to produce an $f \in \mathcal{S}(\mathbb{R}^n)$ such that $Rf = \varphi$. If such an f does exist, the projection-slice theorem tells us what it must be.

Explicitly, let us take the one-dimensional Fourier transform of φ along its second argument. Define the function $\Phi \in \mathcal{S}(\Xi_n)$ by

$$\Phi(\omega, s) = \int_{-\infty}^{\infty} \varphi(\omega, p)\, e^{-ips}\, dp \tag{2.41}$$

Since $\Phi(\omega, s) = \Phi(-\omega, -s)$, there exists a function F on $\mathbb{R}^n \backslash \{0\}$ such that $F(s\omega) = \Phi(\omega, s)$ for all ω and all $s \neq 0$. Since s and $n-1$ of the coordinates of ω act as local coordinates on small open sets on $\mathbb{R}^n \backslash \{0\}$, we see that $F \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

Because of the moment conditions (2.40) for $k = 0$, $\Phi(\omega, 0)$ is homogeneous of degree 0, and hence constant, in ω. Thus we can put $F(0) = \Phi(\omega, 0)$, and the relation $F(s\omega) = \Phi(\omega, s)$ is true for all s and ω.

It is easy to show that F is continuous at 0. In fact, suppose that there is an $\epsilon > 0$ and a sequence $s_j\omega_j$ in $\mathbb{R}^n$ converging to 0 but with $|F(s_j\omega_j) - F(0)| > \epsilon$. By taking a subsequence of the ω_j, we can assume that $\omega_j \to \omega_0$ for some $\omega_0 \in S^{n-1}$. Then since $s_j \to 0$, it follows that $F(s_j\omega_j) = \Phi(\omega_j, s_j) \to \Phi(\omega_0, 0) = F(0)$, a contradiction.

We would now like to prove that $F \in \mathcal{S}(\mathbb{R}^n)$. Once we establish this, the rest of the proof is an easy consequence of the projection-slice theorem. In fact, let f be the inverse Fourier transform of F:

$$f(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} F(x)\, e^{i\langle x, y \rangle}\, dx$$

Then

$$\begin{aligned} F(s\omega) &= \tilde{f}(s\omega) \\ &= \int_{-\infty}^{\infty} Rf(\omega, p)\, e^{-ips}\, dp \end{aligned}$$

Comparing with (2.41), we see that $Rf = \varphi$, as desired.

Now we already know that $F \in C^\infty(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$. Let us now show that F is smooth at 0. This is the most technical part of the proof.

For this, it will suffice to prove that all partial derivatives of F, of all orders, are bounded on the punctured ball $B_1(0)\backslash\{0\}$. We will do so by expressing each mth order partial derivative

$$\frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}} \tag{2.42}$$

in polar coordinates. Let ϵ be a fixed small positive number. We will prove that (2.42) is bounded on the conical sector consisting of all points $x = s\omega$ where $0 < s < 1$ and $\omega_n > \epsilon$. As the punctured ball is the union of finitely many such sectors, this will be enough to prove the smoothness of F at the origin.

On our given sector, we can use $\omega_1, \ldots, \omega_{n-1}, s$ as local coordinates. Then by the chain rule,

$$\frac{\partial}{\partial x_j} = \omega_j \frac{\partial}{\partial s} + \sum_{i=1}^{n-1} \left(\frac{\delta_{ij} - \omega_i \omega_j}{s} \right) \frac{\partial}{\partial \omega_i} \tag{2.43}$$

for $1 \leqslant j \leqslant n-1$, and

$$\frac{\partial}{\partial x_n} = (1 - \omega_1^2 - \cdots - \omega_{n-1}^2)^{\frac{1}{2}} \left(\frac{\partial}{\partial s} - \sum_{i=1}^{n-1} \frac{\omega_i}{s} \frac{\partial}{\partial \omega_i} \right) \tag{2.44}$$

From this, it is straightforward to prove, by induction on m, that in these coordinates, the partial derivative (2.42) is of the form

$$\sum_{J,k} \frac{A_{J,k}(\omega_1, \ldots, \omega_{n-1})}{s^{m-k}} \frac{\partial^{|J|+k} F}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}} \partial s^k} \tag{2.45}$$

In the above, $A_{J,k}$ is a C^∞ function of $(\omega_1, \ldots, \omega_{n-1})$, and the sum ranges over all multiindices $J = (j_1, \ldots, j_{n-1})$ and all nonnegative integers k such that $|J| + k \leqslant m$. From (2.43) and (2.44), and because of the fact that $(1 - \omega_1^2 - \cdots - \omega_{n-1}^2)^{1/2} > \epsilon$, it is not hard to see that the $A_{J,k}$ are bounded in the sector.

Now by the moment conditions we have

$$
\begin{aligned}
F(s\omega) &= \int_{-\infty}^{\infty} \varphi(\omega, p) \left(\sum_{k=1}^{m-1} \frac{(-ips)^k}{k!} + e_m(-ips) \right) dp \\
&= \sum_{k=1}^{m-1} \frac{(-i)^k P_k(s\omega)}{k!} + \int_{-\infty}^{\infty} \varphi(\omega, p)\, e_m(-ips)\, dp
\end{aligned}
$$

where $e_m(-ips) = \sum_{k=m}^{\infty} (-ips)^k/k!$. Since the sum on the right hand side above is a polynomial of degree $\leqslant m-1$ in $x = s\omega$, its mth order derivatives vanish. Thus we need only apply (2.45) to the integral on the right hand side above. But

$$
\frac{1}{s^{m-k}} \frac{\partial^k}{\partial s^k} \int_{-\infty}^{\infty} \varphi(\omega, p)\, e_m(-ips)\, dp = \int_{-\infty}^{\infty} \varphi(\omega, p)\, (-ip)^m \left(\frac{e_{m-k}(-ips)}{(-ips)^{m-k}} \right) dp
$$

Hence

$$
\begin{aligned}
&\frac{1}{s^{m-k}} \frac{\partial^{|J|+k}}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}} \partial s^k} \int_{-\infty}^{\infty} \varphi(\omega, p)\, e_m(-ips)\, dp \\
&\qquad = \int_{-\infty}^{\infty} \frac{\partial^{|J|}}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}}} \varphi(\omega, p)\, (-ip)^m \left(\frac{e_{m-k}(-ips)}{(-ips)^{m-k}} \right) dp
\end{aligned}
$$

Now the fraction $e_{m-k}(it)/(it)^{m-k}$ is a bounded C^∞ function of t and since $\varphi \in \mathcal{S}(S^{n-1} \times \mathbb{R})$, we have the estimate

$$
\left| \frac{\partial^{|J|}}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}}} \varphi(\omega, p)\, (-ip)^m \right| < \frac{C_{J,m}}{1 + |p|^2}
$$

where $C_{J,m}$ is a positive constant independent of the ω_j. This shows that the derivatives (2.42) are bounded on the sector, and completes the proof that $F \in \mathcal{E}(\mathbb{R}^n)$.

Finally we prove that $F \in \mathcal{S}(\mathbb{R}^n)$. For this, it is enough to verify the rapid decrease condition (2.11) on the sector $x = s\omega$ with $s > 0$ and $\omega_n > \epsilon$. On this sector, we can express the mth order partial derivative (2.42) in polar coordinates by (2.45). The functions $A_{J,k}(\omega)$ are bounded in this sector, and for any nonnegative N we have

$$
\begin{aligned}
&s^N \frac{\partial^{|J|+k} F}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}} \partial s^k} \\
&\qquad = (-i)^N \int_{-\infty}^{\infty} \frac{\partial^{|J|} F}{\partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}}} \circ \frac{\partial^N}{\partial p^N} \left((-ip)^k \varphi \right) (\omega, p)\, e^{-ips}\, dp
\end{aligned}
$$

Since $\varphi \in \mathcal{S}(\Xi_n)$, the absolute value of the right hand integral has a bound independent of s. This proves that $F \in \mathcal{S}(\mathbb{R}^n)$.

This completes the proof of Theorem 2.5.1. $\square$

We can now combine Theorem 2.5.1 above and the support Theorem 2.4.1 to give a precise characterization of the range of $\mathcal{D}(\mathbb{R}^n)$ under the classical Radon transform. Let $\mathcal{D}_H(\Xi_n) = \mathcal{D}(\Xi_n) \cap \mathcal{S}_H(\Xi_n)$.

Theorem 2.5.2. *$R\mathcal{D}(\mathbb{R}^n) = \mathcal{D}_H(\Xi_n)$.*

Let us recall from (2.7) that the Radon transform of a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is defined by

$$RT(\varphi) = \Omega_n \, T(R^*\varphi) \tag{2.46}$$

for all $\varphi \in \mathcal{D}(\Xi_n)$. Thus $RT \in \mathcal{E}'(\Xi_n)$. We identify distributions on Ξ_n with even distributions on $S^{n-1} \times \mathbb{R}$; i.e., distributions Ψ on $S^{n-1} \times \mathbb{R}$ such that $\Psi(\varphi) = \Psi(\varphi^*)$, where $\varphi^*(\omega, p) = \varphi(-\omega, -p)$.

Convolution with an approximate identity in the motion group $\mathrm{M}(n)$ allows us to characterize the range of the Radon transform on compactly supported distributions.

Theorem 2.5.3. *Let $\mathcal{E}'_H(\Xi_n)$ be the set of all $\Psi \in \mathcal{E}'(\Xi_n)$ which satisfy the following moment conditions: for each $k \in \mathbb{Z}^+$, there is a homogeneous degree k polynomial P_k on $\mathbb{R}^n$ such that*

$$\int_{S^{n-1}\times\mathbb{R}} h(\omega)\, p^k \, d\Psi(\omega, p) = \int_{S^{n-1}} h(\omega)\, P_k(\omega)\, d\omega \tag{2.47}$$

for all $h \in \mathcal{E}(S^{n-1})$. Then $R\mathcal{E}'(\mathbb{R}^n) = \mathcal{E}'_H(\Xi_n)$.

Proof. Suppose that $T \in \mathcal{E}'(\mathbb{R}^n)$. Let $k \in \mathbb{Z}^+$ and $h \in \mathcal{E}(S^{n-1})$. Then by (2.46), we have

$$\begin{aligned}\int_{S^{n-1}\times\mathbb{R}} h(\omega)\, p^k \, d(RT)(\omega, p) &= \int_{\mathbb{R}^n} \int_{S^{n-1}} h(\omega)\, \langle \omega, x\rangle^k \, d\omega \, dT(x) \\ &= \int_{S^{n-1}} h(\omega) \left(\int_{\mathbb{R}^n} \langle \omega, x\rangle^k \, dT(x) \right) d\omega\end{aligned}$$

by the Fubini theorem for distributions. The inner expression

$$\int_{\mathbb{R}^n} \langle \omega, x\rangle^k \, dT(x)$$

is clearly a homogeneous degree k polynomial in the coordinates of ω. This shows that $RT \in \mathcal{E}'_H(\Xi_n)$.

For the converse, we first observe that if Ψ is a compactly supported C^∞ function on Ξ_n, then the moment conditions (2.40) and (2.47) are equivalent. Now suppose $\Psi \in \mathcal{E}'_H(\Xi_n)$. Then for any $g \in \mathrm{M}(n)$, the translate $\Psi^{\tau(g)}$ also belongs to $\mathcal{E}'_H(\Xi_n)$. In fact, if $\sigma \in \mathrm{O}(n)$, then

$$\int_{S^{n-1}\times\mathbb{R}} h(\omega)\, p^k \, d\Psi^{\tau(\sigma)}(\omega, p) = \int_{S^{n-1}} h(\omega)\, P_k(\sigma^{-1}\cdot\omega)\, d\omega$$

and if $y \in \mathbb{R}^n$, then

$$\begin{aligned}\int_{S^{n-1}\times\mathbb{R}} h(\omega)\, p^k\, d\Psi^{\tau(y)}(\omega, p) &= \int_{S^{n-1}\times\mathbb{R}} h(\omega)\, (p + \langle\omega, y\rangle)^k\, d\Psi(\omega, p) \\ &= \int_{S^{n-1}} h(\omega) \left(\sum_{l=0}^{k} \binom{k}{l} P_l(\omega)\, \langle\omega, y\rangle^{k-l}\right) d\omega\end{aligned}$$

Let $\{h_m\}$ be an approximate identity in $\mathrm{M}(n)$. Then, by the above, the $h_m * \Psi$ are C^∞ functions satisfying the Helgason moment conditions (2.40). Thus by Theorem 2.5.2 there exist functions $f_m \in \mathcal{D}(\mathbb{R}^n)$ such that $h_m * \Psi = Rf_m$. If Ψ has support in $\{\xi \in \Xi_n \mid d(0, \xi) \leqslant R\}$, then for all sufficiently large m, f_m has support in the closed ball $\overline{B}_{R+1}(0)$ in $\mathbb{R}^n$.

By the Radon inversion formula, $f_m = c_n\ R^*(\Lambda(h_m * \Psi))$, where Λ is the operator (2.19). Now Λ can be extended to a weakly continuous map from $\mathcal{E}'(\Xi_n)$ to $\mathcal{D}'(\Xi_n)$. By the remark after Proposition 1.2.3, R^* is weakly continuous from $\mathcal{D}'(\Xi_n)$ to $\mathcal{D}'(\mathbb{R}^n)$. From this we see that the sequence f_m converges weakly in $\mathcal{D}'(\mathbb{R}^n)$ to the distribution $T = c_n\, R^*(\Lambda\Psi))$. Clearly T has support in $\overline{B}_{R+1}(0)$. Since R is weakly continuous from $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{E}'(\Xi_n)$, and since $f_m \to T$ and $h_m * \Psi \to \Psi$, we conclude that $RT = \Psi$, as desired. □

It will now be useful to formulate the moment conditions (2.40) in terms of spherical harmonics. For any $l \in \mathbb{Z}^+$, let us assume that $\{Y_{lm}\}_{m=1}^{d(l)}$ is an orthonormal basis, in $L^2(S^{n-1})$, for the area measure $d\omega$, of the vector space of degree l spherical harmonics. We can assume that the Y_{lm} are real-valued.

Any function $\varphi \in \mathcal{E}(S^{n-1} \times \mathbb{R})$ has a spherical harmonic expansion

$$\varphi(\omega, p) = \sum_{l\geqslant 0} \sum_{m=1}^{d(l)} g_{lm}(p)\, Y_{lm}(\omega) \tag{2.48}$$

where

$$g_{lm}(p) = \int_{S^{n-1}} \varphi(\omega, p)\, Y_{lm}(\omega)\, d\omega \tag{2.49}$$

The series (2.48) converges in the topology of $\mathcal{E}(S^{n-1} \times \mathbb{R})$. Hence it converges absolutely and uniformly on compact sets and can be differentiated term by term in p and ω (see [41]).

Exercise 2.5.4. Suppose that $g \in \mathcal{D}(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} g(p)\, p^k\, dp = 0$$

for all $k < l$. Show that $g(p) = d^l f(p)/dp^l$ for some $f \in \mathcal{D}(\mathbb{R})$. Does this result hold for $g \in \mathcal{S}(\mathbb{R})$?

Proposition 2.5.5. *Suppose that $\varphi \in \mathcal{D}(\Xi_n)$ has spherical harmonic expansion (2.48). Then $\varphi \in R(\mathcal{D}(\mathbb{R}^n))$ if and only if there exist even functions $f_{lm} \in \mathcal{D}(\mathbb{R})$ such that*

$$g_{lm} = \frac{d^l}{dp^l} f_{lm}$$

for all l, m.

Proof. If $\varphi \in R(\mathcal{D}(\mathbb{R}^n))$, then satisfies the moment conditions (2.40). Since $\varphi(\omega, p) = \varphi(-\omega, -p)$, (2.49) gives

$$g_{lm}(-p) = (-1)^l \, g_{lm}(p) \tag{2.50}$$

Now for each $k \in \mathbb{Z}^+$, we have

$$\int_{S^{n-1}} \varphi(\omega, p)\, p^k \, dp = \sum_{l \geqslant 0} \sum_{m=1}^{d(l)} Y_{lm}(\omega) \int_{-\infty}^{\infty} g_{lm}(p)\, p^k \, dp$$

Since the left hand side is a homogeneous polynomial of degree k in ω, it follows that

$$\int_{-\infty}^{\infty} g_{lm}(p)\, p^k \, dp = 0 \tag{2.51}$$

for all $l > k$.

From Exercise 2.5.4, we see that each g_{lm} is the lth derivative of a function $f_{lm} \in \mathcal{D}(\mathbb{R})$, and in view of (2.50), we can take just its even part and assume that f_{lm} is an even function on $\mathbb{R}$.

Conversely, suppose that $\varphi \in \mathcal{D}(\Xi_n)$ has spherical harmonic expansion (2.48), where $g_{lm}(p) = d^l f_{lm}/dp^l$ for even functions $f_{lm} \in \mathcal{D}(\mathbb{R})$. Then (2.51) holds, and so

$$\int_{-\infty}^{\infty} \varphi(\omega, p)\, p^k \, dp = \sum_{\substack{0 \leqslant l \leqslant k \\ k-l \text{ even}}} \sum_{m=1}^{d(l)} a_{lm} \, Y_{lm}(\omega) \tag{2.52}$$

where

$$\begin{aligned} a_{lm} &= \int_{-\infty}^{\infty} g_{lm}(p)\, p^k \, dp \\ &= \frac{k!}{(k-l)!} \int_{-\infty}^{\infty} f_{lm}(p)\, p^{k-l} \, dp \end{aligned}$$

The right hand side of (2.52) is a homogeneous degree k polynomial in ω, since if $k - l$ is even, then

$$Y_{lm}(\omega) = Y_{lm}(\omega)\, (\omega_1^2 + \cdots + \omega_n^2)^{\frac{k-l}{2}}$$

Thus φ satisfies the moment conditions (2.40), so by Theorem 2.5.2, $\varphi \in R(\mathcal{D}(\mathbb{R}^n))$. ∎

The characterization above of the range $R\mathcal{D}(\mathbb{R}^n)$ by spherical harmonics in turn allows us to determine the null space $\mathcal{N}$ of the dual transform on $\mathcal{E}(\Xi_n)$. Suppose now that $\psi \in \mathcal{E}(\Xi_n)$ such that $R^*\psi = 0$. By the duality (2.6) between the Radon and dual transforms, we have

$$\int_{\mathbb{R}^n} f(x)\, R^*\psi(x)\, dx = \frac{1}{\Omega_n} \int_{S^{n-1}\times\mathbb{R}} Rf(\omega,p)\, \psi(\omega,p)\, dp\, d\omega \tag{2.53}$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$. Since $R^*\psi = 0$, the right hand side above equals 0 for all f.

Let us expand ψ according to (2.48):

$$\psi(\omega,p) = \sum_{l\in\mathbb{Z}^+} \sum_{m=1}^{d(l)} h_{lm}(p)\, Y_{lm}(\omega) \tag{2.54}$$

Since the right hand side of (2.53) vanishes for all $\psi = Rf$ (for $f \in \mathcal{D}(\mathbb{R}^n)$), it follows that

$$\int_{-\infty}^{\infty} h_{lm}(p)\, \frac{d^l F}{dp^l}(p)\, dp = 0 \tag{2.55}$$

for all even $F \in \mathcal{D}(\mathbb{R})$, and hence for all $F \in \mathcal{D}(\mathbb{R})$. This implies that $d^l h_{lm}/dp^l = 0$, and so $h_{lm}(p)$ is a polynomial in p of degree $< l$. Since $h_{lm}(-p) = (-1)^l\, h_{lm}(p)$, we see that h_{lm} consists of terms with the same parity as l.

Conversely, if $\psi \in \mathcal{E}(\Xi_n)$ has expansion (2.54) where each h_{lm} is a polynomial of degree $< l$, then the relation (2.55) will hold for all $F \in \mathcal{D}(\mathbb{R})$, and this will in turn imply that the right hand side of (2.53) vanishes. Thus $R^*\psi = 0$.

We summarize our result about $\mathcal{N}$ as follows.

Proposition 2.5.6. *Let $\psi \in \mathcal{E}(\Xi_n)$ have the spherical harmonic expansion (2.54). Then $\psi \in \mathcal{N}$ if and only if $h_{lm}(p)$ is a polynomial of degree $< l$.*

Finally, we can use the description of the null space $\mathcal{N}$ above to show that the range $R\mathcal{E}'(\mathbb{R}^n)$ of the Radon transform on compactly supported distributions on $\mathbb{R}^n$ is a closed subset of $\mathcal{E}'(\Xi_n)$. This range was characterized earlier in Theorem 2.5.3.

Proposition 2.5.7. *Let $\Psi \in \mathcal{E}'(\Xi_n)$. Then $\Psi \in R\mathcal{E}'(\mathbb{R}^n)$ if and only if $\Psi(\psi) = 0$ for all $\psi \in \mathcal{N}$.*

Proof. By Theorem 2.5.3, $R\mathcal{E}'(\mathbb{R}^n) = \mathcal{E}'_H(\Xi_n)$. Suppose that $\Psi \in \mathcal{E}'_H(\Xi_n)$. Let $\psi \in \mathcal{N}$ and consider the expansion (2.54) of ψ in spherical harmonics, where each $h_{lm}(p)$ is a polynomial in p of degree $< l$. This expansion converges in

$\mathcal{E}(\Xi_n)$, so by (2.47),

$$\Psi(\psi) = \int_{S^{n-1}\times\mathbb{R}} Y_{lm}(\omega)\, h_{lm}(p)\, d\Psi(\omega,p)$$

$$= \sum_{l\in\mathbb{Z}^+} \sum_{m=1}^{d(l)} \int_{S^{n-1}} Y_{lm}(\omega)\, P_{lm}(\omega)\, d\omega,$$

where $P_{lm}(\omega)$ is a polynomial in ω of degree $< l$. Since P_{lm} is a sum of spherical harmonics of degree $< l$, we see that the right hand side above equals 0.

Conversely, suppose that $\Psi \in \mathcal{E}'(\Xi_n)$ such that $\Psi(\psi) = 0$ for all $\psi \in \mathcal{N}$. Fix $k \in \mathbb{Z}^+$, and let $h \in \mathcal{E}(S^{n-1})$. We expand h in spherical harmonics

$$h(\omega) = \sum_{l\in\mathbb{Z}^+} \sum_{m=1}^{d(l)} a_{lm}\, Y_{lm}(\omega)$$

Then from the hypothesis we see that

$$\begin{aligned}
\int_{S^{n-1}\times\mathbb{R}} h(\omega)\, p^k\, d\Psi(\omega,p) &= \sum_{\substack{l\leqslant k\\ k-l \text{ even}}} \sum_{m=1}^{d(l)} a_{lm} \int_{S^{n-1}\times\mathbb{R}} Y_{lm}(\omega)\, p^k\, d\Psi(\omega,p)\\
&= \sum_{\substack{l\leqslant k\\ k-l \text{ even}}} \sum_{m=1}^{d(l)} \left(\int_{S^{n-1}} h(\eta)\, Y_{lm}(\eta)\, d\eta\right) \int_{S^{n-1}\times\mathbb{R}} Y_{lm}(\omega)\, p^k\, d\Psi(\omega,p)\\
&= \int_{S^{n-1}} h(\eta)\, P_k(\eta)\, d\eta,
\end{aligned}$$

where

$$P_k(\eta) = \sum_{\substack{l\leqslant k\\ k-l \text{ even}}} \sum_{m=1}^{d(l)} \left(\int_{S^{n-1}\times\mathbb{R}} Y_{lm}(\omega)\, p^k\, d\Psi(\omega,p)\right) Y_{lm}(\eta)$$

For $k - l$ even, each $Y_{lm}(\eta)$ can be thought of as a homogeneous polynomial of degree k in η. Thus $P_k(\eta)$ is a homogeneous polynomial in η. Since h is arbitrary, this shows that $\Psi \in \mathcal{E}'_H(\Xi_n)$. $\square$

Proposition 2.5.7 can also be proved using approximate identities in $\mathrm{M}(n)$.

Theorem 2.5.8. *(Hertle [21])* $R^*\mathcal{E}(\Xi_n) = \mathcal{E}(\mathbb{R}^n)$.

Proof. This is an immediate consequence of Theorem 1.2.5, since R is injective on $\mathcal{E}'(\mathbb{R}^n)$ and $R\mathcal{E}'(\mathbb{R}^n)$ is a closed subspace of $\mathcal{E}'(\Xi_n)$. $\square$

Additional notes on the dual transform. It turns out that while the dual transform is not injective on $\mathcal{E}(\Xi_n)$, it is in fact injective on $\mathcal{S}(\Xi_n)$, and can be inverted by the formula

$$c_n \, \varphi = R(\Lambda R^* \varphi) \tag{2.56}$$

where c_n is given by (2.17). See [39] or the interesting paper be Madych and Solmon [28] for this result.

Chapter 3

The d-plane Transform on $\mathbb{R}^n$

3.1 The Space of d-planes and the Incidence Relation

In this chapter we study the transform which integrates functions on $\mathbb{R}^n$ over d-planes. The classical Radon transform is obviously a special case of this, corresponding to the case $d = n - 1$. In the case $d = 1$, the transform is often of special interest and is called the *X-ray transform*.

The *affine Grassmannian* $G(d, n)$ is the set of unoriented d-planes in $\mathbb{R}^n$. Thus, in particular, $G(n-1, n) = \Xi_n$. The motion group $\mathrm{M}(n)$ acts transitively on $G(d, n)$; if we let σ_0 denote the d-dimensional subspace $\mathbb{R}\, e_1 + \cdots + \mathbb{R}\, e_d$, then the isotropy subgroup H_d of $\mathrm{M}(n)$ at σ_0 is $\mathrm{O}(\sigma_0) \ltimes \sigma_0$, where $\mathrm{O}(\sigma_0)$ is the subgroup of $\mathrm{O}(n)$ fixing σ_0 and the second factor σ_0 in the semidirect product represents the translations in $\mathbb{R}^n$ by points in σ_0. If we identify σ_0 with $\mathbb{R}^d$, then the isotropy subgroup becomes $H_d = [\mathrm{O}(d) \times \mathrm{O}(n-d)] \ltimes \mathbb{R}^d = \mathrm{M}(d) \times \mathrm{O}(n-d)$.

Thus $G(d, n) = \mathrm{M}(n)/(\mathrm{M}(d) \times \mathrm{O}(n-d))$, and so its dimension is $n(n+1)/2 - d(d+1)/2 - (n-d)(n-d-1)/2 = (d+1)(n-d)$. In particular, $\dim \mathbb{R}^n = n < \dim G(d, n)$ unless $d = 0$ or $d = n - 1$.

