## Preface

The fourth scientific colloquium Czech-Japanese Seminar in Applied Mathematics 2008 was held on September 1-7 at Miyazaki, Japan. It represents a continuation of the successful series of the seminars following the first Czech-Japanese Seminar in Applied Mathematics 2003 at Czech Technical University in Prague. This scientific colloquium was organized by the Department of Applied Physics, University of Miyazaki and Faculty of Mathematics, Kyushu University and Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague and was devoted to the meeting of applied mathematicians and researchers from applied disciplines dealing with numerical solution of partial differential equations, mathematical modelling and numerical simulation of problems in technology, environment, biology and computer science. This seminar divided into two parts: the first part is to be held at Takachiho on September 1-4 and the second part is to be held at University of Miyazaki on September 5-7. It featured two invited lectures, 37 contributed lectures and 9 posters which are applicable to the fields in the engineering, agriculture, environment and adjusted to the aim of this colloquium.

The proceedings contains peer reviewed articles which reflect fruitful atmosphere and interesting composition of topics of the meeting.

The organizers gratefully acknowledge the support of the Global COE Program Education-and-Research Hub for Mathematics-for-Industry of Faculty of Mathematics, Kyushu University and the Czech Ministry of Education, Youth and Sport Applications of Mathematics in Technical Sciences and the project Jindrich Nečas Center for Mathematical Modelling and Grant-in-Aid for Scientific Research (No. 18340003, 19540121, 20540122), Japan Society for the Promotion of Science. The organizers would also like to thank people in Takachiho for helping during our stay.

Full information on the meeting as well as the proceeding is available at the address:


The organizers
DISPERSION OF PASSIVE POLLUTANT WITHIN AND ABOVE VARIOUS URBAN CANOPIES

KLAUS RENDPOLUTIONA
AND MASAAKI OHBA

Abstract

Releases of harmful gaseous materials within the lower part of the atmospheric boundary layer are of great concern of the society in general. The understanding of short-range dispersion of air-borne releases of harmful gaseous materials within the urban canopy is crucial to predict the spread of such pollutants. Mathematical modelling is a powerful tool for the simulation of such complex tasks. Physical modeling is an alternative method that can be used for investigation of the dispersion of a passive pollutant. The idealized urban canopy was created by regular cubic cubes which were situated at three different packing densities and two different patterns. The flow and concentration of the passive tracer released from the ground level source was measured from the plume and above the rooftop level is emphasised.

1. Introduction

Dispersion of pollutants in urban areas is still one of the most challenging tasks in environmental sciences. Complex processes like the dispersion of car exhaust in street canyons or the dispersion of accidental releases of harmful substances in built-up areas are not yet fully understood. For a better insight of the driving phenomena it is helpful to study flow and dispersion of pollutants within an idealized urban setting first.

The study of dispersion through large idealized arrays of building-like obstacles is an important method of obtaining a better understanding of dispersion through a real urban environment. Field and laboratory studies of idealized obstacle arrays are necessarily simplifications of the real complex urban environment. However, the idealised canopys can reproduce some of the real urban characteristics as building packing density or building arrangement. These types of geometries, nonetheless, should display some of the characteristics of the more complex, real-world canopies and, shown some generally valid rules. We examined flow and passive tracer dispersion within 6 different configurations of the idealized urban canopy layout during our experiment. The objective of the presented paper is the analysis of the mean concentration field at the street level for different idealized urban canopy set-ups and showing the main mechanisms of the pollutant transport within them. Such knowledge should have significant role in design process of new urban structures or modification of already built-up areas to achieve better ventilation characteristics. One of the first experiments dealing with an idealized urban canopy was conducted by MacDonald et al. [1], who compared the mean concentration characteristics within the idealized urban canopies with different packing densities. There did not find any significant difference between different packing densities and neither did Herrero [2] in his work. Cheng et al. [3] investigated the drag forces and momentum transport for two packing densities and aligned/offseted arrangement. They found significantly increased drag and momentum transport for the staggered arrays. Therefore, an enhanced vertical transport of passive contaminant could be expected.

The objective of the presented paper is the analysis of the mean concentration field and the cross section of the tunnel measures 1.2 m in width and 1 m in height. The experimental conditions were carefully checked by series of measurements. The experimental setup and measurement parameters are shown in Table 1. The scale of the urban setting was made at 1/20, and the cross section of the tunnel measures 1.2 m in width and 1 m in height. Significant differences between different packing densities and alignments were found for experiments with staggered rows. The mean concentration at the street level for the staggered arrangement was reported in [3]. The magnitude of the turbulent part on the mean concentration field was found significant due to the suppression of the mean flow within the plant canopy [Meyers [8]].