For each $\xi \in G(d, n)$, let $\pi(\xi)$ be the parallel d-plane through the origin. The parallel translation $\xi \mapsto \pi(\xi)$ is then a mapping from $G(d, n)$ onto the Grassmann manifold $G_{d,n}$ of d-dimensional subspaces of $\mathbb{R}^n$. If we let $\sigma = \pi(\xi)$, then the orthogonal complement $\sigma^\perp$ intersects ξ at exactly one point x. We put

$$\xi = (\sigma, x), \tag{3.1}$$

and note that since $\pi^{-1}(\sigma)$ can be naturally identified with the vector space $\sigma^{\perp}$, the space $G(d,n)$ is a vector bundle of rank $n-d$ over $G_{d,n}$.

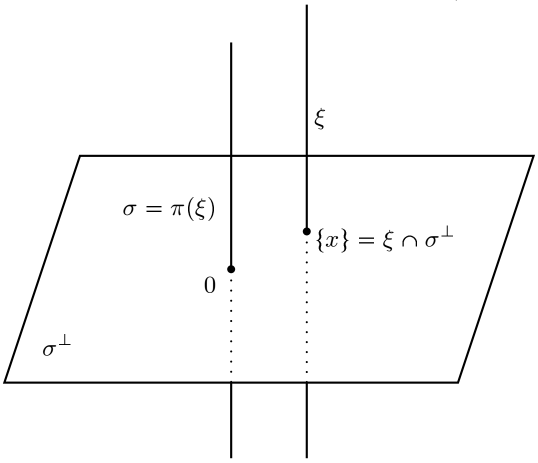

Figure 3.1: $\xi = (\sigma, x)$

Consider the double fibration

$$\begin{tikzcd} & \mathrm{M}(n)/L \arrow[dl] \arrow[dr] & \\ \mathbb{R}^n = \mathrm{M}(n)/\mathrm{O}(n) & & G(d,n) = \mathrm{M}(n)/H_d \end{tikzcd}$$

where $L = \mathrm{O}(n) \cap H_d = \mathrm{O}(d) \times \mathrm{O}(n-d)$. Then the set of all d-planes ξ incident to the origin 0 is the orbit $\mathrm{O}(n) \cdot \sigma_0$, the set of all d-planes through the origin. Thus $\xi \in G(d,n)$ is incident to 0 if and only if 0 lies in ξ. By left translation, we see that $\xi \in G(d,n)$ is incident to $x \in \mathbb{R}^n$ if and only if x lies in ξ. Thus we can identify $\widehat{\xi}$ with ξ itself, while $\check{x}$ is the set of all d-planes ξ containing x. This set is an orbit of the subgroup $\mathrm{O}(x)$ on $\mathrm{M}(n)$ fixing x, and is diffeomorphic to $G_{d,n}$.

The Lebesgue measure on $\widehat{\sigma}_0 = \sigma_0 \subset \mathbb{R}^n$ is invariant under the isotropy subgroup H_d, so we will take this to be the measure dm_{σ_0} in (1.6). If $\xi \in G(d,n)$, then by left translating by an appropriate element of $\mathrm{M}(n)$, we see that dm_ξ will be the Lebesgue measure on ξ.

Thus the Radon transform corresponding to the above incidence relation between $\mathbb{R}^n$ and $G(d,n)$ just integrates a function f on $\mathbb{R}^n$ over all d-planes:

$$Rf(\xi) = \int_\xi f(x)\, dm_\xi(x) \tag{3.2}$$

We call the map $f \mapsto Rf$ the *Radon d-plane transform.* If $\xi = (\sigma, x)$, we can write the left hand side above as $Rf(\sigma, x)$.

From Figure 3.1 we see that

$$Rf(\sigma, x) = \int_\sigma f(x+y)\, dm_\sigma(y) \tag{3.3}$$

for all $\sigma \in G_{d,n}$ and all $x \in \sigma^\perp$. If $f \in L^1(\mathbb{R}^n)$, then Fubini's theorem implies that for each $\sigma \in G_{d,n}$, the d-plane transform $Rf(\sigma, x)$ exists for almost all $x \in \sigma^\perp$.

There is a projection-slice theorem for the d-plane transform, which is given as follows. For any $y \in \mathbb{R}^n$, choose any $\sigma \in G_{d,n}$ such that $y \in \sigma^\perp$. Then

$$\begin{aligned}
\widetilde{f}(y) &= \int_{\mathbb{R}^n} f(w)\, e^{-i\langle w,y\rangle}\, dw \\
&= \int_{\sigma^\perp} \int_\sigma f(x+u)\, e^{-i\langle x+u,y\rangle}\, dm(u)\, dx \\
&= \int_{\sigma^\perp} Rf(\sigma, x)\, e^{-i\langle x,y\rangle}\, dx
\end{aligned} \tag{3.4}$$

where $dm(u) = dm_\sigma(u)$ is Lebesgue measure on σ, and the dx in the last integral refers to Lebesgue measure on $\sigma^\perp$.

It follows immediately from (3.4) that the d-plane transform $f \mapsto Rf$ is injective on $L^1(\mathbb{R}^n)$.

A similar computation shows that if $f \in \mathcal{S}(\mathbb{R}^n)$, then Rf satisfies the *forward moment conditions*: for every $k \in \mathbb{Z}^+$, there is a homogeneous polynomial P_k of degree k on $\mathbb{R}^n$ such that if $y \in \mathbb{R}^n$, then

$$P_k(y) = \int_{\sigma^\perp} Rf(\sigma, x)\, \langle x, y\rangle^k\, dx \tag{3.5}$$

for all $\sigma \in G_{d,n}$ such that $y \in \sigma^\perp$. (See Section 4.1 below.)

While it is possible to recover a function $f \in \mathcal{S}(\mathbb{R}^n)$ from its d-plane transform Rf through (3.4), we can obtain a more direct inversion method when d is even. For this, we will need to make use of the polar coordinate form of the Laplace operator on $\mathbb{R}^n$.

If we write any point $x \neq 0 \in \mathbb{R}^n$ as $x = r\omega$, with $r > 0$ and $\omega \in S^{n-1}$, then

$$L = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} L_{S^{n-1}} \tag{3.6}$$

where $L_{S^{n-1}}$ is the Laplace-Beltrami operator on the Riemannian manifold S^{n-1}. (See Section 5.2 below or [17], Chapter 2.)

In particular, if f is a radial C^2 function on $\mathbb{R}^n$, with $f(r\omega) = F(r)$, then

$$(Lf)(r\omega) = F''(r) + \frac{n-1}{r}\, F'(r) \tag{3.7}$$

Recall now the mean value operator M^r on $\mathbb{R}^n$ defined in (2.35), which takes the average of functions over spheres of radius r:

$$\begin{aligned} M^r f(x) &= \int_{\mathrm{O}(n)} f(x + k\cdot(r\,e_1))\,dk \\ &= \frac{1}{\Omega_n}\int_{S^{n-1}} f(x + r\omega)\,d\omega \end{aligned}$$

We recall, for instance, that $f \in C^2(\mathbb{R}^n)$ is harmonic if and only if $M^r f = f$.

In the first equation above, we can replace re_1 by any $y \in \mathbb{R}^n$ such that $\|y\| = r$, so we can write the left hand side as $M^{d(0,y)} f(x)$.

Proposition 3.1.1. *Given a function $f \in C^2(\mathbb{R}^n)$, let $F(x,y) = M^{d(0,y)} f(x)$. Then*

$$L_x\, F(x,y) = L_y\, F(x,y) \tag{3.8}$$

and hence

$$L_x(M^r\, f(x)) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\,\frac{\partial}{\partial r}\right) M^r f(x) \tag{3.9}$$

The relation (3.9) is known as the *Darboux equation.*

Proof. Note that since L is $\mathrm{M}(n)$-invariant,

$$\begin{aligned} L_x\, F(x,y) &= L_x \int_{\mathrm{O}(n)} f(x + k\cdot y)\,dk \\ &= \int_{\mathrm{O}(n)} L_x\, f(x + k\cdot y)\,dk \\ &= \int_{\mathrm{O}(n)} Lf(x + k\cdot y)\,dk \qquad (3.10) \\ &= \int_{\mathrm{O}(n)} L_y f(x + k\cdot y)\,dk \\ &= L_y \int_{\mathrm{O}(n)} f(x + k\cdot y)\,dk \end{aligned}$$

This proves (3.8). Equation (3.9) then follows from (3.7). ☐

Note that the expression (3.10) shows that $L(M^r f)(x) = M^r(Lf)(x)$.

The following theorem gives an inversion formula for the d-plane transform when d is even.

Theorem 3.1.2. *Suppose that d is even. If $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$c_{d,n}\, f(x) = (L_x)^{\frac{d}{2}}\, (R^* R f)(x) \tag{3.11}$$

where

$$c_{d,n} = (-4\pi)^{\frac{d}{2}}\, \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)} \tag{3.12}$$

Proof. For $x \in \mathbb{R}^n$, let us first calculate $R^* R f(x)$:

$$\begin{aligned} R^* R f(x) &= \int_{\mathrm{O}(n)} Rf(x + k \cdot \sigma_0)\, dk \\ &= \int_{\mathrm{O}(n)} \int_{\sigma_0} f(x + k \cdot y)\, dm(y)\, dk \end{aligned}$$

(since R commutes with the left action of $M(n)$)

$$\begin{aligned} &= \int_{\sigma_0} \int_{\mathrm{O}(n)} f(x + k \cdot y)\, dk\, dy \\ &= \int_{\sigma_0} M^{d(0,y)} f(x)\, dm(y) \\ &= \Omega_d \int_0^\infty M^r f(x)\, r^{d-1}\, dr \end{aligned} \tag{3.13}$$

Put $F(r) = M^r f(x)$. If we apply the Laplace operator to both sides above and use the Darboux equation (3.9), we obtain

$$L_x\, (R^* R f)(x) = \Omega_d \int_0^\infty \left(F''(r) + \frac{n-1}{r}\, F'(r) \right)\, r^{d-1}\, dr$$

If $d = 2$, then an integration by parts shows that the right hand side above equals $(2 - n)\, \Omega_d\, F(0) = (2 - n)\, \Omega_d\, f(x)$. If $d \geqslant 4$, integrating by parts twice yields

$$-(n-d)(d-2)\, \Omega_d \int_0^\infty F(r)\, r^{d-3}\, dr$$

Repeating this procedure gives us

$$L_x^{\frac{d}{2}} (R^* R f)(x) = (-1)^{\frac{d}{2}}\, [(n-d)(n-(d-2)) \cdots (n-2)][(d-2)(d-4) \cdots 2]\, \Omega_d f(x),$$

which is precisely the formula (3.11). ∎

When n is odd, then $n - 1$ is even, and the inversion formula (2.22) for the classical Radon transform is a special case of (3.11).

3.2 Fractional Powers of the Laplacian and Riesz Potentials

In this section we define fractional powers of the negative Laplacian. Our primary purpose is to obtain the inversion formula for the Radon d-plane transform on $\mathbb{R}^n$ when d is odd. The key to defining such fractional powers is the identity

$$((-L)^p f)^\sim(\xi) = \|\xi\|^{2p}\, \tilde{f}(\xi) \qquad (\xi \in \mathbb{R}^n) \tag{3.14}$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $p \in \mathbb{Z}^+$. Note that if $\xi \neq 0$, the map $\xi \mapsto \|\xi\|^{2p} = e^{2p \ln \|\xi\|}$ is holomorphic on $\mathbb{C}$. Thus the idea is to extend the equation (3.14) to an arbitrary complex parameter p by defining an appropriate integral operator on f when p is complex.

For $\alpha \in \mathbb{C}$, consider the function t_+^α on $\mathbb{R}$ given by

$$t_+^\alpha = \begin{cases} t^\alpha & \text{if } t > 0 \\ 0 & \text{if } t \leqslant 0 \end{cases} \tag{3.15}$$

Then t_+^α is locally integrable if and only if $\operatorname{Re}\alpha > -1$. The resulting distribution clearly belongs to $\mathcal{S}'(\mathbb{R})$, and depends holomorphically on α in the sense that for any $h \in \mathcal{S}(\mathbb{R})$, the map

$$\alpha \mapsto t_+^\alpha(h)$$

is holomorphic on the half plane $\{\alpha \in \mathbb{C} \,|\, \operatorname{Re}\alpha > -1\}$. (This is most easily seen using Morera's theorem on the right hand side as a function of α.) We will call t_+^α a *weakly holomorphic* distribution-valued function of α.

Next we show that the map $\alpha \mapsto t_+^\alpha$ can be extended to a weakly meromorphic distribution-valued function on the whole complex plane, with poles at $\alpha = -1, -2, \ldots$ For this, we note that if $h \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned} t_+^\alpha(h) &= \int_0^1 \left(\sum_{k=1}^m \frac{h^{(k-1)}(0)}{(k-1)!}\, t^{k-1} \right) t^\alpha \, dt + \int_0^1 \left(h(t) - \sum_{k=1}^m \frac{h^{(k-1)}(0)}{(k-1)!}\, t^{k-1} \right) t^\alpha \, dt \\ &\qquad + \int_1^\infty h(t) t^\alpha \, dt \\ &= \sum_{k=1}^m \frac{h^{(k-1)}(0)}{(k-1)!(\alpha+k)} + \int_0^1 \left(h(t) - \sum_{k=1}^m \frac{h^{(k-1)}(0)}{(k-1)!}\, t^{k-1} \right) t^\alpha \, dt \\ &\qquad + \int_1^\infty h(t) t^\alpha \, dt \end{aligned} \tag{3.16}$$

By the integral form of the Taylor remainder theorem, we have

$$h(t) - \sum_{k=1}^m \frac{h^{(k-1)}(0)}{(k-1)!}\, t^{k-1} = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}\, h^{(m)}(t)\, dt$$

For small t, the right hand side is clearly bounded in absolute value by $C_h\, t^m$ for some constant C_h (depending on h). Thus the integrand in the first integral in (3.16) is dominated in absolute value by $C_h\, t^{m+\operatorname{Re}\alpha}$ for small t. This integral therefore converges absolutely for $\operatorname{Re}\alpha > -m-1$, and we may define $t_+^\alpha(h)$ by the right hand side of (3.16) for α in this half plane, provided that $\alpha \neq -1, \ldots, -m$.

Since m is arbitrary, we obtain a meromorphic function $\alpha \mapsto t_+^\alpha(h)$ on $\mathbb{C}$ with simple poles at $\alpha = -1, -2, \ldots$. The right hand side of (3.16) also shows that for fixed α, the map $h \mapsto t_+^\alpha(h)$ is continuous on $\mathcal{S}(\mathbb{R})$ and is therefore a tempered distribution on $\mathbb{R}$.

From (3.16) we see that the residue of $\alpha \mapsto t_+^\alpha$ at the simple pole $\alpha = -k$ is the point distribution

$$(-1)^{k-1}\,\frac{\delta_0^{(k-1)}}{(k-1)!} \tag{3.17}$$

where δ_0 is the Dirac delta function at 0.

Next, for a complex parameter α, let us consider the function $\|x\|^\alpha$ on $\mathbb{R}^n\backslash\{0\}$. This function is locally integrable if $\operatorname{Re}\alpha > -n$ and therefore gives rise to a tempered distribution on $\mathbb{R}^n$. If $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} \|x\|^\alpha(f) &= \int_{\mathbb{R}^n} \|x\|^\alpha\, f(x)\, dx \\ &= \int_0^\infty \int_{S^{n-1}} t^\alpha\, f(t\omega)\, t^{n-1}\, d\omega\, dt \\ &= \Omega_n \int_0^\infty t^{\alpha+n-1}\, M^t f(0)\, dt \\ &= \Omega_n\, t_+^{\alpha+n-1}(M^t f(0)) \end{aligned} \tag{3.18}$$

In accordance with (3.16), the right hand side above can be extended to a meromorphic function of α, with simple poles at $\alpha = -n, -n-2, -n-4, \ldots$. The apparent simple poles at $\alpha = -n-1, -n-3, \ldots$ do not exist since the residue at $\alpha + n - 1 = -k$ for k even is given by

$$\frac{1}{(k-1)!}\,\frac{d^{(k-1)}}{dt^{(k-1)}} M^t f(0)\bigg|_{t=0} \tag{3.19}$$

which vanishes since $t \mapsto M^t f(0)$ is an even C^∞ function of t. Now it is not hard to see that linear map $f \mapsto F(t) = M^t f(0)$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}_e(\mathbb{R})$, the subspace of even functions in $\mathcal{S}(\mathbb{R})$. Thus (3.18) shows that $\alpha \mapsto \|x\|^\alpha$ is a weakly meromorphic function on $\mathbb{C}$, with values in $\mathcal{S}'(\mathbb{R}^n)$.

From (3.19) we see that the residue of $\alpha \mapsto \|x\|^\alpha$ at the simple pole $\alpha = -n$ is $\Omega_n\, \delta_0$, where δ_0 now denotes the Dirac delta function at $0 \in \mathbb{R}^n$.

Proposition 3.2.1. *The Fourier transform of the tempered distribution* $\|x\|^\alpha$ *is given by*

$$(\|x\|^\alpha)^\sim = \frac{2^{n+\alpha}\,\pi^{\frac{n}{2}}\,\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)}\,\|\xi\|^{-n-\alpha} \tag{3.20}$$

for $\alpha \in \mathbb{C},\ \alpha \notin -n - 2\mathbb{Z}^+$.

Note that the simple poles of $\|\xi\|^{-n-\alpha}$ on the right hand side above are cancelled by those of $\Gamma\left(-\frac{\alpha}{2}\right)$.

Proof. By the well known fact that the Fourier transform of the Gaussian $e^{-\|x\|^2}$ is $\pi^{\frac{n}{2}}\,e^{-\frac{\|\xi\|^2}{4}}$, we can start with the relation

$$(2\pi)^n \int_{\mathbb{R}^n} f(\xi)\,e^{-t\|\xi\|^2}\,d\xi = t^{-\frac{n}{2}}\,\pi^{\frac{n}{2}} \int_{\mathbb{R}^n} \widetilde{f}(x)\,e^{-\frac{\|x\|^2}{4t}}\,dx,$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$. Let us multiply both sides above by $t^{\frac{\alpha+n}{2}-1}$ and integrate with respect to t on $(0,\infty)$. Then by Fubini's theorem, we obtain

$$2^n\,\pi^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) \int_0^\infty e^{-t\|\xi\|^2}\,t^{\frac{\alpha+n}{2}-1}\,dt\,d\xi = \int_{\mathbb{R}^n} \widetilde{f}(x) \int_0^\infty e^{-\frac{\|x\|^2}{4t}}\,t^{\frac{\alpha}{2}-1}\,dt\,dx$$

The inner integral on the left hand side above equals

$$\Gamma\left(\frac{n+\alpha}{2}\right)\,\|\xi\|^{-n-\alpha}$$

whereas the inner integral on the right equals

$$2^{-\alpha}\,\|x\|^\alpha \int_0^\infty e^{-u}\,u^{-\frac{\alpha}{2}-1}\,du = 2^{-\alpha}\,\Gamma\left(-\frac{\alpha}{2}\right)\,\|x\|^\alpha$$

We obtain

$$\frac{2^{n+\alpha}\,\pi^{\frac{n}{2}}\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} f(\xi)\,\|\xi\|^{-n-\alpha}\,d\xi = \int_{\mathbb{R}^n} \widetilde{f}(x)\,\|x\|^\alpha\,dx$$

The above calculations are valid for $-n < \operatorname{Re}\alpha < 0$. Since f is arbitrary, we have established the relation (3.20) for such α. By analytic continuation, (3.20) then follows in general. $\square$

For $\gamma \in \mathbb{C}_n = \mathbb{C}\backslash\{n + 2\mathbb{Z}^+\}$, the *Riesz potential* I^γ is the operator defined by

$$I^\gamma f = H_n(\gamma)\,f * \|x\|^{\gamma-n} \tag{3.21}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $H_n(\gamma)$ is the constant

$$H_n(\gamma) = \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{2^\gamma\pi^{\frac{n}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \tag{3.22}$$

Note that the simple poles of $\|x\|^{\gamma-n}$ are cancelled out by those of $\Gamma\left(\frac{\gamma}{2}\right)$.

By Proposition 3.2.1 and the definition (3.21), we obtain

$$(I^{\gamma} f)^{\sim}(\xi) = \|\xi\|^{-\gamma} f(\xi) \qquad (\xi \in \mathbb{R}^n) \tag{3.23}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\gamma \in \mathbb{C}_n$. In particular,

$$I^0 f = f, \tag{3.24}$$

a fact which can also be verified directly from the remark above about the residue of $\|x\|^{\alpha}$ at $\alpha = -n$. The Riesz potentials also commute with the Laplacian, and in fact

$$(-L)\, I^{\gamma} f = I^{\gamma}((-L)f) = I^{\gamma-2} f \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

for γ and $\gamma - 2$ in $\mathbb{C}_n$, since the Fourier transform of all three functions above equals $\|\xi\|^{2-\gamma}\, \widetilde{f}(\xi)$.

If $p \in -(1/2)\mathbb{C}_n$, we now define the complex power $(-L)^p$ of the negative Laplacian by

$$(-L)^p = I^{-2p} \tag{3.25}$$

Then by Proposition 3.2.1 we see that

$$((-L)^p f)^{\sim}(\xi) = \|\xi\|^{2p}\, \widetilde{f}(\xi) \tag{3.26}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. This generalizes the same relation when p is a nonnegative integer.

Proposition 3.2.2. *Suppose that $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$ and $\mathrm{Re}(\alpha+\beta) < n$. Then*

$$I^{\alpha}\, I^{\beta}(f) = I^{\alpha+\beta}(f)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Note that because $\mathrm{Re}\,\beta > 0$, the integral

$$\int_{\mathbb{R}^n} f(x-y)\, \|y\|^{\beta-n}\, dy \tag{3.27}$$

converges absolutely for all $f \in \mathcal{S}(\mathbb{R}^n)$.

We now observe that, while $I^{\beta} f$ is not necessarily in $\mathcal{S}(\mathbb{R}^n)$, it does satisfy the estimate

$$|I^{\beta} f(x)| \leqslant C\, (1 + \|x\|)^{\mathrm{Re}\,\beta - n} \tag{3.28}$$

for some constant C. This is because we can break up the integral (3.27) into two parts

$$\int_{\mathbb{R}^n} f(x-y)\, \|y\|^{\beta-n}\, dy = \int_{\|y\| \leqslant \frac{1}{2}\|x\|} f(x-y)\, \|y\|^{\beta-n}\, dy + \int_{\|y\| \geqslant \frac{1}{2}\|x\|} f(x-y)\, \|y\|^{\beta-n}\, dy \tag{3.29}$$

The second integral is bounded above in absolute value by

$$\left\|\frac{x}{2}\right\|^{\operatorname{Re}\beta-n} \int_{\mathbb{R}^n} |f(z)|\, dz \tag{3.30}$$

Now for any $M > 0$, there is a constant C_M such that $|f(z)| \leqslant C_M(1+\|z\|)^{-M}$. Moreover, if $\|y\| \leqslant \frac{1}{2}\|x\|$, then $\|x-y\| \geqslant \frac{1}{2}\|x\|$. Thus the first integral in (3.29) is bounded above in absolute value by

$$C_M\,(1+\|x/2\|)^{-M} \int_{\|y\|\leqslant\frac{1}{2}\|x\|} \|y\|^{\operatorname{Re}\beta-n}\, dy = \frac{C_M\,\Omega_n}{\operatorname{Re}\beta}\,(1+\|x/2\|)^{-M}\,\|x/2\|^{\operatorname{Re}\beta} \tag{3.31}$$

If we put $M = n$ and combine this with (3.30), we obtain the estimate (3.28).

From this and the hypotheses on α and β, it follows that the integral

$$\int_{\mathbb{R}^n} I^\beta f(y)\|x-y\|^{\alpha-n}\, dy$$

converges absolutely for each $x \in \mathbb{R}^n$. Thus $I^\alpha I^\beta f(x)$ is well-defined. Its value is given by

$$\begin{aligned} I^\alpha I^\beta f(x) &= H_n(\alpha)\,H_n(\beta) \int_{\mathbb{R}^n} \|x-y\|^{\alpha-n} \left(\int_{\mathbb{R}^n} f(z)\,\|y-z\|^{\beta-n}\, dz\right) dy \\ &= H_n(\alpha)\,H_n(\beta) \int_{\mathbb{R}^n} f(z) \left(\int_{\mathbb{R}^n} \|x-y\|^{\alpha-n}\,\|y-z\|^{\beta-n}\, dy\right) dz \end{aligned} \tag{3.32}$$

The change of order of integration is justified because the first iterated integral above converges absolutely, because of the estimates (3.29)–(3.31), in which we can replace f by $|f|$.

Suppose that $x \neq z$. If we put $x - z = r\omega$, for $r > 0$ and $\omega \in S^{n-1}$, then the inner integral above equals

$$\begin{aligned} &r^{\alpha+\beta-2n} \int_{\mathbb{R}^n} \left\|\omega - \frac{y-z}{r}\right\|^{\alpha-n} \left\|\frac{y-z}{r}\right\|^{\beta-n} dy \\ &\qquad = r^{\alpha+\beta-n} \int_{\mathbb{R}^n} \|\omega - v\|^{\alpha-n}\,\|v\|^{\beta-n}\, dv, \end{aligned} \tag{3.33}$$

with the last integral above converging due to the assumptions on α and β. Its value, which we denote by $C_n(\alpha,\beta)$, is clearly independent of the unit vector ω. Thus, by (3.32) and (3.33), we have

$$I^\alpha I^\beta f(x) = C_n(\alpha,\beta)\,\frac{H_n(\alpha)\,H_n(\beta)}{H_n(\alpha+\beta)}\, I^{\alpha+\beta} f(x) \tag{3.34}$$

It remains to show that

$$C_n(\alpha,\beta) = \frac{H_n(\alpha+\beta)}{H_n(\alpha)\,H_n(\beta)} \tag{3.35}$$

For this, consider the class $\mathcal{S}^*(\mathbb{R}^n)$ consisting of all $f \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)\, x_1^{j_1} \cdots x_n^{j_n}\, dx = 0 \tag{3.36}$$

for all multiindices $J = (j_1, \ldots, j_n) \in (\mathbb{Z}^+)^n$.