2. Definitions

The advection equation for a scalar ψ, such as concentration of passive contaminant, far from the source of the scalar is expressed mathematically as

$$\frac{∂ψ}{∂t} + \nabla \cdot \mathbf{U} \psi = 0,$$

where \( \mathbf{U} \) is the divergence operator and \( \mathbf{U} \) is the velocity vector field. Any variable within the turbulent field can be divided to the temporally constant mean value (depleted by skewness) and fluctuating part (depleted by prime), which time-averaged mean value is zero. Therefore we can rewrite the temporal mean equation of advection as

$$\bar{\frac{∂ψ}{∂t}} + \nabla \cdot \mathbf{U} \bar{ψ} = \nabla \cdot \mathbf{U}′ \bar{ψ} = \nabla \cdot \mathbf{U}′ \bar{ψ} + \nabla \cdot \mathbf{U} \prime \bar{ψ} = 0,$$

where \( ω ′ \) is the mean and fluctuation part of vertical wind component, respectively, and \( ν ′ \) and \( κ ′ \) is the mean and fluctuation part of concentration of passive contaminant. The first and second term in the braces are called the advection and turbulent vertical flux of the passive contaminant, respectively.

3. Experimental set-up

The experiment was carried out in the Boundary Layer Wind Tunnel at Wind Engineering Center of Tokyo Polytechnic University, Atsugi, Japan. The 14 m long facility provides test sections equipped with turntable and the cross section of the tunnel measures 1.2 m in width and 1 m in height. Measuring instruments (see Fig. 3.1(b)) were placed on a controlled positioning system, which allowed to move the probes in all 3 directions and to rotate the axes. Spires and roughness elements were used to develop model of the suburban atmospheric boundary layer in the 1/20 scale.

The spires are two-dimensional structures placed at the very beginning of the development section of the wind tunnel (just behind the stilling chamber). The roughness elements had dimensions: 50 mm (width), 50 mm (length), 25 mm (Height). The roughness elements were placed in regular pattern. To model roughness elements together with one of the idealized urban area models are shown in Fig.3.1.

The researcher team of Wind Engineering Center of Tokyo Polytechnic University has developed new method for simultaneous measurement of velocity and concentration by the means of constant temperature thermo-anemometry (CTA) and flame ionization detection (FID) of the tracer gas (Snyder et al., [7]). This set-up allows to derive turbulent fluxes related to the momentum and concentration. The flow measurements were conducted using CTA and FID and sometimes by mobile devices such as a fast FID. Density of the tracer gas is 1.18 kg m⁻³ in standard atmospheric conditions and it can be considered as a tracer gas because it does not affect the standard air (δ 1.25 kg m⁻³). The molecular diffusivity is negligible compared to the turbulent diffusivity.

The experimental components are used to assess the accuracy of measurements. The independence of dimensionless properties on the source emission rate Q and building Reynolds number ReB was found for experiments with Q ≤ 50 cm³/min and ReB ≤ 120000 (i.e. Uc ≥ 6 m/s), respectively. The characteristic experimental conditions were: Q ≤ 300 cm³/min (i.e. 18 l per hour) and ReB ≤ 160000 (i.e. Uc = 8 m/s).

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<thead>
<tr>
<th>Table 1</th>
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<td>6</td>
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<td>3</td>
<td>9</td>
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4. Modelled boundary layer

The logarithmic law describing mean wind vertical profile, \( z(U) \), in the lower part of the atmospheric boundary layer can be mathematically written as

$$U(z) = \frac{U_*}{
\ln \left( \frac{z}{H} \right) - \ln \left( \frac{z_0}{H} \right) \right),$$

where \( U_* \) is friction velocity, \( z_0 = 0.05 H \) is Karman constant, \( d_a \) is displacement height, \( d_a = 0.1 H \) is roughness height. Another way to describe the mean wind vertical profile is the power law:

$$U(z) = U(0) \left( \frac{z}{z_0} \right)^{1/ \kappa},$$

where \( U(0) \), reference wind speed at reference height \( z_0 \), and \( \kappa \) is power law exponent. More about the properties of the atmospheric boundary layer can be found in [8].