The Fourier image of $\mathcal{S}^*$ is the class $\mathcal{S}_0(\mathbb{R}^n)$ consisting of all $F \in \mathcal{S}(\mathbb{R}^n)$ all of whose partial derivatives, of all orders, vanish at the origin. If $f \in \mathcal{S}^*$, the Taylor remainder theorem for $\mathbb{R}^n$ shows that $(I^\beta f)^\sim(\xi) = \|\xi\|^{-\beta}\, \widetilde{f}(\xi)$ also belongs to $\mathcal{S}_0$, and so we see that $I^\beta f \in \mathcal{S}^*$. Thus $I^\alpha I^\beta f \in \mathcal{S}^*$, and so by (3.23),

$$(I^\alpha I^\beta f)^\sim(\xi) = \|\xi\|^{-\alpha-\beta}\, \widetilde{f}(\xi) = (I^{\alpha+\beta} f)^\sim(\xi)$$

Thus $I^\alpha I^\beta f = I^{\alpha+\beta} f$ for all $f \in \mathcal{S}^*$. Comparing with (3.34), we obtain (3.35). This proves Proposition 3.2.2. $\square$

The following result allows us to invert the Radon d-plane transform on $\mathbb{R}^n$ for d odd.

Proposition 3.2.3. *Suppose that $0 \leqslant k < n$. Then $I^{-k} I^k f = f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. We can assume that $k > 0$.

From Proposition 3.2.2 we know that $I^\alpha I^k f = I^{\alpha+k} f$ if α lies in the strip $0 < \operatorname{Re}\alpha < n-k$. Fix $x \in \mathbb{R}^n$. By the definition (3.21), the map $\alpha \mapsto I^{\alpha+k} f(x)$ is a holomorphic function of α in the half plane $\operatorname{Re}\alpha < n-k$. Thus it suffices by (3.24) to prove that the map

$$\alpha \mapsto I^\alpha I^k f(x) \tag{3.37}$$

extends to a holomorphic function on the same half plane. For this, we write $I^k f = f_1 + f_2$, where $f_1 \in \mathcal{D}(\mathbb{R}^n)$ and f_2 is a function which vanishes on the ball $B_1(x)$. Thus

$$I^\alpha I^k f(x) = I^\alpha f_1(x) + I^\alpha f_2(x)$$

The first term on the right is holomorphic for $\alpha \in \mathbb{C}_n$. As for the second term, we have

$$I^\alpha f_2(x) = H_n(\alpha) \int_{\|y\| \geqslant 1} f_2(x-y)\, \|y\|^{\alpha-n}\, dy \tag{3.38}$$

By (3.28), $I^k f$ satisfies the estimate

$$|I^k f(z)| \leqslant C\, (1 + \|z\|)^{k-n} \qquad (z \in \mathbb{R}^n) \tag{3.39}$$

for some constant C. Hence f_2 satisfies a similar estimate.

It follows that (for fixed x) the integral (3.38) converges absolutely and uniformly for all α in any compact subset of the half plane $\operatorname{Re}\alpha < n-k$. Thus by Morera's theorem, it is a holomorphic function of α on the half plane. Since $H_n(\alpha)$ is likewise holomorphic there, we see that $\alpha \mapsto I^\alpha f_2(x)$ is a holomorphic function of α on the half plane. $\square$

Remark: According to Ortner ([32], Satz 9), Propositions 3.2.2 and 3.2.3 can be generalized to the following result. If $\gamma, \mu \in \mathbb{C}_n$ such that $\mathrm{Re}(\gamma + \mu) < n$, then

$$I^\gamma I^\mu f = I^{\gamma+\mu} f \tag{3.40}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

We can now provide an inversion formula for the Radon d-plane transform for all d.

Theorem 3.2.4. *Fix an integer d, with $1 \leqslant d \leqslant n-1$. Let R be the Radon d-plane transform on $\mathbb{R}^n$ and let R^* be the its dual transform. Then*

$$c_{d,n}\, f = (-L)^{\frac{d}{2}}\, R^* R f \tag{3.41}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, where

$$c_{d,n} = \frac{(4\pi)^{\frac{d}{2}}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)} \tag{3.42}$$

Proof. If $f \in \mathcal{S}$, then according to (3.13)

$$\begin{aligned} R^* R f(x) &= \Omega_d \int_0^\infty M^r f(x)\, r^{d-1}\, dr \\ &= \frac{\Omega_d}{\Omega_n} \int_0^\infty \int_{S^{n-1}} f(x + r\omega)\, r^{d-1}\, d\omega\, dr \\ &= \frac{\Omega_d}{\Omega_n} \int_{\mathbb{R}^n} f(x+y)\, \|y\|^{d-n}\, dy \\ &= \frac{\Omega_d}{\Omega_n} \frac{1}{H_n(d)}\, I^d f(x) \end{aligned} \tag{3.43}$$

Since $(-L)^{\frac{d}{2}} = I^{-d}$, the result follows from Proposition 3.2.3. $\square$

Note that the inversion formula (3.41) reduces to (3.11) when d is even.

3.3 Shifted Dual Transforms

In this section we develop an alternative method for inverting integral transforms based on "shifting" the incidence relation between dual homogeneous spaces. This arises from a curious interplay between two different incidence relations between X and Ξ. For a systematic study of shifted transforms, see Rouviere's paper [36].

Let us assume, as in Section 1.2 that $X = G/K$ and $\Xi = G/H$ are homogeneous manifolds in duality, and that K is compact. We further assume, as in Lemma 1.1.1, that the maps $x \mapsto \check{x}$ and $\xi \mapsto \widehat{\xi}$ are injective. Let R and R^* be the associated Radon and dual transforms.

Let $o = \{K\}$ and $\xi_0 = \{H\}$ be the identity cosets in G/K and G/H, respectively. Let $\gamma \in G$ and put $\xi_\gamma = \gamma \cdot \xi_0$. Then the isotropy subgroup of G at ξ_γ is $H_\gamma = \gamma H \gamma^{-1}$, and we can write $\Xi = G/\gamma H \gamma^{-1}$. Replacing H by $\gamma H \gamma^{-1}$ gives us a new incidence relation between X and Ξ, which we call the *shifted incidence relation*. Under the shifted incidence relation, the set of all $\xi \in \Xi$ incident to $x = g \cdot o$ is the orbit $gK \cdot \xi_\gamma \subset \Xi$, and the set of all $x \in X$ incident to $\xi = g_1 \cdot \xi_\gamma$ is the orbit $g_1 H_\gamma \cdot o \subset X$.

Example 3.3.1. Let $X = \mathbb{R}^n$ and $\Xi = G(d, n)$, the manifold of d-planes in $\mathbb{R}^n$. If o is the origin 0 and ξ_0 is the d-plane $\mathbb{R}\, e_1 + \cdots + \mathbb{R}\, e_d$ in $\mathbb{R}^n$, then from Section 3.1, we have $K = \mathrm{O}(n)$ and $H = \mathrm{M}(d) \times \mathrm{O}(n-d)$. Now fix $r > 0$, let $x = r\, e_n$, and let $\gamma = (e, x)$, translation by x. Then ξ_γ is the d-plane $r\, e_n + \xi_0$. Under the shifted incidence relation, the set of all ξ incident to $0 \in \mathbb{R}^n$ is the orbit $\mathrm{O}(n) \cdot (r\, e_n + \xi_0) \subset G(d, n)$, the set of all d-planes in $\mathbb{R}^n$ at distance r from 0. By $\mathrm{M}(n)$-invariance, we see that the shifted incidence relation is given by

$$x \text{ is incident to } \xi \iff d(x, \xi) = r$$

Going back to our general setup, we call the Radon transform R_γ and and dual transform R^*_γ corresponding to this shifted incidence relation the *shifted Radon* and *dual* transforms, respectively. If $f \in \mathcal{D}(X)$, let us calculate the shifted dual transform of Rf at the origin o in X:

$$\begin{aligned} R^*_\gamma Rf(o) &= \int_K Rf(k \cdot \xi_\gamma)\, dk \\ &= \int_K \int_{\widehat{\xi}_0} f(k\gamma \cdot y)\, dm(y)\, dk \\ &= \int_{\widehat{\xi}_0} \int_K f(k\gamma \cdot y)\, dk\, dm(y) \end{aligned} \tag{3.44}$$

The function

$$f^{\#}(x) = \int_K f(k \cdot x)\, dk \qquad (x \in X) \tag{3.45}$$

is K-invariant, and since K is compact, its support, which lies in $K \cdot \operatorname{supp} f$, is compact, so $f^{\#} \in \mathcal{D}(X)$. Let $\mathcal{D}^{\#}(X)$ denote the space of all K-invariant functions in $\mathcal{D}(X)$. The map $f \mapsto f^{\#}$ projects $\mathcal{D}(X)$ onto $\mathcal{D}^{\#}(X)$. Now the inner integral in (3.44) equals $f^{\#}(\gamma \cdot y)$, and thus

$$R^*_\gamma Rf(o) = Rf^{\#}(\gamma \cdot \xi_0) \tag{3.46}$$

The relation (3.46) shows that the inversion problem for R reduces to finding an inversion formula (or procedure) for the Radon transform of K-invariant functions. To see why, we first note that if $F \in \mathcal{D}^{\#}(X)$, then $RF \in \mathcal{D}^{\#}(\Xi)$, the subspace of all all K-ivariant functions in $\mathcal{D}(\Xi)$. Thus RF is a function on the set $K\backslash G/H$ of K-orbits in Ξ. If $\xi \in \Xi$, let $[\xi] = K \cdot \xi$ be its K-orbit.

Now suppose that T is an inversion formula or procedure which recovers $F(o)$ from RF if F is any K-invariant function:

$$F(o) = T_{[\xi]}(RF(\xi)) \qquad (F \in \mathcal{D}^{\#}(X)) \tag{3.47}$$

If $f \in \mathcal{D}(X)$, then according to (3.46),

$$\begin{aligned} f(o) &= f^{\#}(o) \\ &= T_{[\gamma\cdot\xi_0]}(Rf^{\#}(\gamma\cdot\xi_0)) \\ &= T_{[\gamma\cdot\xi_0]}(R^*_\gamma Rf(o)) \end{aligned} \tag{3.48}$$

Since both R and R^*_γ are invariant under left translations by elements of G, it follows that

$$f(x) = T_{[\gamma\cdot\xi_0]}(R^*_\gamma Rf(x)) \tag{3.49}$$

for all $x \in X$.

A K-invariant inversion formula of the type (3.47) is sometimes relatively easy to obtain since there may be submanifolds A and B on X and Ξ transversal to the K-orbits in each space. If F is K-invariant on X, then as mentioned earlier its Radon transform RF is K-invariant on Ξ so F and RF are determined on A and B respectively. Thus the transform $F \mapsto RF$ becomes a transform from functions on A to functions on B.

Let us now apply the method of shifted dual transforms to invert the Radon d-plane transform on $\mathbb{R}^n$ when $1 \leqslant d \leqslant n-1$. The method will actually apply to rapidly decreasing functions, and not just compactly supported functions. Let us therefore first consider a radial function $F \in \mathcal{S}(\mathbb{R}^n)$, so that there exists an even function $H \in \mathcal{S}(\mathbb{R})$ such that $F(x) = H(\|x\|)$ for all $x \in \mathbb{R}^n$. Then for any d-plane ξ, $RF(\xi)$ depends only on $d(0,\xi)$; if $d(0,\xi) = r$, we have

$$\begin{aligned} RF(\xi) &= \Omega_d \int_0^\infty H((r^2+t^2)^{\frac{1}{2}})\, t^{d-1}\, dt \\ &= \Omega_d \int_r^\infty H(u)\,(u^2-r^2)^{\frac{d-2}{2}}\, u\, du \end{aligned} \tag{3.50}$$

If we denote the left hand side above by $\widehat{H}(r)$, we obtain an integral equation (for H) which generalizes (2.31). If d is even, $H(r)$ can be recovered by a differential operator:

$$\left(\frac{1}{r}\frac{d}{dr}\right) \circ \left(\frac{1}{2r}\frac{d}{dr}\right)^{\frac{d-2}{2}} \widehat{H}(r) = (-1)^{\frac{d}{2}}\Omega_d \left(\frac{d-2}{2}\right)!\, H(r)$$

for $r > 0$. Thus, if we put $\xi_r = re_n + \xi_0$, this implies that

$$F(0) = \lim_{r \to 0^+} \left(-\frac{1}{2\pi r} \frac{d}{dr} \right)^{\frac{d}{2}} RF(\xi_r) \tag{3.51}$$

For general d, (3.50) is an integral equation of Abel type. Let us first assume that $d \geqslant 2$. As before, denote the left hand side of (3.50) by $\widehat{H}(r)$, put $m = \frac{d-2}{2}$ and $H_1(u) = \Omega_d \, u \, H(u)$. The resulting equation is

$$\widehat{H}(r) = \int_r^\infty H_1(u)\,(u^2 - r^2)^m \, du \tag{3.52}$$

The method of (2.34) then gives the solution

$$H_1(t) = c_m \left(-\frac{d}{dt} \right) \circ \left(-\frac{1}{2t} \frac{d}{dt} \right)^{2m+1} \int_t^\infty \widehat{H}(r)\,(r^2 - t^2)^m \, r \, dr$$

where

$$c_m = \frac{4^{m+1}\,\Gamma\left(m + \frac{3}{2}\right)}{(2m)!\,\Gamma(m+1)\,\Gamma\left(\frac{1}{2}\right)}$$

The function H can thus be recovered from $\widehat{H}$ by

$$H(t) = \frac{2^{d+1}}{(d-2)!\,\Omega_{d+1}} \left(-\frac{1}{2t} \frac{d}{dt} \right)^d \int_t^\infty \widehat{H}(r)\,(r^2 - t^2)^{\frac{d-2}{2}} \, r \, dr$$

Thus

$$F(0) = \frac{2^{d+1}}{(d-2)!\,\Omega_{d+1}} \lim_{r \to 0} \left(-\frac{1}{2t} \frac{d}{dt} \right)^d \int_t^\infty RF(\xi_r)\,(r^2 - t^2)^{\frac{d-2}{2}} \, r \, dr \tag{3.53}$$

When $d = 1$, equation (3.50) becomes

$$\widehat{H}(r) = \int_r^\infty H(u)\,(u^2 - r^2)^{-\frac{1}{2}} \, u \, du$$

and the technique of (2.34) needs to be slightly modified. For $t > 0$, we have

$$\begin{aligned} \int_t^\infty \widehat{H}(r)\, r \, dr &= \int_t^\infty H(u)\, u \left(\int_t^u \frac{r\,dr}{\sqrt{u^2 - r^2}} \right) du \\ &= \int_t^\infty H(u)\, \sqrt{u^2 - t^2}\, u \, du, \end{aligned}$$

and so the right hand side is essentially the same as that for $d = 3$. Denote the left hand side above by $\widehat{G}(t)$. Then by (3.53),

$$\begin{aligned} H(s) &= \frac{8}{\pi^2}\left(-\frac{1}{2s}\frac{d}{ds}\right)^3 \int_s^\infty \widehat{G}(t)\,(t^2-s^2)^{\frac{1}{2}}\,t\,dt \\ &= \frac{8}{\pi^2}\left(-\frac{1}{2s}\frac{d}{ds}\right)^3 \int_s^\infty \int_t^\infty \widehat{H}(r)\,r\,dr\,(t^2-s^2)^{\frac{1}{2}}\,t\,dt \\ &= \frac{8}{3\pi^2}\left(-\frac{1}{2s}\frac{d}{ds}\right)^3 \int_s^\infty \widehat{H}(r)(r^2-s^2)^{\frac{3}{2}}\,r\,dr \end{aligned}$$

This gives

$$F(0) = \frac{8}{3\pi^2}\lim_{s\to 0}\left(-\frac{1}{2s}\frac{d}{ds}\right)^3 \int_s^\infty RF(\xi_r)\,(r^2-s^2)^{\frac{3}{2}}\,r\,dr \tag{3.54}$$

We can now obtain an inversion formula based on (3.48)–(3.49) for the d-plane transform for arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$. Since $K = \mathrm{O}(n)$ here, we see that any K-orbit in $\Xi = G(d,n)$ is just the set of all d-planes at a given distance r from 0. Thus the set of K-orbits is parametrized by r. Each K-orbit contains a unique d-plane $\xi_r = r\,e_n + \xi_0$. If we put $\gamma_r = re_1$, we can denote the shifted dual transform $R^*_{re_1}$ by R^*_r.

The function $f^\#$ in (3.45) then equals the mean value

$$f^\#(x) = M^{\|x\|}f(0)$$

from (2.35). Denoting the right hand side above by $F(x)$, (3.46) becomes

$$R^*_r Rf(0) = RF(\xi_r)$$

From (3.50) we therefore obtain

$$R^*_r Rf(0) = \Omega_d \int_r^\infty M^u f(0)\,(u^2-r^2)^{\frac{d-2}{2}}\,u\,du$$

If d is even, we can invert R using (3.51):

$$c_d\,f(0) = \lim_{r\to 0^+}\left(\frac{1}{r}\frac{d}{dr}\right)\circ\left(\frac{1}{2r}\frac{d}{dr}\right)^{\frac{d-2}{2}} R^*_r Rf(0)$$

where

$$c_d = (-1)^{\frac{d}{2}}\,\Omega_d\,((d-2)/2)!$$

By left translation, we obtain

$$c_d\,f(x) = \lim_{r\to 0^+}\left(\frac{1}{r}\frac{d}{dr}\right)\circ\left(\frac{1}{2r}\frac{d}{dr}\right)^{\frac{d-2}{2}} R^*_r Rf(x) \tag{3.55}$$

For arbitrary $d \geqslant 2$, we can use (3.53) to recover f:

$$\alpha_d\, f(0) = \lim_{t\to 0} \left(-\frac{1}{2t}\,\frac{d}{dt}\right)^d \int_t^\infty (R_r^* Rf)(0)\,(r^2 - t^2)^{\frac{d-2}{2}}\, r\, dr$$

where

$$\alpha_d = \frac{(d-2)!\,\Omega_{d+1}}{2^{d+1}}$$

The general formula is then

$$\alpha_d\, f(x) = \lim_{t\to 0} \left(-\frac{1}{2t}\,\frac{d}{dt}\right)^d \int_t^\infty (R_r^* Rf)(x)\,(r^2 - t^2)^{\frac{d-2}{2}}\, r\, dr \tag{3.56}$$

for all $x \in \mathbb{R}^n$.

Finally, if $d = 1$, equation (3.54) gives

$$c\, f(0) = \lim_{s\to 0} \left(-\frac{1}{2s}\,\frac{d}{ds}\right)^3 \int_s^\infty (R_r^* Rf)(0)(r^2 - s^2)^{\frac{3}{2}}\, r\, dr$$

where $\alpha = 3\pi^2/8$. Thus, for any $x \in \mathbb{R}^n$,

$$c\, f(x) = \lim_{s\to 0} \left(-\frac{1}{2s}\,\frac{d}{ds}\right)^3 \int_s^\infty (R_r^* Rf)(x)(r^2 - s^2)^{\frac{3}{2}}\, r\, dr$$

Chapter 4

John's Equation and the Range of the d-plane Transform

If $1 \leqslant d < n-1$, the d-plane transform R is *overdetermined* in the sense that it maps functions from $\mathbb{R}^n$ to finctions on $G(d,n)$, a higher dimensional manifold.

It is therefore of interest to determine or classify the set of n-dimensional manifolds P of $G(d,n)$ such that the map $f \mapsto Rf(\xi)$, for $\xi \in P$, is injective. Given such "admissible" manifolds, it is possible to reconstruct $f(x)$ by means of a general inversion formula, which is local in the case when d is even. Research on this important topic is still ongoing, and we refer the reader to the important paper [7], as well as a gentler introduction in [37].

In the present chapter, we will go in a different direction. By considerations of dimension, one expects the range $R\mathcal{S}(\mathbb{R}^n)$ of the d-plane transform to consist of functions satisfying certain differential equations. These equations were first obtain by Fritz John in 1938 [23] for the X-ray transform on $\mathbb{R}^3$. For arbitrary d and n, the equations were given in terms of local coordinates on $G(d,n)$ in [7]. However, the proof in this paper omitted many details. A complete proof was obtained by Richter [35], who also provided a range characterization using the infinitesimal left regular representation of $\mathrm{M}(n)$ on $G(d,n)$. While Richter made extensive use of local coordinates, we will prove his result using group-theoretic methods.

4.1 Characterization by Moment Conditions

Recall the parametrization of $G(d,n)$ in Section 3.1. Suppose that $f \in \mathcal{D}(\mathbb{R}^n)$. Then it is easy to show that Rf satisfies certain moment conditions. Fix $k \in \mathbb{Z}^+$ and let $x \in \mathbb{R}^n$. For any $\sigma \in G_{d,n}$ such that $x \perp \sigma$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(y)\,\langle y, x\rangle^k\,dy &= \int_{\sigma^\perp}\left(\int_\sigma f(z+x')\,dz\right)\langle x', x\rangle^k\,dx' \\ &= \int_{\sigma^\perp} Rf(\sigma, x')\,\langle x', x\rangle^k\,dx'\end{aligned}$$

(See Figure 3.1 in the preceding chapter.) Since the left hand side is a polynomial in x independent of σ, we see that the image $\varphi = Rf$ is a function in $\mathcal{D}(G(d,n))$ which satisfies the following moment conditions:

For every $k \in \mathbb{Z}^+$, there is a homogeneous degree k polynomial P_k on $\mathbb{R}^n$ such that

$$\int_{\sigma^\perp} \varphi(\sigma, x')\,\langle x', x\rangle^k\,dx' = P_k(x) \tag{4.1}$$

for all $(\sigma, x) \in G(d,n)$.

Let $\mathcal{D}_H(G(d,n))$ denote the vector subspace of $\mathcal{D}(G(d,n))$ consisting of those functions satisfying the moment conditions above. If R is the Radon d-plane transform, we have thus shown that $R\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}_H(G(d,n))$. We now show that this is an equality.

Theorem 4.1.1.

$$R\mathcal{D}(\mathbb{R}^n) = \mathcal{D}_H(G(d,n))$$

Proof. Suppose that $\varphi \in \mathcal{D}_H(G(d,n))$. The assertion is that there is an $f \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi = Rf$.

As in Chapter 2, let $\Xi_n = G(n-1,n)$, the manifold of codimension one hyperplanes in $\mathbb{R}^n$. The idea is to produce a function $\psi \in \mathcal{D}(\Xi_n)$ which satisfies the Helgason moment conditions (2.40). To this end, let $(\omega, p) \in S^{n-1} \times \mathbb{R}$ and consider any $\sigma \in G_{d,n}$ such that $\omega \perp \sigma$. Define

$$\psi_\sigma(\omega, p) = \int_{\substack{x' \in \sigma^\perp \\ \langle x', \omega\rangle = p}} \varphi(\sigma, x')\,dx' \tag{4.2}$$

The integral on the right is taken over the $(n-d-1)$-plane in $\sigma^\perp$ consisting of all x' such that $\langle x', \omega\rangle = p$. (It is thus a Radon transform of $\psi(\sigma, \cdot)$ on $\sigma^\perp$.) Note that for each σ and ω, the function $p \mapsto \psi_\sigma(\omega, p)$ belongs to $\mathcal{D}(\mathbb{R})$. By the moment conditions (4.1) and (4.2) it follows that for any $k \in \mathbb{Z}^+$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_\sigma(\omega, p)\,p^k\,dp &= \int_{\sigma^\perp} \varphi(\sigma, x')\,\langle x', \omega\rangle^k\,dx' \\ &= P_k(\omega)\end{aligned} \tag{4.3}$$

The right hand side is independent of the choice of σ such that $\sigma \perp \omega$. It implies that if σ and σ' are in $G_{d,n}$ such that $\omega \in \sigma^{\perp} \cap (\sigma')^{\perp}$, then

$$\int_{-\infty}^{\infty} (\psi_{\sigma}(\omega, p) - \psi_{\sigma'}(\omega, p))\, p^k\, dp = 0$$

for all $k \in \mathbb{Z}^+$. But if $F \in C_c(\mathbb{R})$ satisfies $\int_{\mathbb{R}} F(p)\, p^k\, dp = 0$ for all $k \in \mathbb{Z}^+$, then $F \equiv 0$. Hence $\psi_{\sigma}(\omega, p) = \psi_{\sigma'}(\omega, p)$.

We conclude that there is a function ψ on $S^{n-1} \times \mathbb{R}$ such that $\psi(\omega, p) = \psi_{\sigma}(\omega, p)$ for any $\sigma \in G_{d,n}$ such that $\omega \in \sigma^{\perp}$. It is clear that ψ is even in (ω, p), and so gives a function on Ξ_n. Using local cross sections $\omega \mapsto \sigma$ from S^{n-1} to $G_{d,n}$, we can see that $\psi \in \mathcal{E}(\Xi_n)$, and hence $\psi \in \mathcal{D}(\Xi_n)$.