Measured mean wind speed vertical profile is shown in Fig.4.1(a). Logarithmic and power law profiles were applied to the measured profile with following parameters. \( U(z) \) (in full scale): roughness length \( z_0 = 1.0 \), displacement length \( d_a = 0.12 \), friction velocity \( U_* = 0.4 \) m/s, and power law exponent \( \kappa = 0.15 \). These parameters correspond to those observed in atmospheric boundary layer above moderately rough terrain (see [9]). Also high order statistical moments were measured and compared to full scale data. Vertical profile of turbulence intensity compared to the empirical formulae proposed by Snyder [9] and dimensionless power spectra compared with empirical formulae proposed by Kaimal et al. [10] are shown in Fig. 4.1(b) and 4.1(c), respectively.

5. Mean concentration field

Since the experiments were conducted in the scale, we need to define dimensionless or normalized variables to obtain values, which be equivalent to 28 m in the full scale.

The ground-level point source of the tracer gas C2H6 was located in the wake of the cubic at coordinates \( x = -5, y = 0 \) and \( z = 0 \) (coordinates and the origin are shown in Fig.5.1). The location of the point source is near for all set-ups. Density of the tracer gas is 1.18 kg m⁻³ in standard atmospheric conditions and it can be considered as a tracer gas because it does not affect the standard air (δ 1.25 kg m⁻³). The molecular diffusivity is negligible compared to the turbulent diffusivity.

The experimental components are used to assess the accuracy of measurements. The independence of dimensionless properties on the source emission rate Q and building Reynolds number ReB was found for experiments with Q ≤ 50 cm³/min and ReB ≤ 120000 (i.e. Uc ≥ 6 m/s), respectively. The characteristic experimental conditions were: Q ≤ 300 cm³/min (i.e. 18 l per hour) and ReB ≤ 160000 (i.e. Uc = 8 m/s).

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significant difference between different packing densities and neither did Herrero [2] in his work. Cheng et al. [3] investigated the drag forces and momentum transport for two packing densities and aligned/offseted arrangement. They found significantly increased drag and momentum transport for the staggered arrays. Therefore, an enhanced vertical transport of passive contaminant could be expected.

The objective of the presented paper is the analysis of the mean concentration field and the cross section of the tunnel measures 1.2 m in width and 1 m in height. The experimental conditions were carefully checked by series of measurements. The experimental setup and measurement parameters are shown in Table 1. The scale of the model and of the modelled boundary layer was 1/20, i.e. the building height would
The turbulent flux contributes significantly to the total flux at the edges of the passive pollutant released from ground source the pollutant penetrates back to the source emission rate. The mean concentration distribution of the passive tracer gas at the height of 0.29H for the different urban set-ups is shown in Fig. 5.1. The figures have the same exponential concentration scale that examines a clear insight to the plume structure at all positions as well as a direct comparison of the results for different set-ups. There were approximately 130 measurement points per set-up and they are depicted in Fig. 5.1 by small black diamonds. An interpolation was used to create the contour plots.

The differences between plume shapes are evident. The wide spread of the tracer immediately after release, i.e., inside the street canyon, where the point source was located, is distinctive for the aligned set-ups. In this case all the pollutant is kept in the range $y \in (-4; 4)$ independently on the packing densities. On the other hand, the initial spread is not so wide and the concentrations reach high values only behind the central cube and the surrounding street canyons for the staggered set-ups. The lateral spread continues downstream creating a triangle shape plume rather than rectangle shape as in the case of the aligned arrays. The low-packed set-ups showed slightly larger area, where the threshold was exceeded than denser packed set-ups considering the concentration thresholds given in Fig. 5.1. The plume shape differs according to arrangement pattern. The plume is longer but thinner in the case of staggered arrays. We computed the area integral of the exceeding area for all set-ups and the thresholds depicted in Fig. 5.1. The exceeding area for given threshold was always smaller in the case of staggered arrays than within the aligned set-ups. This means that the ground level area threatened by the toxic gas release is larger due to weaker vertical transport in the less dense and aligned set-ups.

The comparison with previous experiments (e.g. by MacDonald et al. [1] and Marcolongo [2]) is difficult because of the significant difference in the release conditions. In the other studies the tracer gas was emitted in front of the building array to the boundary layer free stream. Therefore a wide lateral spreading of the tracer close to the source was not observed. The difference in the plume shape for aligned/staggered set-ups was clearly shown in Fig. 5.1 due to much better spatial resolution of the measurements than the previous experiments had.

6. Vertical flux of passive contaminant. There are two main directions of passive contaminant transport, horizontal and vertical advection. Low dense and aligned set-ups allow higher wind speeds at the street level compared to denser and staggered set-ups. However, smaller concentration at lower elevations in the case of the denser set-ups is caused by the enhanced vertical transport of passive tracer. The vertical wind speed component and the concentration of passive tracer gas were measured simultaneously at one place to obtain the normalised vertical advective and turbulent vertical transport.

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The assignment of the turbulent flux vector to the grid cell where the turbulent wind direction is assumed to be blowing is based on the wind direction at the source location. The turbulence vertical transport is defined as the component of the advective vertical transport of passive tracer. The vertical wind component at the height of 0.29H for the different urban set-ups is shown in Fig. 5.1. The mean vertical wind speed is negative in the windward regions and positive in the leeward regions following the well-known street canyon vortex layout [12]. Therefore, the sign of the vertical turbulent transport is given. Data in Fig. 5.1 show that the turbulent vertical flux has always opposite sign to the advective flux, although the magnitude is much smaller within the urban canopy (note that the scales of vertical fluxes are different for advective and turbulent cases). The turbulent transport reaches the same...
Fig. 6.2. Comparisons of the normalized vertical turbulent, advective, and momentum transports in leeward and windward positions.

DISPERSION OF POLLUTANTS WITHIN VARIOUS URBAN CANOPIES

Vertically aligned buildings enhance both advective and turbulent transports of passive contaminants. The urban canopies should be designed to present aligned layouts, the more variable layouts will be applied the better. Urban canopy should be composed of lists of small structures build close to each other, i.e. with high packing densities, rather than large-blocks buildings.

Acknowledgement. This work was conducted by the financial support of the Japan Society for Promotion in Science. Authors also would like to thanks the project AV0Z50760514 of the Ministry of Education, Youth and Sports of the Czech Republic.

REFERENCES


NUMERICAL STUDY OF THE EFFECT OF DYNAMIC CAPILLARY PRESSURE IN POROUS MEDIUM

Radek Fichek*, Zbynek Smrkovsky, Yoshihiro Sasaki, Tina S. Iljangsamkern

Abstract. In order to investigate effects of the dynamic capillary pressure-saturation relationship used in the modelling of flow in porous mediums, a two-dimensional fully implicit mathematical scheme was proposed and its validity is demonstrated by means of numerical simulations developed by the authors and others. The numerical scheme is based on the simplest experimental observations. The presented fully implicit mathematical scheme is convergent and can be used for simulation of capillary effects in porous media. The numerical scheme is convergent and can be used for simulation of capillary effects in porous media. The numerical scheme is convergent and can be used for simulation of capillary effects in porous media.

1. Introduction. In the understanding and prediction of the flow of immiscible and incompressible fluids in porous mediums, a reliable model of capillary forces acting on the fluids is crucial. In past decades, various capillary models were correlated from laboratory experiments in equilibrium conditions. These static capillary pressure - saturation relationships such as [4] or [28] have been used in almost all mathematical models on modelling of multiphase flow in porous mediums. However, soil physicists found that the laboratory measured capillary pressures do not correspond to capillary pressures in case of large water bodies. Recently, as a result of the empirical approach in [26], theoretical studies [13, 14, 16, 15, 7, 8] were proposed and produced new aspects in the two-phase flow theories. The most important result is that the classical capillary pressure - saturation relationship holds only in the state of thermodynamic equilibrium. Therefore, it is believed that the classical approach cannot be used in the modelling of capillary when the fluid content is in motion and a new model of the capillary pressure - saturation relationship is proposed, i.e., the dynamic capillary pressure [13, 14, 16, 15].

This manuscript focuses on the implications of the dynamic capillary pressure - saturation relationship. The fully implicit numerical scheme is proposed and validated using the (semi-)analytical solution for the static capillary pressures [22, 30], and [41]. By estimating the experimental order of convergence, it is shown that the numerical scheme is convergent and can be used for simulating flow in both homogenous and heterogeneous porous mediums. Consequently, the inclusion of various models of dynamic capillary pressure coefficient is investigated and compared to the static model of capillary pressure.

The two phase flow system can be simplified to the Richards problem, where the pressure of the non-wetting phase (air or oil) is assumed to be constant. This is the case in [38], where the dynamic effects is found to be relevant for the given structure of heterogeneous porous medium. Other numerical approaches using the dynamic capillary pressures have been already studied for instance in [32, 28], or [24]. However, the relevance of using the dynamic capillary pressures in the full two-phase flow system of equations has not been answered yet. The presented fully implicit numerical schemes were presented and discussed in [13, 14, 16, 15].
Dynamic Capillary Pressure in Porous Media

The ongoing research is to study the capillarity in heterogeneous porous media, other capillary pressure - saturation model (like that by van Genuchten [28]) which does not include the barrier effect will not be considered in this manuscript.