Let R_c denote the classical Radon transform. By Theorem 2.4.1 and Theorem 2.5.1, the moment conditions (4.3) for ψ imply that there is a function $f \in \mathcal{D}(\mathbb{R}^n)$ such that $R_c f = \psi$. Hence for each $\sigma \in G_{d,n}$, we obtain

$$\begin{aligned} \psi(\omega, p) &= R_c\, f(\omega, p) \\ &= \int_{\substack{x' \in \sigma^{\perp} \\ \langle x', \omega \rangle = p}} Rf(\sigma, x')\, dx' \end{aligned}$$

Comparing the integral above with (4.2), and noting that the the (classical) Radon transform on $\sigma^{\perp}$ is injective, we conclude that $Rf(\sigma, x) = \psi(\sigma, x)$. $\square$

4.2 Invariant Differential Operators on $G(d, n)$ and the Range Characterization

From the group law (2.1), we see that the Euclidean motion group $\mathrm{M}(n)$ is isomorphic to the Lie subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ consisting of the matrices

$$\begin{pmatrix} k & v \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

where $k \in \mathrm{O}(n)$ and $v \in \mathbb{R}^n$. When convenient, we will identify $M(n)$ with this group. Thus the Lie algebra $\mathfrak{m}(n)$ of $M(n)$ may identified with the Lie subalgebra of $\mathrm{gl}(n+1, \mathbb{R})$ consisting of the matrices

$$\begin{pmatrix} T & w \\ 0 & 0 \end{pmatrix}$$

with $T \in \mathrm{so}(n)$ and $v \in \mathbb{R}^n$. Under these identifications, the adjoint representation is just conjugation: $\mathrm{Ad}(g)\, X = gXg^{-1}$. From this, we see that the adjoint representation on $\mathfrak{m}(n)$ is given by

$$\mathrm{Ad}\,(k, v)\,(T, w) = (kTk^{-1}, -kTk^{-1}v + kw) \tag{4.5}$$

and the Lie bracket by

$$[(T,w)\,,\,(T',w')] = (TT' - T'T, Tw' - T'w) \tag{4.6}$$

If E_{ij} is the elementary $n \times n$ matrix $(\delta_{ik}\delta_{jl})$, then so(n) has basis consisting of the elementary skew-symmetric matrices $X_{ij} = E_{ij} - E_{ji}$ $(1 \leqslant i < j \leqslant n)$, and $\mathfrak{m}(n)$ has basis consisting of the X_{ij} and the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

Let λ be the left regular representation of $\mathrm{M}(n)$ on $\mathcal{E}(\mathbb{R}^n)$, and $d\lambda$ the corresponding infinitesimal representation of the universal enveloping algebra $U(\mathfrak{m}(n))$. Since

$$\exp(-tX_{ij}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ (\cos t)x_i - (\sin t)x_j \\ \vdots \\ (\sin t)x_i + (\cos t)x_j \\ \vdots \\ x_n \end{pmatrix}$$

we see that $d\lambda(X_{ij})$ is the differential operator on $\mathbb{R}^n$ given by

$$d\lambda(X_{ij}) = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \tag{4.7}$$

Moreover,

$$d\lambda(e_j) = -\frac{\partial}{\partial x_j} \tag{4.8}$$

The motion group $\mathrm{M}(n)$ acts on $G(d,n)$ via

$$k \cdot (\sigma, x) = (k \cdot \sigma, k \cdot x) \qquad (k \in \mathrm{O}(n)) \tag{4.9}$$

$$v \cdot (\sigma, x) = (\sigma, x + \mathrm{Pr}_{\sigma^\perp}(v)) \qquad (v \in \mathbb{R}^n) \tag{4.10}$$

where $\mathrm{Pr}_{\sigma^\perp}$ denotes the orthogonal projection of $\mathbb{R}^n$ onto $\sigma^\perp$.

It will now be convenient to define the Schwartz space $\mathcal{S}(G(d,n))$. Let ν be the left regular representation of $\mathrm{M}(n)$ on $\mathcal{E}(G(d,n))$, with $d\nu$ the corresponding infinitesimal representation. By definition, $\mathcal{S}(G(d,n))$ is the vector space consisting of all functions $\varphi \in \mathcal{E}(G(d,n))$ such that for all $U \in U(\mathfrak{m}(n))$ and $N \in \mathbb{Z}^+$,

$$\|\varphi\|_{U,N} := \sup_{\xi \in G(d,n)} |(d\nu(U)f)(\xi)|\;(1 + d(0,\xi))^N < \infty \tag{4.11}$$

The seminorms above give rise to a Fréchet space topology on $\mathcal{S}(G(d,n))$, but we will not be making use of this topology directly. It is clear that $\mathcal{D}(G(d,n)) \subset \mathcal{S}(G(d,n))$.

From (4.11) one can show that $Rf \in \mathcal{S}(G(d,n))$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$. In fact, by (4.7)–(4.8), and since R intertwines $d\lambda(U)$ and $d\nu(U)$,

$$\begin{aligned} |d\nu(U)\, Rf(\sigma,x)|\,(1+\|x\|)^N &\leqslant \int_\sigma |d\lambda(U) f(x+y)|\,(1+\|x\|)^N\,dy \\ &\leqslant \int_\sigma C\cdot(1+\|x+y\|)^{-N-M}\,(1+\|x\|)^N\,dy \end{aligned}$$

For sufficiently large M it is clear that the last expression above is less than some constant independent of $(\sigma,x)\in G(d,n)$.

Since $G(d,n)$ is a vector bundle with a natural inner product on its fibers, we can take the Fourier transform on these fibers. If $\varphi \in \mathcal{S}(G(d,n))$, its *partial Fourier transform* is the function $\mathcal{F}_d\varphi$ on $G(d,n)$ given by

$$\mathcal{F}_d\varphi(\sigma,y) = \int_{\sigma^\perp} \varphi(\sigma,x)\, e^{-i\,\langle x,y\rangle}\,dx \tag{4.12}$$

It is easy to see that $\mathcal{F}_d\varphi \in \mathcal{E}(G(d,n))$.

The following lemma describes how $d\nu(U(\mathfrak{m}(n)))$ transforms under the partial Fourier transform.

Lemma 4.2.1. *Let $X \in so(n)$ and $v \in \mathbb{R}^n$. If $\varphi \in \mathcal{S}(G(d,n))$, then*

(i) $(\mathcal{F}_d(d\nu(X)\varphi))(\sigma,x) = (d\nu(X)\mathcal{F}_d\varphi)(\sigma,x)$

(ii) $(\mathcal{F}_d(d\nu(v)\varphi))(\sigma,x) = -i\langle v,x\rangle\,(\mathcal{F}_d\varphi)(\sigma,x)$

Proof. By (4.10), $\mathcal{F}_d(\nu(k)\varphi) = \nu(k)(\mathcal{F}_d\varphi)$ for any $k \in \mathrm{O}(n)$. Differentiating, we get (i). For (ii), we note that $\exp tv = tv$ in $\mathrm{M}(n)$, so

$$\begin{aligned} \mathcal{F}_d(\nu(tv)\varphi)(\sigma,x) &= \int_{\sigma^\perp} \varphi(\sigma, x' - \mathrm{Pr}_{\sigma^\perp} tv)\, e^{-i\langle x',x\rangle}\,dx' \\ &= e^{-i\langle \mathrm{Pr}_{\sigma^\perp}(tv),x\rangle}\mathcal{F}_d\varphi(\sigma,x) \\ &= e^{-i\langle tv,x\rangle}\,\mathcal{F}_d\varphi(\sigma,x) \end{aligned}$$

Differentiating both sides above proves (ii). ☐

In particular we have the the transformation rules

$$\begin{aligned} (d\nu(X_{jk})\mathcal{F}_d\varphi)(\sigma,x) &= \mathcal{F}_d(d\nu(X_{jk})\varphi)(\sigma,x) \\ \mathcal{F}_d(d\nu(e_j)\varphi)(\sigma,x) &= -i\,x_j\,\mathcal{F}_d\varphi(\sigma,x) \end{aligned} \tag{4.13}$$

From this one can deduce that $\mathcal{F}_d\varphi \in \mathcal{S}(G(d,n))$ whenever $\varphi \in \mathcal{S}(G(d,n))$.

The projection-slice theorem (3.4) can be written

$$\widetilde{f}(x) = \mathcal{F}_d(Rf)(\sigma,x), \qquad ((\sigma,x)\in G(d,n)) \tag{4.14}$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Next let us show briefly why the moment conditions (4.1) fail to characterize the range $R\mathcal{S}(\mathbb{R}^n)$ when $d < n-1$. For simplicity, we let $n = 3$ and $d = 1$ (so that R is the X-ray transform on $\mathbb{R}^3$), although our example easily generalizes. We show that there exists a function $\varphi \in \mathcal{S}(G(1,3))$ satisfying the moment conditions but which is not in the range $R\mathcal{S}(\mathbb{R}^n)$. For this, let ψ be a nonzero function in $\mathcal{D}(G(1,3))$ such that, for some $\epsilon > 0$, $\psi(\xi) = 0$ for all ξ such that $d(0,\xi) < \epsilon$ and such that

$$\psi(\sigma', e_1) \neq \psi(\sigma'', e_1)$$

where σ' is the y-axis $\mathbb{R}e_2$ and σ'' the z-axis $\mathbb{R}e_3$. ψ exists since the set of lines (σ, e_1) is a compact set bounded away from 0 and $\mathcal{D}(G(1,3))$ separates points. Since $\mathcal{F}_1\mathcal{S}(G(1,3)) = \mathcal{S}(G(1,3))$, let φ be the function in $\mathcal{S}(G(1,3))$ such that $\mathcal{F}_d\varphi = \psi$.

Now for each $(\sigma, x) \in G(1,3)$, we have

$$\psi(\sigma, tx) = \int_{\sigma^\perp} \varphi(\sigma, y)\, e^{-i\,t\,\langle x,y\rangle}\, dy$$

Taking the derivative of both sides with respect to t k times and evaluating at $t = 0$, it follows that

$$\begin{aligned} 0 &= \frac{d^k}{dt^k}\psi(\sigma, tx)\bigg|_{t=0} \\ &= \int_{\sigma^\perp} \varphi(\sigma, y)\,(i\,\langle x, y\rangle)^k\, dy \end{aligned}$$

This shows that φ satisfies the moment conditions (4.1) with all P_k equal to 0. Now if $\varphi = Rf$ for some $f \in \mathcal{S}(\mathbb{R}^3)$, then the projection-slice theorem implies that $\widetilde{f}(e_1) = \psi(\sigma', e_1) = \psi(\sigma'', e_1)$, a contradiction.

It turns out that the range $R\mathcal{S}(\mathbb{R}^n)$ can be described by an M(n)-invariant system of second order differential equations. These equations arise from the kernel of the infinitesimal left regular representation of $U(\mathfrak{m}(n))$ on $\mathbb{R}^n$. For any indices i, j, l in $\{1, \ldots, n\}$, let

$$V_{ijl} = e_i\, X_{jl} + e_j\, X_{li} + e_l\, X_{ij} \in U(\mathfrak{m}(n)) \tag{4.15}$$

Note that if any of the indices i, j, or l coincide, then $V_{ijl} = 0$. It is immediate from (4.7)–(4.8) that $d\lambda(V_{ijl}) = 0$ for all i, j, l. By (1.29) the diagram

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}^n) & \xrightarrow{R} & \mathcal{S}(G(d,n)) \\ {\scriptstyle d\lambda(V)}\big\downarrow & & \big\downarrow{\scriptstyle d\nu(V)} \\ \mathcal{S}(\mathbb{R}^n) & \xrightarrow{R} & \mathcal{S}(G(d,n)) \end{array} \tag{4.16}$$

commutes for all $V \in U(\mathfrak{m}(n))$. Thus any function φ in the range $R\mathcal{S}(\mathbb{R}^n)$ satisfies the necessary differential equations

$$d\nu(V_{ijl})\varphi = 0 \tag{4.17}$$

for all i, j, l. It turns out that these equations are sufficient as well.

Let $\mathcal{S}_D(G(d,n))$ denote the subspace of $\mathcal{S}(G(d,n))$ consisting of all functions φ such that $d\nu(V_{ijl})\,\varphi = 0$ for all i, j, l.

Theorem 4.2.2. *If R is the d-plane transform, then*

$$R\mathcal{S}(\mathbb{R}^n) = \mathcal{S}_D(G(d,n))$$

To prove Theorem 4.2.2 we first calculate how the operators V_{ijl} transform under the adjoint action of $\mathrm{M}(n)$.

Lemma 4.2.3. *Let $k = (k_{rs}) \in O(n)$ and $v \in \mathbb{R}^n$. Suppose $1 \leqslant i, j, l \leqslant n$. Then*

$$Ad(k)\,V_{ijl} = \sum_{u<s<r} \det \begin{pmatrix} k_{ui} & k_{uj} & k_{ul} \\ k_{ri} & k_{rj} & k_{rl} \\ k_{si} & k_{sj} & k_{sl} \end{pmatrix} V_{urs} \tag{4.18}$$

$$Ad(v)\,V_{ijl} = V_{ijl} \tag{4.19}$$

Proof. Identifying $\mathrm{M}(n)$ with the group consisting of the matrices (4.4), the adjoint representation is just conjugation: $\mathrm{Ad}(g)\,X = gXg^{-1}$. Thus by a routine computation

$$\mathrm{Ad}(k)\,e_i = \sum_{j=1}^{n} k_{ji}\,e_j, \qquad \mathrm{Ad}(k)\,X_{jl} = \sum_{s,r=1}^{n} k_{sj}k_{rl}\,X_{sr}$$

Hence

$$\begin{aligned} \mathrm{Ad}(k)\,V_{ijl} &= \sum_{u,s,r} k_{ui}k_{sj}k_{rl}\,(e_u\,X_{sr} + e_s\,X_{ru} + e_r\,X_{us}) \\ &= \sum_{u,s,r} k_{ui}k_{sj}k_{rl}\,V_{usr} \end{aligned}$$

For each fixed $u < s < r$ above, we have $V_{urs} = -V_{usr}$, etc., proving (4.18). For (4.19), write $v = \sum_{r=1}^{n} v_r\,e_r$. Then $\mathrm{Ad}(v)\,X_{jl} = X_{jl} + v_j\,e_l - v_l\,e_j$ and $\mathrm{Ad}(v)\,e_i = e_i$ for all i, j, l. Therefore

$$\begin{aligned} \mathrm{Ad}(v)\,V_{ijl} &= e_i(X_{jl} + v_j\,e_l - v_l\,e_j) + e_j(X_{li} + v_l\,e_i - v_i\,e_l) \\ &\qquad + e_l(X_{ij} + v_i\,e_j - v_j\,e_i) \\ &= V_{ijl}. \end{aligned}$$

$\square$

We now proceed with the proof of Theorem 4.2.2. Suppose that $\varphi \in \mathcal{S}_D(G(d,n))$. If φ is to be the Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^n))$, the projection-slice theorem (3.4) tells us that the partial Fourier transform $\mathcal{F}_d\varphi(\sigma, x)$ must equal the Fourier transform $\widetilde{f}(x)$. Thus our next objective is to prove that there is a function $F \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\mathcal{F}_d\varphi(\sigma, x) = F(x) \tag{4.20}$$

for all $(\sigma, x) \in G(d,n)$. Given this function F, let $f \in \mathcal{S}(\mathbb{R}^n)$ be its inverse Fourier transform, so that $\widetilde{f} = F$. Then according to projection-slice,

$$F(x) = \mathcal{F}_d(Rf)(\sigma, x)$$

for all (σ, x). Since $\mathcal{F}_d$ is injective, (4.20) shows that $\varphi = Rf$.

For simplicity, let us denote $\mathcal{F}_d\varphi$ by ψ.

Lemma 4.2.4. *There exists $F \in \mathcal{E}(\mathbb{R}^n\backslash\{0\}) \cap C(\mathbb{R}^n)$ such that $\psi(\sigma, x) = F(x)$ for all $(\sigma, x) \in G(d,n)$.*

Proof. Fix $x \in \mathbb{R}^n$. Define $\mathrm{O}(x^\perp)$ to be the subgroup of all $k \in \mathrm{O}(n)$ such that $k \cdot x = x$. (Note that $\mathrm{O}(0^\perp) = \mathrm{O}(n)$.) Let $\mathrm{so}(x^\perp)$ be its Lie algebra. Then $\mathrm{so}(x^\perp)$ consists of all infinitesimal rotations about 0 fixing x.

Since φ satisfies the differential equations (4.17), we see by (4.13) that ψ satisfies the first order system

$$(x_i\, d\nu(X_{jl}) + x_j\, d\nu(X_{li}) + x_l\, d\nu(X_{ij}))\, \psi(\sigma, x) = 0 \tag{4.21}$$

In particular, if we put $i = 1$ and $j, l > 1$, and $x = re_1$, then we obtain

$$d\nu(X_{jl})\psi(\sigma, re_1) = 0 \tag{4.22}$$

for all $r > 0$ and $\sigma \in G_{d,n}$ orthogonal to e_1. The subgroup $\mathrm{O}(e_1^\perp)$ of $\mathrm{O}(n)$ consists of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \qquad\qquad (k \in \mathrm{O}(n-1))$$

Its Lie algebra $\mathrm{so}(e_1^\perp) \approx \mathrm{so}(n-1)$ has basis $\{X_{ij} \mid 1 < i < j \leqslant n\}$. Moreover, $\mathrm{O}(e_1^\perp)$ acts transitively on the set of all $\sigma \in G_{d,n}$ orthogonal to e_1; this set is a connected submanifold of $G_{d,n}$ diffeomorphic to $G_{d,n-1}$. Hence $\psi(\sigma, re_1)$ is constant in σ in the sense that

$$\psi(\sigma, re_1) = \psi(\sigma', re_1) \tag{4.23}$$

for all σ, σ' orthogonal to e_1.

Next fix $x \in \mathbb{R}^n$, $x \neq 0$. Let $r = \|x\|$ and choose $k \in \mathrm{O}(n)$ such that $x = k \cdot (re_1)$. Now the function $\varphi_1 = \varphi^{\tau(k^{-1})} = \nu(k^{-1})\varphi$ satisfies the system (4.17). In fact by (1.27),

$$\begin{aligned} d\nu(V_{ijl})\varphi_1(\xi) &= d\nu(V_{ijl})\, \varphi^{\tau(k^{-1})}(\xi) \\ &= (d\nu(\mathrm{Ad}(k)V_{ijl})\varphi)\,(k \cdot \xi) \end{aligned}$$

By Lemma 4.2.3, $\mathrm{Ad}(k)V_{ijl}$ is a linear combination of operators V_{urs}, so the right hand side above vanishes.

Now let $\psi_1 = \mathcal{F}_d(\varphi_1)$. Then ψ_1 satisfies (4.23). But by (4.10), $\psi_1 = \psi^{\tau(k^{-1})}$, and thus

$$\psi^{\tau(k^{-1})}(\sigma, re_1) = \psi^{\tau(k^{-1})}(\sigma', re_1)$$

for all σ, σ' orthogonal to re_1. This implies that

$$\psi(\sigma, x) = \psi(\sigma', x) \tag{4.24}$$

for all σ, σ' orthogonal to x.

Next let $\omega \in S^{n-1}$ and for $r > 0$, put $x = r\omega$. We can then take the limit of both sides of (4.24) as $r \to 0$ to obtain

$$\psi(\sigma, 0) = \psi(\sigma', 0) \tag{4.25}$$

for all $\omega \in S^{n-1}$, and all σ and σ' in $G_{d,n}$ orthogonal to ω.

Let us now prove that the relation (4.25) in fact holds for any σ and σ' in $G_{d,n}$. For this, choose any ω and ω' in S^{n-1} such that $\omega \perp \sigma$ and $\omega' \perp \sigma'$. If $\omega = \pm\omega'$, then σ and σ' are both orthogonal to ω so $\psi(\sigma, 0) = \psi(\sigma', 0)$ by (4.25). So let us assume that ω and ω' are linearly independent.

Let W be the linear span of ω and ω' in $\mathbb{R}^n$. Since $d \leqslant n - 2$, we can fix a d-dimensional subspace σ'' of $W^\perp$. Since σ and σ'' are orthogonal to ω, we have $\psi(\sigma, 0) = \psi(\sigma'', 0)$ by (4.25). Likewise, since σ' and σ'' are orthogonal to ω', we obtain $\psi(\sigma', 0) = \psi(\sigma'', 0)$. This shows that $\psi(\sigma, 0) = \psi(\sigma', 0)$.

The relations (4.24) and (4.25) now show that there exists a function F on $\mathbb{R}^n$ such that $F(x) = \psi(\sigma, x)$ for all $(\sigma, x) \in G(d, n)$. This function F is C^∞ on $\mathbb{R}^n \backslash \{0\}$. To show this, we first claim that for any $x_0 \neq 0$ in $\mathbb{R}^n$, there exists a smooth map $x \mapsto \sigma(x)$ from a neighborhood U of x_0 into $G_{d,n}$ such that $x \perp \sigma(x)$ for all $x \in W$.

For this, let $\omega_0 = x_0/\|x_0\|$ and choose a neighborhood Ω_0 of ω_0 in S^{n-1} for which there exists a smooth section $s : \Omega_0 \to \mathrm{O}(n)$ with $s(\omega) \cdot e_n = \omega$. If we recall that $\sigma_0 = \mathbb{R}e_1 + \cdots + \mathbb{R}e_d$, we see that the map

$$\sigma : x \mapsto s(x/\|x\|) \cdot \sigma_0$$

is well-defined and C^∞ from a neighborhood U of x_0 to $G_{d,n}$, with $\sigma(x) \perp x$ for all $x \in U$. Hence the map

$$x \mapsto (\sigma(x), x)$$

is C^∞ from U to $G(d,n)$. It follows from this that $F \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

To prove that F is continuous at 0, suppose, to the contrary, that for some $\epsilon > 0$ there exists a sequence x_j converging to 0 such that $|F(x_j) - F(0)| > \epsilon$ for all j. Choose a sequence σ_j in $G_{d,n}$ with $\sigma_j \perp x_j$; since $G_{d,n}$ is compact, we may assume that σ_j converges to an element $\sigma_0 \in G_{d,n}$. Then $F(x_j) - F(0) = \psi(\sigma_j, x_j) - \psi(\sigma_0, 0) \to 0$, a contradiction.

This finishes the proof of Lemma 4.2.4. $\square$

The following lemma, which is of independent interest, is the most crucial part of the proof of Theorem 4.2.2.

Lemma 4.2.5. *Assume that $d < n-1$. Suppose that F is a function on $\mathbb{R}^n$ such that there exists a function $\psi \in \mathcal{E}(G(d,n))$ with $F(x) = \psi(\sigma, x)$ for all $(\sigma, x) \in G(d,n)$. Then $F \in \mathcal{E}(\mathbb{R}^n)$.*

Remark: The example

$$\psi(\omega, p) = p\,\omega_1^3 = x_1 \left(\frac{x_1}{\|x\|}\right)^2$$

shows that Lemm 4.2.5 does not hold when $d = n-1$.

Proof. The proof of Lemma 4.2.4 shows that $F \in \mathcal{E}(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$. Thus we just need to prove that F is C^∞ on a neighborhood of 0.

For this, we intend to show that for each $v \in \mathbb{R}^n$, there exists a function $\Psi_v \in \mathcal{E}(G(d,n))$ such that the directional derivative

$$D_v F(x) = -d\lambda(v) F(x) = \Psi_v(\sigma, x) \tag{4.26}$$

for all $(\sigma, x) \in G(d,n)$ with $x \neq 0$. Since $D_v F$ and Ψ_v satisfy the hypothesis of the present lemma, the proof of Lemma 4.2.4 shows that $D_v F(x)$ can be extended continuously to the origin. Thus all first order partial derivatives $\partial F / \partial x_i$ can be extended continuously to the origin. An elementary induction then proves that all partial derivatives of F can be extended continuously to the origin.

Now by (4.10),

$$\begin{aligned}
d\nu(v)\psi(\sigma,x) &= \frac{d}{dt}\psi\left(\sigma, x - t\,\mathrm{Pr}_{\sigma^\perp}(v)\right)\Big|_{t=0} \\
&= \frac{d}{dt}F(x - t\,\mathrm{Pr}_{\sigma^\perp}(v))\Big|_{t=0} \\
&= \frac{d}{dt}F(x - tv + t\,\mathrm{Pr}_{\sigma}(v))\Big|_{t=0} \\
&= d\lambda(v)F(x) - d\lambda(\mathrm{Pr}_\sigma(v))F(x)
\end{aligned}$$

Thus the directional derivative $D_vF(x)$ equals

$$\begin{aligned}
-d\lambda(v)F(x) &= -d\nu(v)\psi(\sigma,x) - d\lambda(\mathrm{Pr}_\sigma(v))F(x) \\
&= -d\nu(v)\psi(\sigma,x) + \langle \nabla F(x), \mathrm{Pr}_\sigma(v)\rangle \qquad (4.27)
\end{aligned}$$

We need to introduce some additional notation. First, let $\mathbf{X}$ be the $n \times n$ skew-symmetric matrix with vector entries $X_{ij} \in \mathrm{so}(n)$ for $1 \leqslant i,j \leqslant n$:

$$\mathbf{X} = \begin{pmatrix} 0 & X_{12} & & X_{1n} \\ -X_{12} & 0 & & X_{2n} \\ & & \ddots & \\ -X_{1n} & -X_{2n} & & 0 \end{pmatrix}$$

Next, let $d\nu(\mathbf{X})\psi(\sigma,x)$ denote the $n \times n$ matrix $(d\nu(X_{ij})\psi(\sigma,x))$. Suppose that $(\sigma,x) \in G(d,n)$ with $x \neq 0$. For any $w \in \sigma$, we claim that

$$(d\nu(\mathbf{X})\psi(\sigma,x))\, w = \langle \nabla F(x), w\rangle\, x \qquad (4.28)$$

To prove this, we first note that by (4.10), $d\lambda(X_{ij})F(x) = d\nu(X_{ij})(\sigma,x)$ for all i,j. Thus we have the matrix equation $d\lambda(\mathbf{X})F(x) = d\nu(\mathbf{X})\psi(\sigma,x)$, where the left hand side denotes the $n \times n$ matrix $(d\lambda(X_{ij})F(x))_{1\leqslant i,j\leqslant n}$.

Therefore the ith entry of the left hand side of (4.28) is

$$\begin{aligned}
\left(d\lambda(\mathbf{X})F(x)\,w\right)_i &= \sum_{j=1}^{n} d\lambda(X_{ij})F(x)\,w_j \\
&= \sum_{j=1}^{n}\left(x_i\frac{\partial F}{\partial x_j}(x) - x_j\frac{\partial F}{\partial x_i}(x)\right) w_j \\
&= x_i\sum_{j=1}^{n}\frac{\partial F}{\partial x_j}(x)\,w_j - \frac{\partial F}{\partial x_i}\sum_{j=1}^{n} x_j\,w_j \\
&= x_i\sum_{j=1}^{n}\frac{\partial F}{\partial x_j}(x)\,w_j,
\end{aligned}$$

since $w \perp x$. This is the ith entry on the right hand side of (4.28).

Since we are assuming that $x \neq 0$, let us choose any i such that $x_i \neq 0$. Then if we apply (4.28) to the last term in (4.27) with $w = \mathrm{Pr}_\sigma(v)$, we obtain

$$D_v F(x) = -d\nu(v)\psi(\sigma, x) + \frac{1}{x_i} \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x) \tag{4.29}$$

Let $\Psi_v(\sigma, x)$ denote the right hand side above. Since the first term on the right above belongs to $\mathcal{E}(G(d,n))$, Lemma 4.2.5 will be proved if we can show that the second term above

$$\Phi_v(\sigma, x) = \frac{1}{x_i} \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x) \tag{4.30}$$

extends to a C^∞ function on $G(d,n)$.