The dynamic capillary pressure - saturation relationship is proposed in the following form [44]:

\[ p_{ci} \rightarrow p_{ci-1} = f'\left(\frac{\Delta s_{ci}}{\Delta t}\right) \]

where \( p_{ci} \) is the capillary pressure - saturation relationship in the thermodynamic equilibrium of the system (referred to as the static capillary pressure) and \( f'(\cdot) \), \( f''(\cdot) \), are the dynamic effect coefficient, in a material property of the system.

Early in 1978, before the thermodynamic definition of (1.4) in [44], Stadler [28] observed the dynamic effect in laboratory experiments and proposed a linear dependence in (1.4) with the following definition of \( \tau \)

\[ \tau_{ci} = \frac{\partial p_{ci}}{\partial S_{ci}} V_{ci} \]

where \( \tau_{ci} \) is 0.1 \( \Delta t \) a scalar parameter. Both \( \Delta t \) and \( \tau_{ci} \) are the Brooks and Corey parameters [4] that can be experimentally estimated. Other symbols are described in Table 5.1.

The Stadler model for the dynamic effect coefficient \( \tau_{ci} \) was obtained by correlating experimental data. The value of \( \tau_{ci} \) varies by \( 0.2 \times 10^{-3} P_s a s \) estimated in [15].

Moreover, different models [24,26] take into account for the order of \( 10^{-3} P_s a s \) estimated in [15].

1.3. Interface of porous media. At the interface of two different porous media, the normal components of fluxes of both fluids present in the system are continuous due to conservation of mass [17]. If the fluid phase is present at the interface, its pressure is also continuous. Consequently, following the definition (1.2), the capillary pressure is continuous across the interface. In the case of the dynamic capillary pressure (1.4), this condition yields

\[ \frac{\partial m_{ci}}{\partial x} = 0 \]

where the subscripts I and II enumerate the two different porous media, respectively. As a consequence of the continuity of capillary pressure (both static and dynamic), saturation \( S_{ci} \) can be discontinuous across the interface, see [15] or [3].

2. Mathematical model. The mathematical model describing the two-phase flow in a one-dimensional domain is presented in this section. The aim is to investiga
t the inclusion of the dynamic capillary pressure (1.4) instead of the static relationship (1.3) influences the numerical solution of the resulting system of equa
tions.

If the capillary pressure function \( p_{ci}(S_{ci}) \) is strictly decreasing in the case of the Brooks and Corey model (1.3), the function evaluated at both sides of (2.4)

\[ \frac{\partial f(\xi)}{\partial \xi} = \frac{\Delta i - \gamma}{\Delta \xi} \frac{\Delta s_{ci}}{\Delta \xi} \]

is strictly increasing if \( \gamma \) is a non-decreasing function of \( \xi \). Thus, the existence of the inverse function \( f^{-1} \) is guaranteed and the equation (2.4) do not allow the convergence of the numerical scheme.

The numerical scheme is solved using the Newton-Raphson iteration method, where the Jacobian is block tridiagonal. In each iteration, the upstream sat
tuations in (2.3) is re-computed using the solution of the interface and the interfacial capillary pressure condition (2.4) is solved numerically together with (2.2) and (2.3).
Dynamic Capillary Diffusion in Porous Media

The Ohji sand was used as a porous medium in the following numerical simulations with water and air as a wetting and non-wetting fluid, respectively. Physical properties of the sand shown in Table 3.1 were measured during the laboratory experiment held in the Center for Experimental Study of Subsurface Environmental Processes, Colorado School of Mines. The details of the fluid properties are shown in Table 3.3.

Except for the pure advection problem, the numerical solution is computed also for the models of the dynamic capillary diffusion (1.4). The value of the dynamic effect coefficient \( r = r(\tau) \) was estimated as the result of the laboratory experiment, where capillary pressure and time evolution of the water saturation were measured. These functional models of the dynamic effect coefficient \( r = r(\tau) \) were correlated, see Table 3.2. Additionally, the Starner model (1.5) gave \( r = 3.3 \times 10^3 \) for the Ohji sand. In the following subsections, numerical solutions using these dynamic capillary diffusion models are compared to the reference numerical solutions computed with the static capillary pressure model \( p_0 = p_0^\text{ref} \).

Unfortunately, no laboratory data is available for the case of a simple heterogeneous porous medium described in Sections 3.4 and 3.5. In order to investigate solutions for the different models of the dynamic effect coefficient \( r = r(\tau) \) in a heterogeneous porous medium, a fictive porous media Ohji and is introduced. Its parameters are the same as for the Ohji sand except for the capillary pressure \( p_0 \), the intrinsic permeability \( K \), and \( \tau \), which are all multiplied by the factor 0.9 (see Table 3.1).