Let $u_1, \ldots, u_d$ be any orthonormal basis of σ, considered as column vectors, with $u_j = (u_{ij})_{i=1}^n$. If U is the $n \times d$ matrix whose columns are the u_j, then $\mathrm{Pr}_\sigma(v) = U\,{}^tUv$, and using local cross sections from σ to U (which belongs to the Stiefel manifold $\mathrm{St}(k,n)$) it is easy to see that the map $\sigma \mapsto \mathrm{Pr}_\sigma(v)$ is C^∞ on $G_{d,n}$. The sum on the right hand side of (4.30)

$$\sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x)$$

is therefore a smooth function on $G(d,n)$.

Moreover, (4.28) shows that if x_i and x_m are both nonzero, we have

$$\frac{1}{x_i} \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x) = \frac{1}{x_m} \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{mj})\psi(\sigma, x) \tag{4.31}$$

so that

$$x_i \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{mj})\psi(\sigma, x) = x_m \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x) \tag{4.32}$$

By continuity, the relation (4.32) above holds for all $(\sigma, x) \in G(d,n)$. From this we see that

$$x_i = 0 \implies \sum_{j=1}^{n} \left(\mathrm{Pr}_\sigma(v)\right)_j \, d\nu(X_{ij})\psi(\sigma, x) = 0. \tag{4.33}$$

From (3.1), $G(d,n)$ is a finite union of trivial bundles W_α with the local coordinate representation:

$$\omega_\alpha : (\sigma, x) \mapsto (y(\sigma), x_\alpha)$$

Here α is a choice of $n-d$ indices $i_1, \ldots, i_{n-d}$ in $\{1, \ldots, n\}$, $x_\alpha = (x_{i_1}, \ldots, x_{i_{n-d}})$, and $y(\sigma)$ is a local coordinate system for $\sigma \in \pi(W_\alpha) \subset G_{d,n}$.

To prove that $\Phi_v \in \mathcal{E}(G(d,n))$, it suffices to prove that $\Psi_v|_{W_\alpha} \in \mathcal{E}(W_\alpha)$ for each α. Suppose first that i is one of the indices $i_1, \ldots, i_{n-d}$. Since x_i is a local coordinate in W_α, it follows from (4.30) and (4.33) that Φ_v is a C^∞ function on W_α. Suppose, on the other hand, that i is not one of the indices $i_1, \ldots, i_{n-d}$. Then choose any m in $\{i_1, \ldots, i_{n-d}\}$. According to (4.31) and (4.33), on W_α, Φ_v is given by the smooth function

$$\frac{1}{x_m} \sum_{j=1}^{n} (\mathrm{Pr}_\sigma(v))_j \; d\nu(X_{mj})\psi(\sigma, x)$$

Since x_m is now a coordinate on W_α, this proves that $\Phi_v \in \mathcal{E}(G(d,n))$ and finishes the proof of Lemma 4.2.5. □

Lemma 4.2.6. *Let the function F be as in Lemma 4.2.4. Then $F \in \mathcal{S}(G(d,n))$.*

The proof of this lemma is somewhat technical and the reader may safely skip it. We include it for completeness.

Since $\psi \in \mathcal{S}(G(d,n))$ and $\psi(\sigma, x) = F(x)$ for all $(\sigma, x) \in G(d,n)$, we see that for any $N \in \mathbb{Z}^+$,

$$\sup_{x \in \mathbb{R}^n} \|x\|^N |F(x)| < \infty$$

Let $v \in \mathbb{R}^n$. According to (4.26), there exists a function $\Psi_v \in \mathcal{E}(G(d,n))$ such that directional derivative $D_v F(x)$ equals $\Psi_v(\sigma, x)$ for all (σ, x). Our aim is to show that $\Psi_v \in \mathcal{S}(G(d,n))$. If we can prove this, then we obtain the estimate

$$\sup_{x \in \mathbb{R}^n} \|x\|^N |D_v F(x)| < \infty$$

for all $N \in \mathbb{Z}^+$. We can then apply the same reasoning, replacing F by $D_v F$, and Ψ by Ψ_v, to arrive at similar estimates for the second order partials of F. We can then use induction on successive partials of F to prove the lemma in general.

Now by (4.29) and (4.30), $\Phi_v = -d\nu(v)\psi + \Phi_v$. Since $d\nu(v)\psi \in \mathcal{S}(G(d,n))$, the lemma will be proved if we can show that $\Phi_v \in \mathcal{S}(G(d,n))$.

We now express $G(d,n)$ as a finite union of local trivial bundles. For any d-multiindex $I : 1 \leqslant i_1 < \cdots < i_d \leqslant n$ in $\{1, \ldots, n\}$, let p_I be the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^d$ given by $p_I(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d})$. Then let $G_{d,n,I}$ be the open subset of $G_{d,n}$ consisting of those σ such that $p_I(\sigma) = \mathbb{R}^d$.

Let $\pi : (\sigma, x) \mapsto \sigma$ be the projection of the vector bundle $G(d,n)$ onto its base $G_{d,n}$. For the d-multiindex I above, let $G(d,n)_I = \pi^{-1}(G_{d,n,I})$.

Let I' be the $(n-d)$-multiindex complementary to I, with $I' : 1 \leqslant i'_1 < \cdots < i'_{n-d} \leqslant n$, and let $p_{I'}(x) = x_{I'}$ be the corresponding orthogonal projection of

$\mathbb{R}^n$ onto $\mathbb{R}^{n-d}$. Then the map

$$\begin{aligned} \tau_I : G(d,n)_I &\to G_{d,n,I} \times \mathbb{R}^{n-d} \\ (\sigma, x) &\mapsto (\sigma, x_{I'}) \end{aligned} \tag{4.34}$$

is a local trivializing map of the vector bundle $G(d,n)$.

The open sets $G_{d,n,I}$ cover $G_{d,n}$. Since $G_{d,n}$ is compact, there exists an open cover $\{W_I\}$ of $G_{d,n}$ such that $\overline{W}_I \subset G_{d,n,I}$ for each I. Put $\Xi_I = \pi^{-1}(W_I)$. Ξ_I is of course a sub-bundle of the trivial bundle $G(d,n)_I$.

For any I, we now define the functions of rapid decrease on Ξ_I as follows. Let $\mathcal{S}(\Xi_I)$ be the set of all $\Psi \in \mathcal{E}(\Xi_I)$ such that for any $N \in \mathbb{Z}^+$, for any differential operator D on $G_{d,n}$, and any constant coefficient differential operator E on $\mathbb{R}^{n-d}$, the following condition holds

$$\sup_{(\sigma, x_{I'}) \in W_I \times \mathbb{R}^{n-d}} \|x_{I'}\|^N \left| E_y\, D_\sigma (\Psi \circ \tau_I^{-1})(\sigma, x_{I'}) \right| < \infty \tag{4.35}$$

We now claim that

$$\mathcal{S}(G(d,n)) = \{\, \Psi \in \mathcal{E}(G(d,n)) \mid \Psi|_{\Xi_I} \in \mathcal{S}(\Xi_I) \text{ for all } I \} \tag{4.36}$$

This is an alternative characterization of $\mathcal{S}(G(d,n))$ and is in fact the one used in [8] and [35]. We will postpone for later the proof of the equivalence (4.36), which involves a few tedious calculations.

Assuming the equivalence (4.36), we note that the sum on the right hand side in (4.30)

$$\Phi_i(\sigma, x) = \sum_{j=1}^{n} (\mathrm{Pr}_\sigma(v))_j \; d\nu(X_{ij}) \psi(\sigma, x) \tag{4.37}$$

belongs to $\mathcal{S}(G(d,n))$. This is because $d\nu(X_{ij})\psi \in \mathcal{S}(G(d,n))$ for all i, j and, for fixed v, the map $\sigma \mapsto (\mathrm{Pr}_\sigma(v))_j$ is a smooth, hence bounded, function on the compact set $G_{d,n}$.

Now fix a d-multiindex I, choose any i, $1 \leqslant i \leqslant n$, and consider the restriction $\Phi_i|_{\Xi_I}$. By the equivalence (4.36), this restriction belongs to $\mathcal{S}(\Xi_I)$. Thus, for any differential operator D on $G_{d,n}$, $D_\sigma \Phi_i(\sigma, x)$ belongs to $\mathcal{S}(\Xi_I)$. From this and from the estimate (4.35), we see that the family of functions on $\mathbb{R}^{n-d}$ given by $\{D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, x) \mid \sigma \in W_I\}$ forms a bounded set in $\mathcal{S}(\mathbb{R}^{n-d})$.

Next let i and m be any two indices such that $i, m \in I'$. The relation (4.32) can be written

$$x_i\, (\Phi_m \circ \tau_I^{-1})(\sigma, x_{I'}) = x_m\, (\Phi_i \circ \tau_I^{-1})(\sigma, x_{I'}) \tag{4.38}$$

for all $(\sigma, x_{I'}) \in W_I \times \mathbb{R}^{n-d}$. Applying the differential operator D (which acts on the first argument σ) to both sides, we get

$$x_i\, D_\sigma(\Phi_m \circ \tau_I^{-1})(\sigma, x_{I'}) = x_m\, D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, x_{I'}) \tag{4.39}$$

This last relation inplies that

$$x_i = 0 \implies D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, x_{I'}) = 0$$

on $W_I \times \mathbb{R}^{n-d}$.

At this point we invoke a simple version of a result by Langebruch ([27], Theorem 1.6), which we state as follows:

Theorem 4.2.7. *Fix $1 \leqslant i \leqslant n$ and let $\mathcal{S}_i(\mathbb{R}^n)$ be the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of all $f \in \mathcal{S}(\mathbb{R}^n)$ which vanish on the hyperplane $x_i = 0$. Let L_i be multiplication by x_i. Then L_i is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}_i(\mathbb{R}^n)$.*

Note that one consequence of Langenbruch's theorem is the non-obvious result that if $x_i f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in \mathcal{S}(\mathbb{R}^n)$.

Going back to the proof of Lemma 4.2.6, we know that since $D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, y) \in \mathcal{S}(\Xi_I)$, the family of functions

$$\{D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, \cdot) \mid \sigma \in W_I\}$$

is a bounded set in $\mathcal{S}(\mathbb{R}^{n-d})$.

By (4.33), the family above is in fact a bounded subset of $\mathcal{S}_i(\mathbb{R}^{n-d})$. Now according to Theorem 4.2.7, multiplication by $1/x_i$ is a continuous map from $\mathcal{S}_i(\mathbb{R}^{n-d})$ onto $\mathcal{S}(\mathbb{R}^{n-d})$. Hence the collection

$$\{(1/x_i)\, D_\sigma(\Phi_i \circ \tau_I^{-1})(\sigma, \cdot) \mid \sigma \in W_I\}$$

is a bounded subset of $\mathcal{S}(\mathbb{R}^{n-d})$. But $(1/x_i)\, D_\sigma(\Phi_i \circ \tau_I^{-1}(\sigma, x) = D_\sigma((1/x_i)\Phi_i \circ \tau_I^{-1}(\sigma, x_{I'})$. If we apply the usual seminorms defining $\mathcal{S}(\mathbb{R}^d)$, we conclude that

$$\sup_{(\sigma,x)\in W_I\times\mathbb{R}^d} \|x\|^N \left|E_x\, D_\sigma((1/x_i)\Phi_i \circ \tau_I^{-1})(\sigma, x_{I'})\right| < \infty \tag{4.40}$$

Finally we note that

$$\frac{1}{x_i}\,(\Phi_i \circ \tau_I^{-1})(\sigma, x_{I'}) = (\Phi_v|_{\Xi_I}) \circ \tau_I^{-1}(\sigma, x_{I'})$$

(4.40) therefore shows that the restriction $\Phi_v|_{\Xi_I}$ belongs to $\mathcal{S}(\Xi_I)$. Hence by the correspondence (4.36), $\Phi_v \in \mathcal{S}(G(d,n))$.

It finally remains to prove the correspondence (4.36). As a first step, we will show that for any C^∞ function Ψ on Ξ_I, the derivatives $d\nu(U)\Psi$, for $U \in \mathfrak{g}$, can be expressed as a finite sum

$$(d\nu(U)\Psi) \circ \tau_I^{-1}(\sigma, x_{I'}) = \sum_\alpha p_\alpha(x_{I'})\,(E_\alpha)_{x_{I'}}\,(D_\alpha)_\sigma\,(\Psi \circ \tau_I^{-1})(\sigma, x_{I'}) \tag{4.41}$$

where $p_\alpha(x_{I'})$ is a polynomial, E_α is a constant coefficient differential operator on $\mathbb{R}^{n-d}$, and D_α is a differential operator on $G_{d,n,I}$.

Now we note that there is a constant C_I such that if $(\sigma, x) \in \Xi_I$, then

$$\|x\| \leqslant C_I \, \|x_{I'}\| \tag{4.42}$$

To prove (4.42), we may, for simplicity, assume that $I = \{1, \ldots, d\}$ and $I' = \{d+1, \ldots, n\}$. Let F be any $n \times d$ matrix whose columns form an orthonormal basis of σ, and write

$$F = \begin{pmatrix} F_I \\ F_{I'} \end{pmatrix}$$

where F_I is the upper $d \times d$ part of F. Since $x \in \sigma^\perp$, we have

$$\begin{aligned} 0 &= {}^tF\,x \\ &= \left({}^tF_I \; {}^tF_{I'}\right) \begin{pmatrix} x_I \\ x_{I'} \end{pmatrix} \\ &= {}^tF_I \, x_I + {}^tF_{I'} \, x_{I'} \end{aligned}$$

Now since $(\sigma, x) \in \Xi_I$, the matrix F_I is nonsingular. Hence

$$x_I = -\left(({}^tF_I)^{-1} \, {}^tF_{I'}\right) \, x_{I'}$$

The matrix $\left(({}^tF_I)^{-1} \, {}^tF_{I'}\right)$ depends only on σ; since $\sigma \in W_I$, which has compact closure, its operator norm of $\left(({}^tF_I)^{-1} \, {}^tF_{I'}\right)$ is bounded, and the estimate (4.42) follows.

Suppose then that $\Psi \in \mathcal{E}(G(d,n))$ such that, for each I, the restriction $\Psi|_{\Xi_I}$ belongs to $\mathcal{S}(\Xi_I)$. Since $\overline{W}_I \subset G_{d,n,I}$, the equation (4.41) and the estimate (4.42) shows that $\Psi \in \mathcal{S}(G(d,n))$.

Next we show that if Ψ is any C^∞ function on $G(d,n)_I$, then the derivatives $ED_\sigma(\Psi \circ \tau_I^{-1})(\sigma, x_{I'})$, for any differential operator D on $G(d,n)_I$, can be expressed as a finite sum in terms of the action of the universal enveloping algebra as follows:

$$\left\{ED_\sigma(\Psi \circ \tau_I^{-1})\right\} \circ \tau_I(\sigma, y) = \sum_\beta h_\beta(\sigma, y) \, d\nu(U_\beta)\Psi(\sigma, y) \tag{4.43}$$

Here $U_\beta \in U(\mathfrak{m}(n))$ and each h_β is a C^∞ function on $G(d,n)_I$, whose restriction restriction to each fiber $(\sigma, \cdot)$ is a polynomial of fixed degree k in y, with coefficients depending smoothly on σ.

Suppose now the $\Psi \in \mathcal{S}(G(d,n))$. We apply the conversion equation (4.43) to the restriction of Ψ on Ξ_I. Now W_I has compact closure in $G_{d,n,I}$, and so there exists a exists a constant R such that $|h_\beta(\sigma, y)| \leqslant R(1 + \|y\|)^k$ for all $(\sigma, y) \in \Xi_I$. This estimate on h shows that the restriction of Ψ to each Ξ_I belongs to $\mathcal{S}(\Xi_I)$.

Thus we need only verify the conversion equations (4.41) and (4.43). They can be derived by tedious but straightforward calculations. First, let us derive (4.41). To do this, we just need to derive it for $U \in \mathfrak{m}(n)$. If we express any $U \in U(\mathfrak{m}(n))$ as an algebraic combination of elements of $\mathfrak{m}(n)$, (4.41) will follow by iteration.

For simplicity, we again assume that $I = \{1, \ldots, d\}$ and $I' = \{d+1, \ldots, n\}$.

Let $U = v \in \mathbb{R}^n$. Note that

$$\tau_I^{-1}(\sigma, x_{I'}) = (\sigma, \begin{pmatrix} -\left({}^tF_I^{-1}\,{}^tF_{I'}\right) x_{I'} \\ x_{I'} \end{pmatrix}) \tag{4.44}$$

and if we write

$$v = \begin{pmatrix} v_I \\ v_{I'} \end{pmatrix}$$

the projection of v on $\sigma^\perp$ is

$$v - F\,({}^tF)v = \begin{pmatrix} v_I - F_I\,{}^tF_I v_I - F_I\,{}^tF_{I'} v_{I'} \\ v_{I'} - F_{I'}\,{}^tF_I v_I - F_{I'}\,{}^tF_{I'} v_{I'} \end{pmatrix}$$

It follows that

$$\begin{aligned}
&(d\nu(U)\Psi) \circ \tau_I^{-1}(\sigma, x_{I'}) \\
&\quad = \frac{d}{dt}\left(\Psi \circ \tau_I^{-1}\right)\left(\sigma, x_{I'} - t\left(v_{I'} - (F_{I'}\,{}^tF_I)v_I - (F_{I'}\,{}^tF_{I'})v_{I'}\right)\right)\Big|_{t=0}
\end{aligned}$$

Now both $F_{I'}\,{}^tF_I$ and $F_{I'}\,{}^tF_{I'}$ depend only on σ. It is easy to see that the right hand side above equals the expression

$$\sum_{j=d+1}^{n} a_j(\sigma)\frac{\partial}{\partial x_j}(\Psi \circ \tau_I^{-1})(\sigma, x_{I'}),$$

where the $a_{i'_j}$ are C^∞ functions of σ. The expression above is of course of the form (4.41).

Next let $U = X \in \mathrm{so}(n)$. Then

$$\begin{aligned}
&(d\nu(U)\Psi) \circ \tau_I^{-1}(\sigma, x_{I'}) \\
&\quad = \frac{d}{dt}\Psi(\exp(-tX)\cdot\sigma, \exp(-tX)\cdot \begin{pmatrix} -\left({}^tF_I^{-1}\,{}^tF_{I'}\right) x_{I'} \\ x_{I'} \end{pmatrix})\Big|_{t=0} \\
&\quad = \frac{d}{dt}(\Psi \circ \tau_I^{-1})(\exp(-tX)\cdot\sigma, x_{I'})\Big|_{t=0} \\
&\qquad + \frac{d}{dt}(\Psi \circ \tau_I^{-1})(\sigma, p_{I'}(\exp(-tX)\cdot \begin{pmatrix} -\left({}^tF_I^{-1}\,{}^tF_{I'}\right) x_{I'} \\ x_{I'} \end{pmatrix}))\Big|_{t=0}
\end{aligned}$$

where $p_{I'}$ denotes the projection $x \mapsto x_{I'}$ of $\mathbb{R}^n$ onto $\mathbb{R}^{n-d}$. The first term on the right is obviously of the form $D_\sigma(\Psi \circ \tau_I^{-1})(\sigma, x_{I'})$, where D is a differential operator acting on the first argument. The second term corresponds to a vector field acting on $x_{I'}$, with coefficients depending smoothly on σ and linearly on $x_{I'}$. Both terms can thus be seen to be of the form of the right hand side of (4.41).

To derive (4.43), we can again assume that $I = \{1, \ldots, d\}$. It is sufficient to consider the operators D and E separately, and in fact, to assume that these are C^∞ vector fields on $G_{d,n,I}$ and $\mathbb{R}^{n-d}$. The reason is that we can apply induction on the order of the operator $E\,D_\sigma$, keeping in mind that if $h_\beta(\sigma, y)$ is a polynomial in y with smooth coefficients depending on σ, then so is $d\nu(U)\,h(\sigma, y)$, for any $U \in U(\mathfrak{m}(n))$.

So let $\Psi \in \mathcal{E}(\Xi_I)$. Fix a vector $v \in \mathbb{R}^{n-d}$, and let $E = E_v$ be the corresponding directional derivative in $\mathbb{R}^{n-d}$. Let us again assume that F is any $n \times d$ Stiefel matrix whose columns form an orthonormal basis of σ. Then

$$\begin{aligned}
\left\{E(\Psi \circ \tau_I^{-1})\right\} \circ \tau_I(\sigma, y) &= \frac{d}{ds}\Psi(\sigma, y_{I'} + sv)\bigg|_{s=0} \\
&= \frac{d}{ds}\,\Psi(\sigma, \begin{pmatrix} {}^tF_I^{-1}\,F_{I'}(y_{I'} + sv) \\ y_{I'} + sv \end{pmatrix})\bigg|_{s=0} \\
&= \frac{d}{ds}\,\Psi(\sigma, y + s\begin{pmatrix} {}^tF_I^{-1}\,F_{I'}v \\ v \end{pmatrix})\bigg|_{s=0} \\
&= \frac{d}{ds}\,\Psi(\sigma, y + s\,\gamma_\sigma(v))\bigg|_{s=0} \qquad (4.45)
\end{aligned}$$

Now the orthogonal projection $P_{\sigma^\perp}$ from $\mathbb{R}^{n-d} = \{(0, \ldots, 0, x_{d+1}, \ldots, x_n) \,|\, x_j \in \mathbb{R}\}$ onto $\sigma^\perp$ is a bijection. If G is a Stiefel $n \times (n-d)$ matrix whose columns form an orthonormal basis of $\sigma^\perp$, then if $w \in \mathbb{R}^{n-d}$, we have

$$P_{\sigma^\perp}\begin{pmatrix} 0 \\ w \end{pmatrix} = G\,{}^tG\begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} G_I\,{}^tG_{I'}w \\ G_{I'}\,{}^tG_{I'}w \end{pmatrix} \qquad (4.46)$$

where we have written

$$G = \begin{pmatrix} G_I \\ G_{I'} \end{pmatrix},$$

where $G_{I'}$ is the lower $(n-d) \times (n-d)$ part of G. By our choice of I, $G_{I'}$ is nonsingular on $G_{d,n,I}$, so if we put

$$\gamma_\sigma(v) = P_{\sigma^\perp}\begin{pmatrix} 0 \\ w \end{pmatrix}$$

we obtain $w = (G_{I'}\,{}^tG_{I'})^{-1}\,v$. We can write the linear operator $(G_{I'}\,{}^tG_{I'})^{-1}$ on $\mathbb{R}^{n-d}$ as $A(\sigma)$, with the matrix coefficients depending smoothly on σ. Thus the right hand side of (4.45) equals

$$\frac{d}{ds}\Psi(\sigma, y + s\,P_{\sigma^\perp}(A(\sigma)w))\bigg|_{s=0} = d\nu(A(\sigma)w)\,\Psi(\sigma, y),$$

which is an expression of the form (4.43).

Next we verify (4.43) for $E = 1$ and D_σ a vector field on $G_{d,n,I}$. Such a vector field is a smooth linear combination of the vector fields induced by the infinitesimal $\mathrm{O}(n)$-action on $G_{d,n}$. Thus for fixed $X \in \mathrm{so}(n)$, it suffices to calculate

$$(D_\sigma(\Psi \circ \tau_I^{-1})) \circ \tau_I(\sigma, y) = \frac{d}{ds}(\psi \circ \tau_I^{-1})(\exp(sX)\cdot\sigma, y_{I'})\Big|_{s=0}$$

for $(\sigma, y) \in \Xi_I$. By (4.44), the right hand side above equals

$$\frac{d}{ds}\Psi(\exp(sX)\cdot\sigma, \begin{pmatrix} -\,{}^t(\exp(sX)F)_I^{-1}\,{}^t(\exp(sX)F)_{I'}\,y_{I'} \\ y_{I'} \end{pmatrix})\Big|_{s=0} \tag{4.47}$$

Put

$$A(\sigma, s)\,y = \begin{pmatrix} -\,{}^t(\exp(sX)F)_I^{-1}\,{}^t(\exp(sX)F)_{I'}\,y_{I'} \\ y_{I'} \end{pmatrix}$$

and $B(\sigma, s)y = \exp(-sX)\,A(\sigma, s)y$. Since $A(\sigma, s)y \in (\exp(sX)\cdot\sigma)^\perp$, we have $B(\sigma, s)y \in \sigma^\perp$. Moreover, $B(\sigma, 0)y = y$. Thus (4.47) equals

$$\begin{aligned}\frac{d}{ds}\Psi(\exp(sX)\cdot\sigma, \exp(sX)\cdot B(\sigma, s)y)\Big|_{s=0}& \\ = \frac{d}{ds}\psi(\exp(sX)\cdot\sigma, \exp(sX)\cdot y)\Big|_{s=0} &+ \frac{d}{ds}\Psi(\sigma, B(\sigma, s)y)\Big|_{s=0}\end{aligned}$$

The first term on the right above is just $-d\nu(X)\Psi(\sigma, y)$, which is certainly of the form (4.43). By use of (4.46), we can write the second term as

$$\frac{d}{ds}\Psi(\sigma, P_{\sigma^\perp}(C(\sigma, s)\,y))\Big|_{s=0} \tag{4.48}$$

where $C(\sigma, s)y \in \mathbb{R}^{n-d}$. Now $C(\sigma, s)y$ depends smoothly on σ and s, and linearly on y. Thus its derivative with respect to s at $s = 0$ is a vector in $\mathbb{R}^{n-d}$ of the form $\sum_j C_j(\sigma)(y)\,e_j$, where each $C_j(\sigma)(y)$ is a linear function of y, with coefficients depending smoothly on σ. Thus (4.48) equals

$$-\sum_j C_j(\sigma)(y)d\nu(e_j)\Psi(\sigma, y),$$

which is also of the form (4.43).

This finishes the proof of Lemma 4.2.6, and completes the proof of Theorem 4.2.2.

We can also formulate the range theorem for the d-plane transform in terms of a single fourth order $\mathrm{M}(n)$-invariant differential equation. Consider the element

$$D = \sum_{1\leqslant i<j<l\leqslant n} V_{ijl}^2 \tag{4.49}$$

of $U(\mathfrak{m}(n))$. We claim that D is an $\mathrm{Ad}\,(\mathrm{M}(n))$-invariant element of $U(\mathfrak{m}(n))$. From this it will follow that $d\nu(D)$ is an $\mathrm{M}(n)$-invariant differential operator on $G(d,n)$.