### Table 3.1

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<tr>
<th>Parameter</th>
<th>Ohji sand</th>
<th>Ohji x sand</th>
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<tr>
<td>Density</td>
<td>( \rho_w )</td>
<td>( \rho_w )</td>
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<td>Water viscosity</td>
<td>( \eta_w )</td>
<td>( \eta_w )</td>
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<td>Air viscosity</td>
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<tr>
<td>Dynamic viscosity</td>
<td>( \eta_{dy} )</td>
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### 3.1. Pure advection

Assuming \( p_0 = 0 \), the system of equations (2.1) can be simplified into a single hyperbolic equation with \( \partial_0 c = \partial_0 u = 0 \). As for the pure advection problem for the Ohji sand and it is a spherical solution obtained with static capillary pressure. The experimental order of convergence indicates the inter-

Dynamic Capillary Diffusion in Porous Media

\( \frac{d}{dx} [p(x, t)] = 0 \)

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<td>Sand</td>
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### 3.2. Pure capillary diffusion in homogeneous medium

### 3.3. Advection and capillary diffusion in homogeneous medium

### Table 3.4

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### 3.4. Pure capillary diffusion in heterogeneous medium

### Table 3.5

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<td>Ohji sand</td>
<td>Ohji x sand</td>
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### 3.5. Advection and capillary diffusion in heterogeneous medium

### Figure 3.6

### Figure 3.7

### Figure 3.8

### Figure 3.9

### Figure 3.10

### Figure 3.11

obtain semi-analytical solution, which has been discussed by the authors in [10] and [11]. The description of the problem is given in Table 3.6. The main benefit of such a closed-form solution is direct comparison of effects of both advection and capillarity on the two-phase flow. The numerical solutions compared to the McWhorter and Sunada semi-analytical solution are shown in Figure 3.4 and, again, the experimental errors of convergence show convergence of the numerical solution towards the exact solution.

The necessity of solution of the PDE (1.4) requires careful handling of the boundary conditions. Despite simplicity at \( t = 0 \), the entry flux of air, denoted as \( u_0 \), is integrable and thus the simplification of such a condition is possible. The numerical solutions for the dynamic capillary pressure models are shown in Figure 3.5. Compared to the static, linear, or log-linear cases, the inclusion of the constant model of \( r \) in the capillary pressure leads to a non-monotone profile of \( p_0 = p_0^\text{ref} \).

3.4. Pure capillary diffusion in heterogeneous medium. If the porous medium described in Section 3.2 has a single discontinuity in the material properties, i.e., it consists of two homogeneous porous media, the system of equation (2.1) can be reduced to the two-dimensional domain is governed only by capillarity and the system of equation (2.1) can be reformulated to satisfy the McWhorter and Sunada problem formulation for the case of a bi-directional fluid displacement, see [22], [11]. Therefore, the McWhorter and Sunada semi-analytical solution for the pure capillary diffusion problem can be obtained as a benchmark analytically in the numerical solution. The problem details are shown in Table 3.5.

As the numerical grid gets finer, the agreement of the numerical solution with respect to the semi-analytical solution is apparent as it is shown in Figure 3.3. The numerical solutions to the dynamic capillary pressure problem are shown in Figure 3.3.
3.5. Advection and capillary diffusion in heterogeneous medium. In 2008, Falk et al. [11] generalized the van Duijn and de Neef problem formulation for the inclusion of the McWhorter and Sunada solution in homogeneous porous medium. The resulting problem formulation requires that the van Duijn and de Neef initial saturation distribution and the McWhorter and Sunada boundary fluxes are prescribed, see Table 3.8.

In Figure 3.8, the numerical solutions are compared to the semi-analytical solution and the experimental error of convergence is shown.

The numerical solutions of the problem for the dynamic capillary diffusion models are shown in Figure 3.9.

case of the heterogeneous porous medium is determined correctly and, moreover, the presence of a heterogeneity does not influence the experimental order of convergence.

The inclusion of the dynamic capillary diffusion models (constant, linear, and loglinear model of \( \tau \)) is important in cases where there is a significant temporal change in the saturation \( S_n \) since the temporal derivative of \( S_n \) is multiplied by the dynamic effect coefficient \( \tau \), see (1.4). This occurs when the advection together with the capillary diffusion dominates the displacement as it is shown in Figures 3.3 and 3.5.

As shown in Figure 3.5, the use of the constant model for the dynamic effect coefficient \( \tau \) changes the monotonicity of the capillary pressure profile which may be physically unrealistic. Therefore, the constant model requires further investigation of its validity. On the other hand, the use of the linear and the loglinear models of \( \tau \) does not seem to be important in the homogeneous porous medium since the air saturation and capillary pressure profiles are similar to the profiles computed with the static capillary pressure (compare Figures 3.3 and 3.5).

In the case of a heterogeneous medium, the inclusion of the dynamic capillary pressure may substantially change the simulated evolution of the flow since the entry pressure of the finer porous media can be achieved sooner or later than in the static case as it is shown in Figures 1.7 and 3.9. This indicates that the conclusions published for the case of Richards equation in [18] are not valid for the full system of equations of the two-phase flow.

4. Conclusion. This manuscript presents a one-dimensional numerical scheme of two-phase incompressible and immiscible flow that enables for simulating non-stationary capillary pressure models in both homogeneous and heterogeneous porous media. The numerical scheme is validated and its order of convergence is estimated using the analytical and semi-analytical solutions for advection, diffusion and, diffusion dominated problems, respectively.

Laboratory measured parameters were used in the numerical simulation of the dynamic capillary pressure including three main models of the dynamic effect coefficient \( \tau \) - constant, linear, and loglinear. The numerical solutions for the dynamic capillary pressure show that the dynamic effect has significant impact on the magnitude of the capillary pressure while the change in the saturation profiles may be considered negligible in some cases. The constant model of \( \tau \) showed rather unrealistic profile of the numerical approximation of the capillary pressure because the spatial monotonicity was different with respect to the results obtained with the static capillary pressure model.

Results of the simulation indicate that the dynamic effect may not be so important in drainage problems in a homogeneous porous medium, but, on the other hand, it is of a great importance in highly heterogeneous media where the capillary governs flow through material interfaces.

5. List of symbols.

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Dynamic Capillary Flow in Porous Media

REFERENCES

NUMERICAL SIMULATION OF 3D TRANSONIC INVISCID FLOW OVER A SWEEP WING

PETR FVRMÁNEK1, JIRI DOREJ2, JIRI PURMT2, JAROSLAV FORT2, MILAN KLADRUBSKÝ1, Karel KOZEL1

Abstract. The aim of this work is to investigate some of the phenomena occurring in the numerical solution of transonic flows in order aerodynamics. A comparison of various computational schemes based on the Finite-Volume Method (FVM) is presented, particularly their ability to capture discontinuities and to ensure the conservation of physical quantities. The flow over a swept wing is considered. The results are evaluated both in simulations and with the experimental data.

1. Introduction. This paper compares four different FVM schemes based on various approaches. The first two are derived from classical Lax-Wendroff scheme - a modification of predictor-corrector MacCormack scheme in TVD (Total Variation Diminishing) cell-centered form and Ron-Ho-Ni scheme in cell-vertex form.

2. Mathematical Model. Inviscid compressible flow in three dimensions is described by the system of Euler equations, which can be written in the following conservative vector form:

\[
\begin{align*}
W_t + F_x + G_y + H_z &= 0, \\
\end{align*}
\]

where:

\[
\begin{align*}
W &= (\rho, \rho u, \rho v, \rho w, \rho e)^T, \\
F &= (\rho u, \rho u^2 + p, \rho u v, \rho u w, (\rho e + p) u)^T, \\
G &= (\rho v, \rho u v, \rho v^2 + p, \rho v w, (\rho e + p) v)^T, \\
H &= (\rho w, \rho u w, \rho v w, \rho w^2 + p, (\rho e + p) w)^T.
\end{align*}
\]

W is a vector of conservation variables with components: \(\rho\) - density, \(u, v, w\) - velocity vector, \(e\) - total energy per unit volume and \(p\) - static pressure. \(F, G, H\) are inviscid fluxes. System (2.1) is enclosed by the Equation of State:

\[
\gamma = \frac{c^2}{\rho},
\]

where \(c\) and \(\rho\) are specific heat capacities under constant pressure (at constant volume).

3. Numerical Methods. A classic computational fluid dynamics (CFD) validation case - the Onera M6 wing was chosen for the numerical testing. Initial conditions were the same as in the AGARD AR31 report [7] i.e. initial Mach number \(M_0 = 0.805\) and angle of attack \(\alpha = 3.06\)°. System (2.1) was solved using following FVM schemes.

Method 1: The first method is a classical 3D MacCormack predictor-corrector scheme in so called Modified Cauchy's form was chosen [8]. It's influence in each spatial dimension is given by a switch dependent on the value of gradient of the vector \(W\). Although this scheme does not possess the TVD property, it is able to deliver even better results than TVD variant of MacCormack scheme.