It is clear from Lemma 4.2.3 that $\mathrm{Ad}\,(v)D = D$ for all $v \in \mathbb{R}^n$. Thus to prove the claim, we just need to show that $\mathrm{Ad}\,(k)\,D = D$ for all $k \in \mathrm{O}(n)$. Now let W be the subspace of $U(\mathfrak{m}(n))$ spanned by the V_{ijl} and let $\widetilde{W} \subset U(\mathfrak{m}(n))$ the subalgebra generated by W. Let T be the (well-defined) linear map of $V = \Lambda^3\mathbb{R}^n$ onto W given by $T(e_i \wedge e_j \wedge v_l) = V_{ijl}$ $(1 \leqslant i,j,l \leqslant n)$. Next let $\Psi = \Lambda^3 k : \Lambda^3\mathbb{R}^n \to \Lambda^3\mathbb{R}^n$ be the orthogonal transformation on $V = \Lambda^3\mathbb{R}^n$ induced by k. Then by (4.18),

$$T \circ \Psi = \mathrm{Ad}(k) \circ T \tag{4.50}$$

Extend T to a homomorphism of the tensor algebra $\otimes V$ onto $\widetilde{W}$, and extend Ψ to an algebra homomorphism on $\otimes V$. Then (4.50) also holds for $T : \otimes V \to \widetilde{W}$. Letting $w_{ijl} = e_i \wedge e_j \wedge e_l \in \Lambda^3\mathbb{R}^n$, we have

$$T\Bigg(\sum_{j<k<l<m} w_{ijl} \otimes w_{ijl}\Bigg) = \sum_{i<j<l} V_{ijl}^2.$$

Since Ψ is an orthogonal transformation on V, the sum $\sum w_{ijl} \otimes w_{ijl}$ is Ψ-invariant in $\otimes V$. Hence by (4.50), $\sum V_{ijl}^2$ is $\mathrm{Ad}(k)$-invariant in $\widetilde{W}$.

Since $G(d,n) = \mathrm{M}(n)/(\mathrm{M}(d) \times \mathrm{O}(n-d))$ is the quotient of unimodular groups, there exists a measure μ on $G(d,n)$, unique up to constant multiple and invariant under the action of $\mathrm{M}(n)$. We fix the measure μ to satisfy

$$\int_{G(d,n)} \varphi(\xi)\, d\mu(\xi) = \int_{G_{d,n}} \int_{\sigma^\perp} \varphi(\sigma, x)\, d\sigma^\perp(x) d\sigma, \qquad (\varphi \in \mathcal{D}(G(d,n))) \tag{4.51}$$

Then by the formal duality (1.11) of R and R^*, or by direct computation,

$$\int_{G(d,n)} Rf(\xi)\, \varphi(\xi)\, d\mu(\xi) = \int_{\mathbb{R}^n} f(x)\, R^*\varphi(x)\, dx \tag{4.52}$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \in \mathcal{E}(G(d,n))$. If D is a differential operator on $G(d,n)$, the adjoint operator with respect to μ will be denoted D^*. Since μ is preserved under the $\mathrm{M}(n)$-action, we have $d\nu(X)^* = -d\nu(X)$ for all $X \in \mathfrak{m}(n)$. From this it follows that $d\nu(V_{ijl})^* = d\nu(V_{ijl})$.

Let $\varphi \in \mathcal{S}(G(d,n))$. If φ satisfies $d\nu(V_{ijl})\varphi = 0$, then of course $d\nu(D)\varphi = 0$. Conversely, if $d\nu(D)\varphi = 0$, then

$$\begin{aligned} 0 &= \sum_{i<j<l} \int_{G(d,n)} \left(d\nu(V_{ijl}^2\varphi)\,(\xi)\right) \left(\overline{\varphi(\xi)}\right) d\mu(\xi) \\ &= \sum_{i<j<l} \int_{G(d,n)} \left|(d\nu(V_{ijl})\varphi)(\xi)\right|^2\, d\mu(\xi) \end{aligned}$$

This implies that $d\nu(V_{ijl})\varphi = 0$. Thus we have proved that

$$\varphi \in \mathcal{S}_D(G(d,n)) \iff d\nu(D)\,\varphi = 0$$

It follows from Theorem 4.2.2 that the range of R consists precisely of the functions annihilated by the single fourth order operator $d\nu(D)$:

Theorem 4.2.8.

$$R\mathcal{S}(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(G(d,n)) \,|\, d\nu(D)\varphi = 0\}.$$

Theorem 4.2.8 has a natural generalization to Radon transforms on affine Grassmannians. (See [11].)

4.3 The Range of the Dual d-plane Transform

In Theorem 2.5.8 we proved that if R^* is the classical dual transform, then $R^*\mathcal{E}(G(n-1,n)) = \mathcal{E}(\mathbb{R}^n)$. Our objective in this section is to generalize this result to the dual d-plane transform. Thus we wish to prove:

Theorem 4.3.1. *Let R^* be the dual d-plane transform. Then*

$$R^*\mathcal{E}(G(d,n)) = \mathcal{E}(\mathbb{R}^n).$$

The proof of Theorem 4.3.1 is easier when d is even, since R^*R is inverted by a power of the Laplacian. To prove the theorem for all d, we will need to invoke a few results on Riesz potentials from Ortner's paper [32], which we will mention along the course of the proof.

For each $\sigma \in G_{d,n}$, let $L_{\sigma^\perp}$ denote the Laplacian on the fiber $\sigma^\perp$. Define the operator $\square$ on $G(d,n)$ by

$$(\square\varphi)|_{\sigma^\perp} = L_{\sigma^\perp}\left(\varphi|_{\sigma^\perp}\right)$$

for $\varphi \in \mathcal{E}(G(d,n))$, where $\varphi|_{\sigma^\perp}$ denotes the restriction of φ on $\sigma^\perp$.

Then it is easy to check directly that

$$R(Lf)(\sigma,x) = \square(Rf)(\sigma,x) \tag{4.53}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. For instance, if we put $U = \sum_{j=1}^n e_j^2 \in U(\mathfrak{m}(n))$, then the relation above follows from (4.16) and the fact that $L = d\lambda(U)$, $\square = d\nu(U)$.

We note that is easy to check that U is $\mathrm{Ad}\,\mathrm{M}(n)$-invariant. This implies that $\square$ is an $\mathrm{M}(n)$-invariant differential operator on $G(d,n)$.

The paper [10] shows that the algebra $\mathbb{D}(G(d,n))$ of $\mathrm{M}(n)$-invariant differential operators on $G(d,n)$ has $l = \min(d+1, n-d)$ algebraically independent generators, of orders 2, 4, ... The operator $\Box$ and the range-characterizing operator $D = \sum V_{ijl}^2$ defined in (4.49) turn out to be the generators of order 2 and 4, respectively.

Now by Propositions 1.2.3, 1.2.4 and the remark after, we know that the d-plane transform can be extended to distributions. More precisely, the basic duality of R and R^* gives us continuous map $R : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(G(d,n))$ given by

$$RT(\varphi) = T(R^*\varphi) \qquad\qquad (\varphi \in \mathcal{E}(G(d,n)))$$

and $R^* : \mathcal{D}'(G(d,n)) \to \mathcal{D}'(\mathbb{R}^n)$ given by

$$(R^*\Psi)(f) = \Psi(Rf) \qquad\qquad (f \in \mathcal{D}(\mathbb{R}^n))$$

We now define a "fiber Riesz potential" on $G(d,n)$ as follows. Suppose that $\varphi \in \mathcal{S}(G(d,n))$. In accordance with (3.21), the fractional power $(-\Box)^k\varphi$ is defined by

$$\begin{aligned}(-\Box)^k\varphi(\sigma,x) &= I^{-2k}\varphi(\sigma,x)\\ &= H_{n-d}(-2k)\int_{\sigma^\perp}\varphi(\sigma,y)\,\|x-y\|^{-2k-(n-d)}\,d\sigma^\perp(y) \qquad (4.54)\end{aligned}$$

interpreted for $k \geqslant 0$ by analytic continuation. Here the constant $H_{n-d}(-2k)$ is given by (3.22).

Suppose that $\mathcal{O}$ is an open subset of $G_{d,n}$ which admits a local cross section $\sigma \mapsto \eta(\sigma)$ of $\mathrm{O}(n)$, so that $\eta(\sigma)\cdot\sigma_0 = \sigma$. If we identify $\sigma_0^\perp$ with $\mathbb{R}^{n-d}$, then we obtain a local trivialization of the vector bundle $G(d,n)$ over $\mathcal{O}$:

$$\begin{aligned}\Gamma : \mathcal{O}\times\mathbb{R}^{n-d} &\to \pi^{-1}(\mathcal{O})\\ (\sigma,x) &\mapsto (\sigma,\eta(\sigma)\cdot x) \qquad (4.55)\end{aligned}$$

The Riesz potential (4.54) can thus be written

$$(-\Box)^k\varphi(\Gamma(\sigma,x)) = H_{n-d}(-2k)\int_{\mathbb{R}^{n-d}}\varphi\circ\Gamma(\sigma,u)\,\|x-u\|^{-2k-(n-d)}\,du$$

Thus, under the parametrization Γ, the operator $(-\Box)^k$ is given by convolution with a tempered distribution in $\mathbb{R}^{n-d}$. Hence $(-\Box)^k\varphi \in \mathcal{E}(G(d,n))$ and the map $\varphi \mapsto (-\Box)^k\varphi$ is continuous from $\mathcal{S}(G(d,n))$ to $\mathcal{E}(G(d,n))$.

Lemma 4.3.2. *Let $\varphi \in \mathcal{S}(G(d,n))$ and $k \geqslant 0$. Then there exists a constant C such that*

$$\left|(-\Box)^k\varphi(\sigma,x)\right| \leqslant C\,(1+\|x\|)^{-2k-(n-d)}$$

for all $(\sigma,x) \in G(d,n)$.

This lemma is a consequence of Lemma 1 of [32], which generalizes the estimate (3.28).

From the lemma, it follows that the partial Fourier transform $\mathcal{F}_d((-\Box)^k\varphi)$ is therefore given by an absolutely convergent integral for $\varphi \in \mathcal{S}(G(d,n))$. By (3.26) it satisfies

$$\mathcal{F}_d((-\Box)^k\varphi)(\sigma,u) = \|u\|^{2k}\,(\mathcal{F}_d\varphi)(\sigma,u) \tag{4.56}$$

Lemma 4.3.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for any $k \geqslant 0$,*

$$R((-L)^k f) = (-\Box)^k(Rf) \tag{4.57}$$

Proof. If $k \in \mathbb{Z}^+$, this follows from (4.53), so we assume $k \notin \mathbb{Z}^+$. By Lemma 1 of [32], $(-L)^k f(x) = \mathrm{O}(\|x\|^{-n-2k})$ as $\|x\| \to \infty$, so $R((-L)^k f)$ is well-defined. The relation 4.57 follows, since the partial Fourier transform of both sides is $\|u\|^{2k}\,\widetilde{f}(u)$. $\square$

The lemma above allows us to express the inversion formula (3.41) for the d-plane transform in the following way. If $f \in \mathcal{S}(\mathbb{R}^n)$, then by (3.40) and (3.43), we have

$$\begin{aligned}
f(x) &= I^d(I^{-d}f)(x)\\
&= H_n(d)\int_{\mathbb{R}^n}(I^{-d}f)(x-y)\,\|y\|^{d-n}\,dy\\
&= \frac{\Omega_n}{\Omega_d}\,H_n(d)\,R^*R(I^{-d}f)(x)\\
&= \frac{1}{c_{d,n}}\,R^*R((-L)^{\frac{d}{2}}f)(x)\\
&= \frac{1}{c_{d,n}}\,R^*((-\Box)^{\frac{d}{2}}Rf)(x)
\end{aligned} \tag{4.58}$$

where $c_{d,n}$ is the constant $\Omega_d(H_n(d)\,\Omega_n)^{-1}$.

Since $(-\Box)^k$ is a radial convolution operator (or an integral power of the Laplacian) on the fibers of $G(d,n)$, it is easy to see from (4.51) that

$$\int_{G(d,n)}(-\Box)^k\,\varphi(\xi)\,\psi(\xi)\,d\mu(\xi) = \int_{G(d,n)}\varphi(\xi)\,(-\Box)^k\psi(\xi)\,d\mu(\xi) \tag{4.59}$$

for all $k \geqslant 0$ and all φ, $\psi \in \mathcal{D}(G(d,n))$. Thus for any $T \in \mathcal{E}'(G(d,n))$, we can define the distribution $(-\Box)^kT \in \mathcal{D}'(G(d,n))$ by the formula $((-\Box)^kT)(\varphi) = T((-\Box)^k\varphi)$, for any $\varphi \in \mathcal{D}(G(d,n))$. Since the map $\varphi \mapsto (-\Box)^k\varphi$ is continuous from $\mathcal{D}(G(d,n))$ to $\mathcal{E}(G(d,n))$, the adjoint map $T \mapsto (-\Box)^kT$ is continuous from $\mathcal{E}'(G(d,n))$ to $\mathcal{D}'(G(d,n))$.

Lemma 4.3.4. *For any $g \in M(n)$, $\varphi \in \mathcal{D}(G(d,n))$, and $T \in \mathcal{E}'(G(d,n))$, we have*

$$\nu(g)\,((-\Box)^k \varphi) = (-\Box)^k (\nu(g)\,\varphi) \tag{4.60}$$

and

$$\nu(g)\,((-\Box)^k T) = (-\Box)^k (\nu(g)\,T) \tag{4.61}$$

Proof. If $g = k \in \mathrm{O}(n)$, then (4.60) is obvious from the definition of $(-\Box)^k$. If $g \in \mathbb{R}^n$, then (4.60) will follow if we take the partial Fourier transform of both sides and use Lemma 4.2.1 (ii) and (4.56). (4.61) then follows by applying both sides to a test function $\varphi \in \mathcal{D}(G(d,n))$. ☐

Let D be the element $\sum_{i<j<l} V_{ijl}^2 \in U(\mathfrak{m}(n))$. The following lemma is a distribution version of Theorem 4.2.8.

Lemma 4.3.5.

$$R\mathcal{E}'(\mathbb{R}^n) = \{\Psi \in \mathcal{E}'(G(d,n)) \,|\, d\nu(D)\,\Psi = 0\}$$

Proof. We use an approximation argument as follows. Let $\{\phi_m\}_{m=1}^{\infty}$ be an approximate identity in $\mathrm{M}(n)$. If T and Ψ are distributions on $\mathbb{R}^n$ and $G(d,n)$, respectively, we define the regularizations $\lambda(\phi_m)\,T$ and $\nu(\phi_m)\,\Psi$ by (1.37). In particular, if $T \in \mathcal{E}'(\mathbb{R}^n)$, then $\lambda(\phi_m)\,T \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda(\phi_m)\,T \to T$ weakly in $\mathcal{E}'(\mathbb{R}^n)$ as $m \to \infty$. Since $R : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(G(d,n))$ is continuous (with respect to either the weak and strong topologies), we obtain

$$\begin{aligned} d\nu(D)\,(RT) &= \lim_{m\to\infty} d\nu(D) R(\nu(\phi_m)\,T) \\ &= 0 \end{aligned}$$

where we have used (4.17).

On the other hand, suppose that $\Psi \in \mathcal{E}'(G(d,n))$ satisfies $d\nu(D)\,\Psi = 0$. Then Ψ is the weak limit of $\nu(\phi_m)\,\Psi$ in $\mathcal{E}'(G(d,n))$, and by the $\mathrm{M}(n)$-invariance of $d\nu(D)$, we obtain

$$\begin{aligned} d\nu(D)\,\nu(\phi_m)\,\Psi &= \nu(\phi_m)\,d\nu(D)\,\Psi \\ &= 0. \end{aligned}$$

Thus by Theorem 4.2.8 and the support theorem for the d-plane transform (which follows easily from the classical support theorem), we conclude that $\nu(\phi_m)\,\Psi = Rf_m$, for some $f_m \in \mathcal{D}(\mathbb{R}^n)$. From the inversion formula (4.58) we have

$$c_{d,n}\, f_m = R^*((-\Box)^{\frac{d}{2}}(\nu(\phi_m)\,\Psi)) \tag{4.62}$$

Now by Lemma 4.3.4, $(-\Box)^{\frac{d}{2}}(\nu(\phi_m)\,\Psi) = \nu(\phi_m)((-\Box)^{\frac{d}{2}}\,\Psi)$, which converges to $(-\Box)^{\frac{d}{2}}\,\Psi$ in $\mathcal{D}'(G(d,n))$. Since R^* is continuous, f_m converges to the distribution $T = c_{d,n}^{-1}\,R^*((-\Box)^{\frac{d}{2}}\,\Psi)$ in $\mathcal{D}'(\mathbb{R}^n)$.

Now the functions $\nu(\phi_m)\,\Psi$ are all supported on a common compact subset of $G(d,n)$, so by the support theorem, the functions f_m are all supported in a common compact subset of $\mathbb{R}^n$. Thus $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\{f_m\}$ converges to T in the (weak) topology of $\mathcal{E}'(\mathbb{R}^n)$. Since R is continuous from $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{E}'(G(d,n))$, it follows that

$$\Psi = \lim_{m\to\infty} \nu(\phi_m)\,\Psi = \lim_{m\to\infty} Rf_m = RT$$

the convergence being in $\mathcal{E}'(G(d,n))$. This completes the proof of the lemma. $\Box$

Lemma 4.3.5 shows that the range $R\mathcal{E}'(\mathbb{R}^n)$ is a closed subset of $\mathcal{E}'(G(d,n))$. It is not hard to show, using approximate identities in $\mathbb{R}^n$ for example, that R is injective on $\mathcal{E}'(\mathbb{R}^n)$. From this, we see that Theorem 4.3.1 is a consequence of Theorem 1.2.5.

Chapter 5

Radon Transforms on Spheres

5.1 The Duality

In this chapter, we study the generalization of Funk's transform to the n-dimensional sphere S^n. The problem is to invert, and find the range of, the Radon transform which integrates a function $f \in \mathcal{E}(S^n)$ over all d-dimensional great spheres. (We could assume that f is merely continuous, but we get a richer set of results by assuming smoothness.) Now the integral of an odd function on S^n over any great sphere is zero. Thus we can recover only the even part of a function from its great sphere integrals. We will therefore assume from the outset that our functions are even.

We endow S^n with the Riemannian metric inherited from $\mathbb{R}^{n+1}$. If we consider the elements of $\mathbb{R}^{n+1}$ as column vectors

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

the matrix multiplication $(k, x) \mapsto k \cdot x$ is a transitive action of the orthogonal group $\mathrm{O}(n+1)$ on S^n. Each $k \in \mathrm{O}(n+1)$ acts as an isometry on S^n, and the isotropy subgroup K of $\mathrm{O}(n+1)$ at the north pole e_0 is the subgroup consisting of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} \qquad (k' \in \mathrm{O}(n))$$

which we identify with $\mathrm{O}(n)$. Thus $S^n = \mathrm{O}(n+1)/\mathrm{O}(n)$.

Since we'll be integrating even functions on S^n, we can just as well work with functions on real projective space $\mathbb{RP}^n$, but we will stick with S^n.

Now any d-dimensional great sphere in S^n is the intersection of S^n with a unique $(d+1)$-dimensional subspace of $\mathbb{R}^n$, so the space of such great spheres is the Grassmannian $G_{d+1,n+1}$. This is acted upon transitively by $\mathrm{O}(n+1)$, and the isotropy subgroup Q at the subspace $\sigma_0 = \mathbb{R}e_0 + \cdots + \mathbb{R}e_d$ consists of the matrices

$$\begin{pmatrix} k' & 0 \\ 0 & k'' \end{pmatrix} \qquad (k \in \mathrm{O}(d+1),\ k' \in \mathrm{O}(n-d)) \tag{5.1}$$

which we identify with $\mathrm{O}(d+1) \times \mathrm{O}(n-d)$. Thus $G_{d+1,n+1} = \mathrm{O}(n+1)/(\mathrm{O}(d+1) \times \mathrm{O}(n-d)$.

Our Radon transform should therefore correspond to the double fibration

$$\begin{tikzcd} & \mathrm{O}(n+1)/(K \cap Q) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ S^n & & G_{d+1,n+1} \end{tikzcd} \tag{5.2}$$

where $K \cap Q = \mathrm{O}(d) \times \mathrm{O}(n-d)$.

Exercise 5.1.1. Prove that $x = gK \in S^n$ and $\sigma = \gamma H \in G_{d+1,n+1}$ are incident if and only if x lies in σ.

The orbit $\widehat{\sigma}_0 = Q \cdot e_0$ is the d-dimensional great sphere $\sigma_0 \cap S^n$. Since the elements of H act as isometries on S^n, the Riemannian measure on this great sphere is invariant under the action of H. By left translation by an appropriate element of $\mathrm{O}(n)$, it is clear that if $\sigma \in G_{d+1,n+1}$, then $\widehat{\sigma} = \sigma \cap S^n$ and that the Riemannian measure on $\widehat{\sigma}$ is invariant under the subgroup of $\mathrm{O}(n)$ fixing σ. Thus the Radon transform associated with the double fibration (5.2) integrates any function $f \in C(S^n)$ over d-dimensional great spheres, with respect to the Riemannian measure on those spheres.

For convenience we put $G_{d+1,n+1} = \Xi_d$ and view the elements of Ξ_d as the d-dimensional great spheres in S^n.

If $x = g \cdot e_0 \in S^n$, then the orbit $\check{x} = g\,\mathrm{O}(n) \cdot \sigma_0$ consists of all d-dimensional great spheres containing x. It is clear that the mapping $x \mapsto \check{x}$ is two to one, with $(-x)^\vee = \check{x}$.

Let f be an even continuous function on S^n. Then its Radon transform is given by

$$Rf(\sigma) = \int_\sigma f(x)\,dm(x) \qquad (\sigma \in \Xi_d) \tag{5.3}$$

where $dm(x)$ denotes the Riemannian measure on the great d-sphere σ. Thus when $d = 1$, $dm(x)$ is just the arc length on the great circle σ. If $\varphi \in C(\Xi_d)$, then its dual transform at $x \in S^n$ is

$$R^*\varphi(x) = \int_{\sigma \supset x} \varphi(\sigma)\, d\mu(\xi) \tag{5.4}$$

where $d\mu(\sigma)$ is the normalized measure on the set of all $\sigma \in \Xi_d$ containing x invariant under the subgroup K_x of $\mathrm{O}(n)$ fixing x. If $x = g \cdot e_0$, we have

$$R^*\varphi(x) = \int_K \varphi(gk \cdot \sigma_0)\, dk \tag{5.5}$$

where dk is the normalized Haar measure on $\mathrm{O}(n)$.

5.2 The Laplace-Beltrami Operator on S^n

Let us first recall some general facts about the Laplace-Beltrami operator on a Riemannian manifold M. For details, see, for example Helgason's book *Groups and Geometric Analysis* [17], Chapter II, Section 2.

Let g be the Riemannian metric tensor on M. If $f \in \mathcal{E}(M)$, then its *gradient* is the unique smooth vector field $\operatorname{grad} f$ on M such that

$$g(\operatorname{grad} f, Y) = Yf \tag{5.6}$$

for any smooth vector field on M.

Next, let ∇ be the Riemannian connection corresponding to g. ∇ is the unique affine connection on M such that (i) the torsion tensor is zero:

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{5.7}$$

for all smooth vector fields X and Y in M, and (ii) g is invariant under parallel translations, which amounts to

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{5.8}$$

for all smooth vector fields X, Y, Z on M.

We arrive at ∇ as follows. Assuming (i) and (ii) for the moment, permute X, Y and Z cyclically in (5.8):

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{5.9}$$
$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \tag{5.10}$$

subtract (5.10) from the sum of (5.8) and (5.9) and use (5.7) to obtain

$$\begin{aligned} 2\, g(\nabla_Z X, Y) = {} & X(g(Y,Z)) - Y(g(Z,X)) + Z(g(X,Y)) \\ & - g([X,Y],Z) + g([Y,Z],X) + g([Z,X],Y) \end{aligned} \tag{5.11}$$

So define $\nabla_Z X$ by (5.11). By direct computation, it can be checked that ∇ is an affine connection on M satisfying properties (i) and (ii); (5.11) itself, which follows from (i) and (ii), guarantees the uniqueness of ∇.

The relation $\nabla_{fY} X = f\,\nabla_Y X$ for all $f \in \mathcal{E}(M)$ and all smooth vector fields X and Y on M shows that, for fixed X, $(\nabla_Y X)(p)$ depends only on the tangent vector Y_p.

If X is any smooth vector field on M, then its *divergence* $\operatorname{div} X$ is the function on M given at $p \in M$ by

$$\operatorname{div} X(p) = \operatorname{trace}(v \mapsto \nabla_v X) \qquad (v \in T_p M) \tag{5.12}$$

where T_pM denotes the tangent space to M at p.

The gradient and the divergence have the following local expressions. Let $(U, x_1, \dots, x_n)$ be a coordinate chart in M. If $p \in U$, then T_pM has basis $\{\partial/\partial x_j\}_{j=1}^n$. Put $g_{ij}(p) = g_p(\partial/\partial x_i, \partial/\partial x_j)$. The $n \times n$ symmetric matrix $(g_{ij}(p))$ is positive definite, and we put $\overline{g}(p) = \det(g_{ij}(p))$, $(g^{ij}(p)) = (g_{ij}(p))^{-1}$. It is easy to check that on U

$$\operatorname{grad} f = \sum_{1 \leqslant i,j \leqslant n} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \tag{5.13}$$

by noting that the inner product of both sides with $\partial/\partial x_k$ is $\partial f/\partial x_k$.

The *Christoffel symbols* Γ_{ij}^k are the smooth functions on U defined by

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tag{5.14}$$

From (5.7), we have $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k, and (5.11) gives

$$\frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} + \frac{\partial g_{lj}}{\partial x_k} = 2 \sum_r \Gamma_{jk}^r\, g_{rl} \tag{5.15}$$

Let X be a smooth vector field on U, and write $X = \sum_{i=1}^n f_i\, \partial/\partial x_i$. Then by (5.12) and (5.14),

$$\operatorname{div} X = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i,j} f_j \Gamma_{ij}^i \tag{5.16}$$

Proposition 5.2.1. *If $X = \sum_i f_i\, \partial/\partial x_i$, then*

$$\mathrm{div\,X} = \frac{1}{\sqrt{\mathrm{g}}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\partial}{\partial \mathrm{x_i}} (\sqrt{\mathrm{g}}\, \mathrm{f_i}) \tag{5.17}$$

Proof. The right hand side of (5.17) equals

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \frac{1}{2\overline{g}} \sum_{i=1}^{n} f_i \frac{\partial \overline{g}}{\partial x_i}$$

Thus by (5.17) we just need to prove that

$$\frac{1}{2\overline{g}} \frac{\partial \overline{g}}{\partial x_j} = \sum_{i=1}^{n} \Gamma^i_{ij} \tag{5.18}$$

Now according to Exercise 5.2.2 below,

$$\frac{1}{\overline{g}} \frac{\partial \overline{g}}{\partial x_j} = \sum_{k,l} g^{kl} \frac{\partial g_{kl}}{\partial x_j} \tag{5.19}$$

To relate the right hand side above to (5.18), we make use of (5.15). In particular, we replace the indices j, k, l in (5.15) by l, j, k, respectively, we obtain

$$\frac{\partial g_{jk}}{\partial x_l} - \frac{\partial g_{lj}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} = 2 \sum_r \Gamma^r_{lj}\, g_{rk}$$

Adding this to (5.15) gives

$$\frac{\partial g_{kl}}{\partial x_j} = \sum_r \left(\Gamma^r_{jk}\, g_{rl} + \Gamma^r_{lj}\, g_{rk}\right)$$

which in turn yields

$$\sum_{k,l} g^{kl} \frac{\partial g_{kl}}{\partial x_j} = \sum_k \Gamma^k_{jk} + \sum_l \Gamma^l_{lj} = 2 \sum_i \Gamma^i_{ij}$$

Using (5.19), this proves (5.18). $\square$

Exercise 5.2.2. Prove Equation (5.19).

The *Laplace-Beltrami* operator in M is the second order differential operator

$$f \in \mathcal{E}(M) \mapsto Lf = \operatorname{div}(\operatorname{grad} f) \tag{5.20}$$

On a chart $(U, x_1, \ldots, x_n)$, (5.13) and (5.17) shows that the Laplacian is given by

$$Lf = \frac{1}{\sqrt{\overline{g}}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{\overline{g}}\, g^{ij} \frac{\partial f}{\partial x_j} \right) \tag{5.21}$$

For example, if $M = \mathbb{R}^n$, then $g_{ij} = \delta_{ij}$ and Lf is the usual Laplacian of f.

If φ is a diffeomorphism of M, the push-forward of the affine connection ∇ by φ is the affine connection ∇^φ on M given by $\nabla^\varphi_X Y = (\nabla_{X^{\varphi^{-1}}} Y^{\varphi^{-1}})^\varphi$, where

$X^{\varphi^{-1}} = (\varphi^{-1})_* X$. If φ is an isometry, then (5.7), (5.8), and the uniqueness of ∇ show that $\nabla^\varphi = \nabla$. Then (5.12) shows that $\operatorname{div}(X^\varphi) = (\operatorname{div} X)^\varphi$. In addition, (5.6) shows that $\operatorname{grad} f^\varphi = (\operatorname{grad} f)^\varphi$. From this it follows that the Laplace-Beltrami operator is invariant under any isometry φ of M:

$$L(f^\varphi) = (Lf)^\varphi$$

It turns out that any diffeomorphism of M preserving the Laplacian is in fact an isometry; see [19], Chapter II.

Fix $p \in M$. From general theory, we know that the Exponential map is a diffeomorphism of a neighborhood V of 0 in T_pM onto a neighborhood U of p in M. We can assume that V is a ball $\|v\| < R$, with R sufficiently small, so that $U = \{q \in M \mid d(p,q) < R\}$. Fix any orthonormal basis $v_1, \ldots, v_n$ of T_pM and consider the isometry $\eta : e_j \mapsto v_j$ of $\mathbb{R}^n$ onto T_pM. Then the map $\psi : (r, \omega) \to \operatorname{Exp}(\eta(r\omega))$ is a diffeomorphism of $(0, R) \times S^{n-1}$ onto $U \backslash \{p\}$. If $q \in U \backslash \{p\}$, we call $\psi^{-1}(q) = (r, \omega)$ the geodesic polar "coordinates" of q.

It is a well-known fact that if R is small and if v is any unit vector in T_pM, then the geodesic $t \mapsto \operatorname{Exp}(tv)$, for $0 < t < R$, intersects the sphere $S_r(p)$, for $0 < r < R$, at a right angle. Hence if we choose local coordinates $\theta_1, \ldots, \theta_{n-1}$ on S^n, then the Riemannian metric tensor with respect to the coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ on U is of the form

$$dr^2 + \sum_{i,j} g_{ij}(r, \theta_1, \ldots, \theta_{n-1})\, d\theta_i\, d\theta_j$$

Let us now consider the case when $M = S^n$. If $p \in S^n$, then the geodesics through p are the great circles passing through p, and if $0 < r < \pi$, the sphere $S_r(p) = \{\omega \in S^n \mid d(p, \omega) = r\}$ is the $(n-1)$-dimensional sphere of constant latitude $\langle p, \omega \rangle = \cos r$. The Riemannian metric tensor in geodesic polar coordinates at p is thus of the form

$$g = dr^2 + \sin^2 r \sum_{i,j} h_{ij}(\theta_1, \ldots, \theta_{n-1})\, d\theta_i\, d\theta_j$$

Thus, if $f \in \mathcal{E}(S^{n-1})$ is constant on the spheres $S_r(p)$, then by (5.21),

$$Lf = \left(\frac{\partial^2}{\partial r^2} + (n-1) \cot r \frac{\partial}{\partial r} \right) f \tag{5.22}$$

The differential operator on the right hand side above, which we denote by $\Delta(L)$, is called the *radial part* of L:

$$\Delta(L) = \frac{\partial^2}{\partial r^2} + (n-1) \cot r \frac{\partial}{\partial r} \qquad (0 < r < \pi) \tag{5.23}$$

In the next section we will see that it figures prominently in the inversion of the Radon transform on S^n.

Now since the elements of $\mathrm{O}(n+1)$ act as isometries (specifically rotations and reflections) on $\mathbb{R}^n$, and since $\mathrm{O}(n+1)$ preserves S^n, the elements of $\mathrm{O}(n+1)$ are also isometries on the submanifold S^n. It turns out that any isometry σ of S^n is given by $\omega \mapsto k \cdot \omega$ for some $k \in \mathrm{O}(n+1)$. We shall skip the proof of this, which is easy, but we won't need it later. We shall also skip the proof of the following result, which may be found in [17], Chapter II.

Proposition 5.2.3. *Let D be a differential operator on S^n, invariant under all elements $\sigma \in O(n+1)$. Then D is a polynomial in the Laplace-Beltrami operator:*

$$D = a_m L^m + a_{m-1} L^{m-1} + \cdots + a_1 L + a_0$$

where the a_j are complex constants.

On S^n, the Laplacian is a differential operator coming from the infinitesimal left action of an invariant element in the universal enveloping algebra of so(n).

For this we temporarily generalize a bit and consider a semisimple (and not necessarily compact) Lie group G with Lie algebra $\mathfrak{g}$. Then by definition, the *Killing form* on $\mathfrak{g}$, given by

$$B(X,Y) = \mathrm{tr}\,(\mathrm{ad}X \circ \mathrm{ad}Y) \qquad\qquad (X,Y \in \mathfrak{g})$$

is nondegenerate. Fix a basis $X_1, \ldots, X_m$ of $\mathfrak{g}$, put $b_{ij} = B(X_i, X_j)$, and let $(b^{ij}) = (b_{ij})^{-1}$. The *Casimir operator* of $\mathfrak{g}$ is the element

$$\Omega = \sum_{i,j} b^{ij} X_i X_j$$

of $U(\mathfrak{g})$.

Exercise 5.2.4. Show that Ω is independent of the choice of basis of $\mathfrak{g}$.

The Adjoint representation of G on $\mathfrak{g}$ extends to a representation of G on the universal algebra $U(\mathfrak{g})$, given on monomials by

$$\mathrm{Ad}(g)(Y_1 \cdots Y_r) = (\mathrm{Ad}(g)Y_1) \cdots (\mathrm{Ad}(g)Y_r)$$

for $g \in G$ and $Y_1, \ldots, Y_r \in \mathfrak{g}$.

Let

$$U(\mathfrak{g})^G = \{D \in U(\mathfrak{g} \,|\, \mathrm{Ad}(g)D = D \text{ for all } g \in G\}$$

For G semisimple, the algebra $U(\mathfrak{g})^G$ is well known. (See [16], Chapter II.) For us, we will just need the fact that $\Omega \in U(\mathfrak{g})^G$. This will, of course, follow from Exercise 5.2.4, but we can also prove this directly as follows. Put

$$\mathrm{Ad}\,(g)\,X_j = \sum_{i=1}^m c_{ij}\,X_i$$

Then since $B(X,Y) = B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y)$, we have

$$b_{ij} = \sum_{k,l} c_{ki}\, b_{kl}\, c_{lj}$$

If $C = (c_{ij})$ and $B = (b_{ij})$, this means that ${}^tC\,B\,C = B$. Thus $B^{-1} = CB^{-1}\,{}^tC$, and so

$$\begin{aligned}
\mathrm{Ad}\,(g)\,\Omega &= \sum_{i,j} b^{ij} \sum_{k,l} c_{ki}c_{lj}\, X_kX_l \\
&= \sum_{k,l}(\sum_{i,j} c_{kj}b^{ij}c_{lj})\, X_kX_l \\
&= \sum_{k,l} b^{kl}\, X_kX_l \\
&= \Omega
\end{aligned}$$

Suppose that G acts smoothly on a manifold M on the left via $(g,m) \mapsto g \cdot m$. As in Section 1.3, let λ denote the left regular representation of G on $\mathcal{E}(M)$:

$$\lambda(g)\, f(m) = f(g^{-1} \cdot m)$$

and let $d\lambda$ be the corresponding infinitesimal left regular representation of $U(\mathfrak{g})$, which on any monomial $Y_1 \cdots Y_r \in U(\mathfrak{g})$ is given by

$$d\lambda(Y_1 \cdots Y_r)\, f(m) = \frac{\partial^r}{\partial t_1 \cdots \partial t_r} f(\exp(-t_rY_r) \cdots \exp(-t_1Y_1) \cdot m)\bigg|_{(t_j=0)}$$

If $\tau(g)$ is the left translation $m \mapsto g \cdot m$ by $g \in G$, it follows that

$$d\lambda(U)^{\tau(g)} = d\lambda(\mathrm{Ad}(g)\, U)$$

for all $g \in G$ and $U \in U(\mathfrak{g})$. Thus if $U \in U(\mathfrak{g})^G$, $d\lambda(U)$ is a differential operator on M invariant under the left action of G.

It is a well-known fact that the Killing form on $\mathfrak{g} = \mathrm{so}(n+1)$ is $B(X,Y) = 2(n+1)\,\mathrm{tr}(XY)$. Hence the basis X_{ij} of $\mathrm{so}(n+1)$ is an orthogonal basis with respect to B, and so

$$\Omega_1 = \sum_{0 \leqslant i < j \leqslant n} X_{ij}^2 \tag{5.24}$$

is a multiple of the Casimir operator Ω. Thus, if λ is the left regular representation of $\mathrm{O}(n+1)$ on $S^n = \mathrm{O}(n+1)/\mathrm{O}(n)$, then $d\lambda(\Omega_1)$ is an $\mathrm{O}(n+1)$-invariant second order differential operator on S^n. We claim that $d\lambda(\Omega_1)$ coincides with the Laplace-Beltrami operator L on S^n. We will prove this directly, although it also follows from Proposition 5.2.3 and the fact that both L and $d\lambda(\Omega_1)$ annihilate constant functions.

For this we first consider the local coordinates $x = (x_0, \ldots, x_n) \in S^n \mapsto (x_1, \ldots, x_n)$ on the open hemisphere $x_0 > 0$. Since

$$\exp(-tX_{ij}) \cdot x = \begin{pmatrix} x_0 \\ \vdots \\ x_i \cos t - x_j \sin t \\ \vdots \\ x_i \sin t + x_j \cos t \\ \vdots \\ x_n \end{pmatrix}$$

we see that

$$d\lambda(X_{ij}) = \begin{cases} x_i \dfrac{\partial}{\partial x_j} - x_j \dfrac{\partial}{\partial x_i} & \text{if } 0 < i < j \leqslant n \\ \left(1 - \sum_{j=1}^n x_j^2\right)^{\frac{1}{2}} \dfrac{\partial}{\partial x_j} & \text{if } i = 0 < j \leqslant n \end{cases}$$

in the given local coordinates. If $f \in \mathcal{E}(S^n)$, this implies that

$$d\lambda(\Omega_1)\, f(e_0) = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(0, \ldots, 0) + \text{lower order terms}$$

On the other hand, the vectors $\partial/\partial x_j|_{e_0}$ form an orthonormal basis of the tangent space $T_{e_0} S^n$. Thus at e_0, (g_{ij}) is the identity $n \times n$ matrix, and by (5.21), we see that

$$Lf(e_0) = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(0, \ldots, 0) + \text{lower order terms}$$

Thus $d\lambda(\Omega_1)f(e_0)$ and $Lf(e_0)$ agree up to lower order derivatives of f at e_0. Since $d\lambda(\Omega_1)$ and L annihilate constants, the difference $(d\lambda(\Omega_1) - L)f(e_0)$ must be a directional derivative of f at e_0; i.e., a tangent vector applied to f. But since $d\lambda(\Omega_1)$ and L are left-$\mathrm{O}(n+1)$-invariant, $d\lambda(\Omega_1) - L$ is an $\mathrm{O}(n+1)$-invariant vector field on S^n. Such a vector field is, in particular, invariant under all rotations preserving each point in S^n. The only such vector field is 0. This shows that $L = d\lambda(\Omega_1)$.

Let $r \geqslant 0$. The *mean value operator* M^r on functions on S^n is defined as follows. Choose any point $y \in S^n$ such that $d(e_0, y) = r$. Then the set of all points at distance r from the point $x = g \cdot e_0$ is the orbit $g\,\mathrm{O}(n) \cdot y$. If $f \in C(S^n)$, the function $M^r f$ is defined by

$$M^r f(x) = \int_{\mathrm{O}(n)} f(gk \cdot y)\, dk \tag{5.25}$$

where the right hand integral is taken with respect to the normalized Haar measure on $\mathrm{O}(n)$.

The following is the analogue of the Darboux equation (3.9) on the n-sphere.

Proposition 5.2.5. *If $f \in \mathcal{E}(S^n)$, then*

$$L_x\left(M^{d(e_0,y)} f(x)\right) = L_y(M^{d(e_0,y)} f(x)) \tag{5.26}$$

and

$$L(M^r f)(x) = (M^r L f)(x) = \left(\frac{\partial^2}{\partial r^2} + (n-1)\,\cot r\,\frac{\partial}{\partial r}\right) M^r f(x) \tag{5.27}$$

Proof. Let $x = g \cdot e_0$. Then since $L = d\lambda(\Omega_1)$,

$$\begin{aligned}
L_x(M^{d(e_0,y)} f)(x) &= d\lambda(\Omega_1)(M^{d(e_0,y)} f)(g \cdot e_0) \\
&= \int_{\mathrm{O}(n)} d\lambda(\Omega_1)\, f(gk \cdot y)\, dk \qquad (5.28) \\
&= \int_{\mathrm{O}(n)} (d\lambda(\Omega_1)\, f)^{\tau((gk)^{-1})}(y)\, dk \\
&= \int_{\mathrm{O}(n)} \big(d\lambda(\Omega_1)\, f^{\tau((gk)^{-1})}\big)(y)\, dk \\
&= \int_{\mathrm{O}(n)} L\big(f^{\tau((gk)^{-1})}\big)(y)\, dk \\
&= L_y \int_{\mathrm{O}(n)} f^{\tau((gk)^{-1})}(y)\, dk \\
&= L_y\left(M^{d(e_0,y)} f(x)\right),
\end{aligned}$$

proving (5.26). Again, since $L = d\lambda(\Omega_1)$, we note that (5.28) can also be written as

$$\int_{\mathrm{O}(n)} Lf(gk \cdot y)\, dk = (M^r L f)(x),$$

proving the first equation in (5.27). The second equation in (5.27) follows from applying (5.22) to the right hand side of (5.26), since $y \mapsto M^{d(e_0,y)} f(x)$ is $\mathrm{O}(n)$-invariant. ☐

Let $\Delta(L)$ denote the differential operator

$$\frac{\partial^2}{\partial r^2} + (n-1)\,\cot r\,\frac{\partial}{\partial r}.$$

This is the "radial part" of the Laplace-Beltrami operator on S^n acting on any great circle through the north pole, acting as a transversal manifold to the action by $\mathrm{O}(n)$.

5.3 Radon Inversion

Let us now invert the Radon transform R in (5.3), which integrates even functions $f \in \mathcal{E}(S^n)$ over d-dimensional great spheres. We shall see that when d

is even, f can be recovered by applying a polynomial in the Laplace-Beltrami operator L to R^*Rf.

Theorem 5.3.1. *Assume that d is even. Let $P_{n,d}(t)$ be the polynomial $P_{n,d}(t) = (t-1\cdot(n-2))(t-3\cdot(n-4))\cdots(t-(d-1)\cdot(n-d))$ Then for any even function $f \in \mathcal{E}(S^n)$, we have*

$$c_{n,d}\, f(x) = P_{n,d}(L)\, R^*Rf(x) \qquad (x \in S^n) \tag{5.29}$$

where

$$\begin{aligned} c_{n,d} &= (-1)^{\frac{d}{2}}\, 2(2\cdot 4\cdots(d-2))((n-2)(n-4)\cdots(n-d))\Omega_d \\ &= 2(-2\pi)^{\frac{d}{2}}(n-2)(n-4)\cdots(n-d) \end{aligned}$$

Proof. Since R, R^*, and L are $\mathrm{O}(n+1)$-invariant, it suffices to derive the inversion formula above for $x = e_0$. For this, let us first calculate $R^*Rf(e_0)$. Now

$$\begin{aligned} R^*Rf(e_0) &= \int_{\mathrm{O}(n)} Rf(k\cdot\sigma_0)\,dk \\ &= \int_{\mathrm{O}(n)}\int_{\sigma_0} f(k\cdot y)\,dm(y)\,dk \\ &= \int_{\sigma_0}\int_{\mathrm{O}(n)} f(k\cdot y)\,dk\,dm(y) \\ &= \int_{\sigma_0} M^{d(e_0,y)} f(e_0)\,dm(y) \\ &= \Omega_d \int_0^{\pi} M^r f(e_0)\,\sin^{d-1} r\,dr \\ &= \int_0^{\pi} F(r)\,\sin^{d-1} r\,dr, \end{aligned} \tag{5.30}$$

where we have put $F(r) = \Omega_d\, M^r f(e_0)$. Now according to Proposition 5.2.5,

$$\begin{aligned} L(R^*Rf)(e_0) &= R^*R(Lf)(e_0) \\ &= \Omega_d \int_0^{\pi} M^r Lf(e_0)\,\sin^{d-1} r\,dr \\ &= \int_0^{\pi} \left(\frac{\partial^2}{\partial r^2} + (n-1)\cot r\,\frac{\partial}{\partial r}\right) F(r)\,\sin^{d-1} r\,dr \end{aligned} \tag{5.31}$$

Repeated use of (5.31) thus shows that for any polynomial $P(t)$,

$$P(L)(R^*Rf)(e_0) = \int_0^{\pi} P(\Delta(L))\,F(r)\,\sin^{d-1} r\,dr$$

Now by integration by parts,

$$\begin{aligned} LR^*Rf(e_0) &= \int_0^\pi (F''(r) + (n-1)\,\cot r\, F'(r))\,\sin^{d-1} r\,dr \\ &= (n-d)\int_0^\pi F'(r)\,\sin^{d-2} r\,\cos r\,dr \end{aligned} \tag{5.32}$$

If $d = 2$, the right hand side above equals

$$(n-2)\left[(F(r)\,\cos r)|_0^\pi + \int_0^\pi F(r)\,\sin r\,dr\right]$$

and thus

$$\begin{aligned} (L-(n-2))R^*Rf(e_0) &= -(n-2)(F(0)+F(\pi)) \\ &= -2(n-2)\,\Omega_d\,f(e_0) \end{aligned}$$

which is (5.29). If $d > 2$, (5.32) equals

$$-(n-d)(d-2)\int_0^\pi F(r)\,\sin^{d-3} r\,dr + (n-d)(d-1)\int_0^\pi F(r)\,\sin^{d-1} r\,dr$$

so that

$$((L-(n-d)(d-1))\,R^*Rf)(e_0) = -(n-d)(d-2)\int_0^\pi F(r)\,\sin^{d-3} r\,dr \tag{5.33}$$

By (5.30), the right hand side is a multiple of $R_0^*R_0f(e_0)$, where R_0 is the Radon transform on S^n which integrates over $(d-2)$-dimensional great spheres. If $d = 4$, we obtain

$$(L-3(n-4))(L-(n-2))R^*Rf(e_0) = 4(n-4)(n-2)\,\Omega_d\,f(e_0),$$

which gives (5.29); if $d > 4$, we have

$$\begin{aligned} &((L-(n-d)(d-1))(L-(n-d+2)(d-3))\,R^*Rf)(e_0) \\ &\qquad = (n-d)(n-d+2)(d-2)(d-4)\int_0^\pi F(r)\,\sin^{d-5} r\,dr \end{aligned}$$

Repeating this calculation proves the formula (5.29) in general. $\square$

We can also obtain an inversion formula for R by using the method of shifted dual transforms. In accordance with the script in (3.46) let us first consider the problem of recovering $f(o)$ from its Radon transform R when f is a smooth function on S^n invariant under $\mathrm{O}(n)$. For such a function f, it is clear that $f(x)$ depends only on the distance $d(o,x)$, or more precisely on its restriction to any great circle passing through the north pole e_0. Hence there exists a smooth 2π-periodic even function h on $\mathbb{R}$ such that $f(x) = h(d(o,x))$. Now the Radon transform Rf is also $\mathrm{O}(n)$-invariant. Since $\mathrm{O}(n)$ is transitive on all d-dimensional great spheres at a fixed distance from the north pole o, $Rf(\sigma)$ depends only on $d(e_0, \sigma)$. Hence there exists a continuous function g on $[0, \pi/2]$, smooth on $(0, \pi/2)$, such that $Rf(\sigma) = g(d(o, \sigma))$ for all $\sigma \in \Xi_d$.

Let $\sigma \in \Xi_d$, let $\alpha = d(e_0, \sigma)$, and pick a point x in σ closest to e_0. The point x is unique if $\alpha < \pi/2$. If $y \in \sigma$, let $\beta = d(x, y)$ and let $\gamma = d(e_0, y)$. Any great circle from e_0 to x is orthogonal, at x, to any great circle from x to y. Thus according to the spherical law of cosines,

$$\cos\gamma = \cos\alpha\,\cos\beta$$

Using geodesic polar coordinates on σ centered at x, it is easy to see that

$$\begin{aligned} g(\alpha) &= \Omega_d \int_0^\pi h(\gamma)\,\sin^{d-1}\beta\,d\beta \\ &= \Omega_d \int_0^\pi h(\arccos(\cos\alpha\,\cos\beta))\,\sin^{d-1}\beta\,d\beta \end{aligned} \tag{5.34}$$

To simplify things, we define the functions G and H by

$$G(\cos\alpha) = g(\alpha), \qquad H(\cos\gamma) = \Omega_d\, h(\gamma)$$

Note that G is continuous on $[0, 1]$ and smooth on $(0, 1)$, whereas H is continuous on $[-1, 1]$ and smooth on $(-1, 1)$. Note also that H is even since f is an even $\mathrm{O}(n)$-invariant function on S^n. The relation (5.34) then becomes

$$G(\cos\alpha) = \int_0^\pi H(\cos\alpha\,\cos\beta)\,\sin^{d-1}\beta\,d\beta$$

and putting $u = \cos\alpha$,

$$\begin{aligned} G(u) &= \int_{-1}^1 H(ut)\,(1-t^2)^{\frac{d}{2}-1}\,dt \\ &= u^{1-d} \int_{-u}^u H(v)\,(u^2 - v^2)^{\frac{d}{2}-1}\,dv \end{aligned} \tag{5.35}$$

for $0 \leqslant u \leqslant 1$. We would like to recover $H(1) = \Omega_d\, h(0) = \Omega_d\, f(o)$ from this integral equation. When d is even, we obtain

$$H(u) + H(-u) = \frac{1}{\Gamma\left(\frac{d}{2}\right)}\,\frac{d}{du} \circ \left(\frac{1}{2u}\frac{d}{du}\right)^{\frac{d}{2}-1} \left(u^{d-1}\,G(u)\right)$$

Since H is an even function,

$$H(1) = \frac{1}{2\,\Gamma\left(\frac{d}{2}\right)} \lim_{u \to 1^-} \left[\frac{d}{du} \circ \left(\frac{1}{2u}\frac{d}{du}\right)^{\frac{d}{2}-1} \left(u^{d-1}\,G(u)\right)\right]$$

and thus

$$f(o) = \frac{1}{4\pi^{\frac{d}{2}}} \lim_{u\to 1^-} \left[\frac{d}{du} \circ \left(\frac{1}{2u}\frac{d}{du}\right)^{\frac{d}{2}-1} \left(u^{d-1}\, g(\arccos u)\right)\right] \tag{5.36}$$

For general d, the method of (2.33)–(2.34) gives

$$\int_0^s u^d\, G(u)\, (s^2-u^2)^{\frac{d}{2}-1}\, du = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2^d\,\Gamma\left(\frac{d+1}{2}\right)} \int_{-s}^s H(v)\,(s^2-v^2)^{d-1}\, dv.$$

Hence

$$H(1) = \frac{2^{d-1}\,\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\,\Gamma(d)\,\Gamma\left(\frac{1}{2}\right)} \lim_{s\to 1^-} \frac{d}{ds} \circ \left(\frac{1}{2s}\frac{d}{ds}\right)^{d-1} \int_0^s u^d\, G(u)\,(s^2-u^2)^{d-1}\, du$$

and thus

$$\begin{aligned} f(o) &= c_d \lim_{s\to 1^-} \frac{d}{ds} \circ \left(\frac{1}{s}\frac{d}{ds}\right)^{d-1} \int_0^s u^d\, g(\arccos u)\,(s^2-u^2)^{d-1}\, du \\ &= c_d \lim_{s\to 1^-} \frac{d}{ds} \circ \left(\frac{1}{s}\frac{d}{ds}\right)^{d-1} \int_{\arccos s}^{\pi/2} g(\alpha)\,(s^2-\cos^2\alpha)^{d-1}\, \cos^d\alpha\, \sin\alpha\, d\alpha \end{aligned} \tag{5.37}$$

where

$$c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\,\Gamma(d)\,\pi^{\frac{3}{2}}}$$

Now for each real r, let a_r be the matrix

$$a_r = \exp(-r\,X_{01}) = \begin{pmatrix} \cos r & -\sin r & & 0 \\ -\sin r & \cos r & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \in \mathrm{O}(n+1)$$

and let $\sigma_r = a_r\cdot\sigma_0$. The isotropy subgroup of σ_r in $\mathrm{O}(n+1)$ is $H_r = a_r H a_r^{-1}$. Let consider the "shifted" incidence relation corresponding to the double fbration

$$\begin{tikzcd} & \mathrm{O}(n+1)/L_r \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ S^n & & G_{d+1,n+1} = \mathrm{O}(n)/H_r \end{tikzcd} \tag{5.38}$$

where $L_r = \mathrm{O}(n)\cap H_r$. Under this new incidence relation, the set of all $\sigma\in\Xi_d$ incident to o is the orbit $\mathrm{O}(n)\cdot\sigma_r$. This coincides with the set of all $\sigma\in\Xi_d$ such that $d(o,\sigma)=r$. By $\mathrm{O}(n+1)$-invariance, we see that

$$x \text{ is incident to } \sigma \iff d(x,\sigma) = r$$

Let R_r^* denote the shifted dual transform corresponding to this shifted incidence relation.

Now since $\mathrm{O}(n)$ is transitive on the set of all $\sigma \in \Xi_d$ at a fixed distance r from o, we see that the $\mathrm{O}(n)$-orbits are parametrized by r. Suppose that $f \in \mathcal{E}(S^n)$. Let d be even. If we put $u = \cos r$, then according to the general shift formula (3.49) and (5.36)

$$f(x) = \frac{1}{8\pi^{\frac{d}{2}}} \lim_{u \to 1^-} \left[\frac{d}{du} \circ \left(\frac{1}{2u} \frac{d}{du} \right)^{\frac{d}{2}-1} \left(u^{d-1} R^*_{\arccos u} Rf(x) \right) \right] \tag{5.39}$$

If d is odd, then from (5.37) we obtain

$$f(x) = c_d \lim_{s \to 1^-} \frac{d}{ds} \circ \left(\frac{1}{s} \frac{d}{ds} \right)^{d-1} \int_{\arccos s}^{\pi/2} R^*_\alpha Rf(x) \, (s^2 - \cos^2 \alpha)^{d-1} \cos^d \alpha \, \sin \alpha \, d\alpha \tag{5.40}$$

5.4 The Range of the Great Sphere Transform

In this section we characterize the range of the transform R in (5.3), which integrates even functions on S^n over d-dimensional great spheres. We will assume some knowledge of spherical highest weight modules. There are many books which introduce this topic, such as [17], [25], [42].

Now when $d = n - 1$, it is not hard to show that R is a linear bijection from the subspace $\mathcal{E}_e(S^n)$ of even functions in $\mathcal{E}(S^n)$ onto itself, or from $\mathcal{E}(\mathbb{RP}^n)$ onto itself.

In fact each codimension one great sphere ξ in S^n corresponds to an antipodal point pair $\{\omega, -\omega\}$ in S^n at maximum distance from ξ (or perpendicular to the linear span of ξ). This identifies $G_{n,n+1}$ with $G_{1,n+1} = \mathbb{RP}^n$. The inclusion incidence relation between points and codimension one great spheres is then equivalent to an incidence relation between points in S^n and points in $\mathbb{RP}^n$:

$$x \in S^n \text{ is incident to } \{\omega, -\omega\} \iff x \perp \omega$$

This incidence relation still corresponds to the double fibration (5.2) with $d = n - 1$ and with Q the subgroup consisting of all elements of $\mathrm{O}(n+1)$ preserving the x_n axis. Under this equivalent incidence relation, the Radon transform is given by

$$Rf(\omega) = \int_{x \perp \omega} f(x) \, dm(x) \qquad (f \in C(S^n)) \tag{5.41}$$

where we can think of Rf as an even function on S^n. The dual transform is given by

$$R^*\varphi(x) = \frac{1}{\Omega_n} \int_{\omega \perp x} \varphi(\omega) \, dm(\omega) \qquad (\varphi \in C(\mathbb{RP}^n)) \tag{5.42}$$

In the above we can likewise think of φ as an even function on S^n. The factor Ω_n^{-1} is present because the set $\check{x}$ is assumed to have measure one. Thinking of the transform R this way, we see that $R^* = \Omega_n^{-1}\, R$.

Now if n is odd, we can write the inversion formula (5.29) as

$$c_n\, f(x) = (R^* R(P(L)f))(x) \tag{5.43}$$

where c_n is a constant depending on n and $P(L)$ is a polynomial in the Laplace-Beltrami operator L. Here we have used the fact that R^*R commutes with $P(L)$, since $L = d\lambda(\Omega_1)$, with Ω_1 given by (5.24). The formula (5.43) together with (5.42) shows that R maps $\mathcal{E}_e(S^n)$ onto itself; since R is already injective and continuous, the open mapping theorem shows that it is a linear homeomorphism from the Fréchet space $\mathcal{E}_e(S^n)$ onto itself.

More generally, for each $l \in \mathbb{Z}^+$, let $\mathcal{H}_l$ denote the vector space of degree l spherical harmonics on S^n. We have the orthogonal decomposition

$$L^2(S^n) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l \tag{5.44}$$

The elements of $\mathcal{H}_l$ are the restrictions of the degree l harmonic polynomials (with complex coefficients) in $\mathbb{R}^{n+1}$, and the the left regular action of $\mathrm{O}(n+1)$ on each $\mathcal{H}_l$ is irreducible. Moreover, since $\dim \mathcal{H}_l \neq \dim \mathcal{H}_m$ whenever $l \neq m$, the $\mathcal{H}_l$ are inequivalent $\mathrm{O}(n+1)$ modules. If $f \in \mathcal{E}_e(S^n)$, then f can be written as a "Fourier" series of even degree harmonics

$$f = \sum_{l=0}^{\infty} f_{2l} \tag{5.45}$$

where f_{2l} is the component of f in $\mathcal{H}_{2l}$ and by [41] the sum above converges in the topology of $\mathcal{E}(S^n)$. Now since R maps $\mathcal{E}_e(S^n)$ to $\mathcal{E}_e(S^n)$ and commutes with the left action of $\mathrm{O}(n+1)$, we see from Schur's lemma that R is a constant multiplication operator c_{2l} on each $\mathcal{H}_{2l}$. To calculate c_{2l}, let φ_{2l} be the zonal spherical harmonic in $\mathcal{H}_{2l}$. Since φ_{2l} is constant on $(n-1)$-spheres of constant latitude, and since $\varphi(e_0) = 1$, (5.41) shows that

$$\begin{aligned} c_{2l} &= c_{2l}\, \varphi_{2l}(e_0) \\ &= (R\varphi_{2l})(e_0) \\ &= \varphi_{2l}(0, x_1, \dots, x_n), \end{aligned}$$

where $(0, x_1, \dots, x_n)$ is any point in $S^n \bigcap \{x_0 = 0\}$. But according to [29] §5.3,

$$\varphi_{2l}(x_0, x_1, \dots, x_n) = \frac{\Gamma(n-1)\Gamma(2l+1)}{\Gamma(n+2l-1)}\, C_{2l}^{\frac{n-1}{2}}(x_0),$$

where $C_m^{\lambda}(x)$ is the Gegenbauer polynomial of degree m and type λ. From the same reference, we have

$$C_{2l}^{\frac{n-1}{2}}(0) = (-1)^l\, \frac{\Gamma(l + \frac{n-1}{2})}{\Gamma(\frac{n-1}{2})\, \Gamma(l+1)}$$

Using the duplication formula for the Gamma function, we then obtain

$$\begin{aligned} c_{2l} &= (-1)^l \, \frac{\Gamma\left(\frac{n}{2}\right)\, \Gamma(2l+1)}{2^{2l}\Gamma\left(l+\frac{n}{2}\right)\, \Gamma(l+1)} \\ &= (-1)^l \, \frac{2\,\Gamma\left(\frac{n}{2}\right)\, \Gamma\left(l+\frac{3}{2}\right)}{\pi^{\frac{1}{2}}\,(2l+1)\,\Gamma\left(l+\frac{n}{2}\right)} \end{aligned}$$

This shows that $c_{2l} \neq 0$ and that c_{2l}^{-1} is bounded above in absolue value by a polynomial in l.

If f is given by (5.45), then the function

$$g = \sum_{l=0}^{\infty} \frac{1}{c_{2l}}\, f_{2l}$$

belongs to $\mathcal{E}_e(S^n)$ and satisfies $Rg = f$. Thus R is onto. Since R is also one-to-one, it is a bijection, and by the open mapping theorem, a homeomorphism.

Now if $d < n-1$, then $\dim G_{d+1,n+1} = (d+1)(n-d) > n$, so R is "overdetermined" in the sense that it maps functions of n variables into functions of $> n$ variables. It turns out that the range $R\mathcal{E}(S^n)$ is characterized by a system of second order differential equations akin to (4.17), or by a single fourth order $\mathrm{O}(n+1)$-invariant differential equation. Let us now proceed to show this.

For simplicity, let $U = \mathrm{O}(n+1)$ and $\mathfrak{u} = \mathrm{so}(n+1)$. As usual, $U(\mathfrak{u})$ will denote the universal enveloping algebra of $\mathfrak{u}$. We will also shift our coordinates in what follows, so that any point x in $\mathbb{R}^{n+1}$ or S^n will have coordinates $(x_1, \ldots, x_{n+1})$.

For $1 \leqslant j,k \leqslant n+1$, let E_{jk} denote the $(n+1)\times(n+1)$ elementary matrix $(\delta_{rj}\delta_{sk})_{1\leqslant r,s\leqslant n+1}$. Let X_{jk} be the elementary skew-symmetric matrix $E_{jk}-E_{kj}$. Then $\mathfrak{u}$ has basis $\{X_{jk} \mid 1 \leqslant j < k \leqslant n+1\}$. For each quadruple of indices j,k,l,m in $\{1,\ldots,n+1\}$, let $V_{jklm} = X_{jk}X_{lm} - X_{jl}X_{km} + X_{jm}X_{kl} \in U(\mathfrak{u})$. Note that $V_{jklm} = 0$ if any of the indices j,k,l,m coincide, and that $V_{jklm} = -V_{jlkm}$, etc. Now let

$$E = \sum V_{jklm}^2 \in U(\mathfrak{u}),$$

where the sum ranges over all $1 \leqslant j < k < l < m \leqslant n+1$. We let $\mathfrak{z}(U(\mathfrak{u}))$ denote the subalgebra of $U(\mathfrak{u})$ consisting of its $\mathrm{Ad}(U)$-invariant elements.

Lemma 5.4.1. *The element E satisfies the following properties:*

(1). $E \in \mathfrak{z}(U(\mathfrak{u}))$

(2). $E \in \mathit{ker}(d\lambda)$

Proof. The proof of assertion (1) above is completely analogous to the proof at the end of Section 4.2 that $\sum_{i<j<l} V_{ijl}^2$ is $\mathrm{Ad}\,\mathrm{M}(n)$-invariant.

To prove assertion (2), it suffices to prove that each $V_{jklm} \in \ker(d\lambda)$. Let $i : S^n \to \mathbb{R}^{n+1}$ be the inclusion map, and for $g \in C(\mathbb{R}^{n+1})$, let $\overline{g} = g \circ i$ be its restriction to S^n. Let ω denote the left regular representation of $U = O(n+1)$ on $\mathbb{R}^{n+1}$. Since the map i commutes with the left action of $O(n+1)$ on $\mathbb{R}^{n+1}$ and S^n, we have $d\lambda(D)\overline{g} = \overline{d\omega(D)g}$ for all $g \in \mathcal{E}(\mathbb{R}^{n+1})$, $D \in U(\mathfrak{u})$. The map $g \mapsto \overline{g}$ is onto, so it suffices to prove that $V_{jklm} \in \ker(d\omega)$. But this is immediate, since $d\omega(X_{jk}) = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}$ □

Let $L^2_e(S^n) = L^2(\mathbb{RP}^n)$ denote the Hilbert space of even L^2 functions on S^n and consider its L^2 Fourier decomposition

$$L^2_e(S^n) = \sum_{\pi \in \Lambda_1} \mathcal{H}_\pi \tag{5.46}$$

as well as the decomposition

$$L^2(G_{d+1,n+1}) = \sum_{\pi' \in \Lambda_2} \mathcal{F}_{\pi'} \tag{5.47}$$

Here $\Lambda_1 = 2\mathbb{Z}^+$; the components in (5.46) are of course just the spaces of even degree spherical harmonics. The index set Λ_2 is the set of equivalence classes of irreducible unitary Q-spherical representations of $\mathrm{O}(n+1)$, where Q is the subgroup $\mathrm{O}(d+1) \times \mathrm{O}(n-d)$ consisting of the matrices of the form (5.1). Since $G_{d+1,n+1}$ is a compact symmetric space, each component $\mathcal{F}_{\pi'}$ occurs with multiplicity one.

Since R is injective, R must be a bijection of $\mathcal{H}_\pi$ onto $R\mathcal{H}_\pi$ for each $\pi \in \Lambda_1$. Thus $R\mathcal{H}_\pi$ must be one of the $\mathcal{F}_{\pi'}$, and so we can take Λ_1 to be a subset of Λ_2 and, for simplicity, put $R\mathcal{H}_\pi = \mathcal{F}_\pi$.

Suppose that $D \in \mathfrak{z}(U(\mathfrak{u}))$. If Λ is the set of *all* equivalence classes of irreducible unitary representations of $\mathrm{O}(n+1)$ and $\pi \in \Lambda$, then according to Schur's lemma $d\pi(D)$ acts as a scalar $c_\pi(D) \in \mathbb{C}$ on the representation space V_π:

$$d\pi(D) = c_\pi(D) \cdot I_{V_\pi}$$

As usual, let λ and ν be the left regular representations of U on functions on S^n and $G_{d+1,n+1}$, respectively. Since $E \in \ker(d\lambda)$, we must have $c_\pi(E) = 0$ for all $\pi \in \Lambda_1$.

Lemma 5.4.2. *$c_{\pi'}(E) \neq 0$ for all $\pi' \in \Lambda_2 \backslash \Lambda_1$.*

Proof. We first parametrize Λ_1 and Λ_2 by means of compatible sequences of integers as follows. Set $\gamma = \operatorname{rank}(G_{d+1,n+1}) = \min(d+1, n-d)$ and $m = \operatorname{rank}(\mathrm{so}(n+1)) = \left[\frac{n+1}{2}\right]$. For $d+1 \leqslant m$, we set

$$H_j = \begin{cases} X_{j,d+1+j} & j = 1, \ldots, d+1 \\ X_{2j-1,2j} & j = d+2, \ldots, m; \end{cases}$$

and for $d+1>m$ we set

$$H_j = \begin{cases} X_{j,d+1+j} & j=1,\dots,n-d \\ X_{2j-(n-d)-1,2j-(n-d)} & j=n-d+1,\dots,m \end{cases}$$

Then $H_1,\dots,H_m$ span a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}=\mathrm{so}(n+1)$. For a real or complex vector space, let V' denote its dual and V^c its complexification if V is real. Let $\mathfrak{g},\mathfrak{h}$ denote the complexifications of $\mathfrak{u}$ and $\mathfrak{t}$, respectively, and let $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ denote the set of roots of the complex semisimple Lie algebra $\mathfrak{g}=\mathrm{so}(n+1,\mathbb{C})$ with respect to $\mathfrak{h}$. We write the corresponding root space decomposition as $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$.

Defining $e_j\in\mathfrak{h}'$ by $e_j(H_k)=-i\delta_{jk}$, the root system $\Delta(\mathfrak{g},\mathfrak{h})$ is the set $\{\pm e_j\pm e_k \mid 1\leqslant j<k\leqslant m\}$ if $n+1=2m$ and the set $\{\pm e_j\pm e_k \mid 1\leqslant j<k\leqslant m\}\cup\{\pm e_j \mid 1\leqslant j\leqslant m\}$ if $n+1=2m+1$. We order the roots in Δ so that $e_1>e_2>\cdots>e_m$ (>0 if $n+1=2m+1$).

Now let $\mathfrak{k},\mathfrak{q}$ denote the Lie algebras of K and Q respectively. We have the Cartan decompositions

$$\mathfrak{u}=\mathfrak{k}+\mathfrak{p}_*, \qquad \mathfrak{u}=\mathfrak{q}+\mathfrak{m}_*$$

where $\mathfrak{p}_*$ and $\mathfrak{m}_*$ are the orthogonal complements of $\mathfrak{k}$ and $\mathfrak{m}$, respectively, with respect to the Killing form on $\mathfrak{u}$. The spaces $\mathfrak{a}_*=\mathbb{R}H_1$ and $\mathfrak{b}_*=\sum_{j=1}^{\gamma}\mathbb{R}H_j$ are maximal abelian in $\mathfrak{p}_*$ and $\mathfrak{q}_*$, respectively. From [42], the highest weights corresponding to representations $\pi\in\Lambda_1$ are given by $\lambda=\lambda_1e_1$, where $\lambda_1\in 2\mathbb{Z}^+$ (these correspond to the spherical harmonics of degree λ_1); the highest weights corresponding to representations $\pi'\in\Lambda_2$ are given by

$$\mu=\sum_{j=1}^{\gamma}\lambda_j e_j, \tag{5.48}$$

where $\lambda_j\in 2\mathbb{Z}$ and $\lambda_1\geqslant\lambda_2\geqslant\cdots\lambda_\gamma\geqslant 0$ (unless $\gamma=m=\frac{n+1}{2}$, in which case $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant|\lambda_m|$.) Thus, in particular, μ is the highest weight of a representation $\pi'\in\Lambda_2\backslash\Lambda_1$ if and only if $\lambda_2\neq 0$.

Now let α,β denote the positive roots $\alpha=e_1-e_2$, $\beta=e_1+e_2$. It can be readily verified that $\mathfrak{g}^\alpha=\mathbb{C}X_\alpha$, where

$$X_\alpha=(X_{12}+X_{k+2,k+3})+i(X_{1,k+3}+X_{2,k+2})$$

Similarly, $\mathfrak{g}^{-\alpha},\mathfrak{g}^\beta$ and $\mathfrak{g}^{-\beta}$ are generated by the vectors $X_{-\alpha},X_\beta$ and $X_{-\beta}$, respectively, where

$$\begin{aligned} X_{-\alpha} &= (-X_{12}-X_{k+2,k+3})+i(X_{1,k+3}+X_{2,k+2}) \\ X_\beta &= (X_{12}-X_{k+2,k+3})+i(-X_{1,k+3}+X_{2,k+2}), \\ X_{-\beta} &= \overline{X}_\beta=(X_{12}-X_{k+2,k+3})+i(X_{1,k+3}-X_{2,k+2}). \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned}
X_{12} &= \frac{1}{4}\big[X_\alpha - X_{-\alpha} + X_\beta + X_{-\beta}\big] \\
X_{1,k+3} &= -\frac{i}{4}\big[X_\alpha + X_{-\alpha} - X_\beta + X_{-\beta}\big] \\
X_{2,k+2} &= -\frac{i}{4}\big[X_\alpha + X_{-\alpha} + X_\beta - X_{-\beta}\big] \\
X_{k+2,k+3} &= \frac{1}{4}\big[X_\alpha - X_{-\alpha} - X_\beta - X_{-\beta}\big]
\end{aligned}$$

Since $X_{\pm\alpha}$ commutes with $X_{\pm\beta}$, it follows that

$$\begin{aligned}
V_{1,2,k+2,k+3} &= X_{12}X_{k+2,k+3} - X_{1,k+2}X_{2,k+3} + X_{1,k+3}X_{2,k+2} \\
&= \frac{1}{16}\big[X_\alpha - X_{-\alpha} + X_\beta + X_{-\beta}\big]\big[X_\alpha - X_{-\alpha} - X_\beta - X_{-\beta}\big] - H_1H_2 \\
&\qquad - \frac{1}{16}\big[X_\alpha + X_{-\alpha} - X_\beta + X_{-\beta}\big]\big[X_\alpha + X_{-\alpha} + X_\beta - X_{-\beta}\big] \\
&= -H_1H_2 - \frac{1}{8}\big[X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha\big] - \frac{1}{8}\big[X_\beta X_{-\beta} + X_{-\beta}X_\beta\big] \\
&= -H_1H_2 - \frac{1}{8}\big[X_\alpha, X_{-\alpha}\big] - \frac{1}{8}\big[X_\beta, X_{-\beta}\big] - \frac{1}{4}X_{-\alpha}X_\alpha - \frac{1}{4}X_{-\beta}X_\beta
\end{aligned}$$

Now let π' be a representation in Λ_2 with highest weight $\mu = \sum_{j=1}^{\gamma}\lambda_j e_j$, and let v be a nonzero highest weight vector in $V_{\pi'}$. By direct calculation from (4), we have

$$\begin{aligned}
\big[X_\alpha, X_{-\alpha}\big] &= 4i\big(X_{1,k+2} - X_{2,k+3}\big) = 4i\big(H_1 - H_2\big), \\
\big[X_\beta, X_{-\beta}\big] &= -4i\big(X_{1,k+2} + X_{2,k+3}\big) = 4i\big(H_1 + H_2\big).
\end{aligned}$$

Thus

$$\begin{aligned}
d\pi'(V_{1,2,k+2,k+3})v &= -d\pi'(H_1H_2)v - \frac{i}{2}d\pi'(H_1 - H_2)v + \frac{i}{2}d\pi'(H_1 + H_2)v \\
&= \big[-\mu(H_1)\mu(H_2) - \frac{i}{2}(\mu(H_1) - \mu(H_2)) + \frac{i}{2}(\mu(H_1) + \mu(H_2))\big]v \\
&= \lambda_2(\lambda_1 + 1)v \qquad (5.49)
\end{aligned}$$

Let $\langle\ ,\ \rangle$ denote the unitary structure of $V_{\pi'}$. Since $d\pi'(V_{jklm})$ is self-adjoint with respect to $\langle\ ,\ \rangle$,

$$\begin{aligned}
c_{\pi'}(E)\langle v, v\rangle &= \langle d\pi'(E)v, v\rangle \\
&= \sum_{j<k<l<m} \|d\pi'(V_{jklm})v\|^2
\end{aligned}$$

By (5.49), $\|d\pi'(V_{1,2,k+2,k+3})v\| = |\lambda_2|(\lambda_1 + 1)\|v\|$. Hence $c_{\pi'}(E)\langle v, v\rangle > 0$ if $\lambda_2 \neq 0$, so $c_{\pi'}(E) \neq 0$ whenever $\pi' \in \Lambda_2 \backslash \Lambda_1$. (In particular, $E \notin \ker(d\nu)$.) This completes the proof of the lemma. $\square$

We are now ready to give a characterization of the range $R\mathcal{E}(S^n)$.

Theorem 5.4.3. $R(\mathcal{E}(S^n)) = R(\mathcal{E}_e(S^n)) = \{\phi \in \mathcal{E}(G_{d+1,n+1}) \mid d\nu(E)\phi = 0\}$.

Proof. Since $E \in \ker(d\lambda)\backslash\ker(d\nu)$, the condition $d\nu(E)\phi = 0$ is clearly necessary for the range. Conversely, let us assume $\phi \in \mathcal{E}(G_{d+1,n+1})$ satisfies $d\nu(E)\phi = 0$. By [41], the series $\phi = \sum_{\pi' \in \Lambda_2} \phi_{\pi'}$ converges in the topology of $\mathcal{E}(G_{d+1,n+1})$. Since $c_\pi(E) = 0$ for all $\pi \in \Lambda_1$,

$$0 = d\nu(E)\phi = \sum_{\pi' \in \Lambda_2} c_{\pi'}(E)\phi_{\pi'} = \sum_{\pi' \in \Lambda_2 \backslash \Lambda_1} c_{\pi'}(E)\phi_{\pi'}$$

Each of the coefficients $c_{\pi'}(E)$ in the last sum above is nonzero by Lemma 2, so in fact $\phi_{\pi'} = 0$ for all $\pi' \in \Lambda_2\backslash\Lambda_1$. Hence $\phi = \sum_{\pi \in \Lambda_1} \phi_\pi$. Now for each $\pi \in \Lambda_1$, let f_π be the function in $\mathcal{H}_\pi$ such that $Rf_\pi = \phi_\pi$. Supposing π has highest weight $\lambda_1 e_1$, then by [40] and Lemmas 4.1 and 4.2, the L^2 norm of R on $\mathcal{H}_\pi$ is

$$c_d \sqrt{\frac{\Gamma(\frac{\lambda+1}{2})\Gamma(\frac{\lambda+n-d}{2})}{\Gamma(\frac{\lambda+d+1}{2})\Gamma(\frac{\lambda+n}{2})}} \qquad (\lambda = \lambda_1), \tag{5.50}$$

where c_d is a constant depending only on d. In turn, (5.50) is bounded below by $C\lambda^{-d}$. Thus from [41] it follows that the sum $f = \sum_{\pi \in \Lambda_1} f_\pi$ is convergent in $\mathcal{E}_e(S^n)$. Since R is continuous in $\mathcal{E}(S^n)$, we have

$$Rf = \sum_{\pi \in \Lambda_1} Rf_\pi = \sum_{\pi \in \Lambda_1} \phi_\pi = \phi.$$

This completes the proof of the theorem.

□

Corollary 5.4.4. $R(\mathcal{E}(S^n)) = R(\mathcal{E}_e(S^n)) = \{\phi \in \mathcal{E}(G_{d+1,n+1}) \mid d\nu(V_{jklm})\phi = 0$ *for all* $j, k, l, m\}$.

Proof. The proof of Lemma 5.4.1 shows that $V_{jklm} \in \ker(d\lambda)$, so it follows that $d\nu(V_{jklm})Rf = 0$. Conversely, if $\phi \in \mathcal{E}(G_{d+1,n+1})$ satisfies $d\nu(V_{jklm})\phi = 0$ for all j, k, l, m, then of course $d\nu(E)\phi = 0$ so $\phi = Rf$ for some $f \in \mathcal{E}(S^n)$. □

Theorem 5.4.3 is generalized in [24] and [11], where a range theorem for the Radon transform $R : \mathcal{E}(G_{p,n}) \to \mathcal{E}(G_{q,n})$ (with respect to the inclusion incidence relation) is given in terms of Pfaffian-type elements of the universal enveloping algebra $U(\mathrm{so}(n))$.

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