Method 2: In this case the cell-vertex Ron-Ho-Ni scheme described in [2], [5] was used. It is one-step explicit Lax-Wendroff scheme with \(\frac{3}{2}\)th order Jameson's artificial dissipation. The same computational mesh as for Method 1 was used.

Method 3: Considering the first method, the computational area was discretized by an unstructured mesh with quadrilateral computational cells. The problem was solved by FVM in a cell-centered formulation. The Roe-Benjamin solver [5] was used to solve the Riemann problem on each side of each finite volume. Spatial accuracy of the method was increased by linear reconstruction using the least square method [6]. For the time discretization the linearized backward Euler method was used. Final system of equations was solved by GMRES method with ILU(0) preconditioning [9].

Method 4: Last used method is an extension of the high-order FVM weighted least-squares (WLSQR) scheme mentioned in [4] into three dimensions. The high order WLSQR reconstruction is combined with the HLLC [15] flux and the resulting semi-discrete system of equations is solved by the linearized backward Euler method. Bonding sparse system of linear equations is then solved with GMRES method [6] with modified GMRES preconditioner.

The boundary conditions were considered in a standard way for transonic compressible flow. At the inflow boundary the variables \((u, v, p)\) were given by initial values \((u, v, p) = (U, V, P)\), at the outflow boundary the outlet pressure was given by \(p = p_{\infty}\) and on the wall the slip condition was prescribed as \((u, v, p) = 0\).

Fig. 4.1: Computational meshes.

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Fig. 4.2: Method 2 - isolines of the \(c_p\) coefficient and Mach number.
Fig. 4.4: Methods 3 and 4, unstructured mesh. Isolines of $c_p$ coefficient and Mach number alongside the wing.

Fig. 4.5: Methods 1 and 2, comparison with the experimental results. Behaviour of the $c_p$ in the cut ($x$ - spatial coordinate, $c$ - chord): a) 20 %, b) 44 %, c) 65 %, d) 80 %, e) 90 %, f) 95 % of the wing spread.

Numerical Simulation of 3D Transonic Inviscid Flow over a Swept Wing

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4. Numerical Results. As can be seen from the comparison between the numerical and experimental results in the case of 3D flow over the Omega M6 wing (Figures 4.1 - 4.6), the schemes based on classical Lax-Wendroff approach still deliver satisfactory results, and can be used as a reliable numerical simulation.

Both the position and intensity of shockwaves are captured reasonably well, only the Method No.2 pushes the shockwave closer to the trailing edge (Fig.4.5) - more than the other methods. The observed differences are highly probable consequence of inviscid nature of the chosen model, which collides with viscous turbulent behaviour of the physical reality. Another constraint is explicit form of the classical schemes, which restrain their usability in three dimensional computations only to inviscid flows (both steady and unsteady). But considering the possibility of these conversion into an implicit variant, both can be successfully used also for the turbulent simulations.

The modern schemes deliver a good results too, particularly the Method 4 (WLSQR scheme) shows a very good correspondence with the experimental data. Method 3 smoothes the shockwaves more than the other methods, most probably because of insufficient density of the computational mesh.

5. Conclusion. Proposed FVM schemes for numerical solution of stationary 3D transonic inviscid flow show reasonably good accuracy and efficiency. Although the inviscid mathematical model has been chosen, the schemes were able to capture important flow characteristics as the position and intensity of the shockwaves and proved themselves as a reliable numerical simulations of investigated cases. Some of the schemes however would need some further improvement (implicit form in the case of Modified Cusen scheme, matrix-free GMRES in the case of WLSQR scheme).

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For a comprehensive lemma for evaluating differentials of a stochastic portfolio that is balanced by buying parabolic PDE is derived by constructing a riskless portfolio of bonds and using Itô's lemma for evaluating differentials of a stochastic process. This is called a Wiener process if $w_t$ is a normal distribution.

The volatility of the short rate process is given in a stochastic differential equation: $d\sigma (t,r) = \lambda \gamma \sigma (t,r) \, dt + \sigma (t,r) \, dw$, where $\sigma > 0$ is a positive constant. Here $\sigma$ is a proxy for volatility and $\lambda$ is a parameter that controls the rate of mean reversion.

For a constant market price of risk, the conditional PDE for bond prices has an explicit solution $P(t,r) = \exp(-\int_t^T \kappa (s) \, ds)$ for $r > 0$. In this model, for a given market price of risk, the bond price is determined by the parameters $\kappa$, $\theta$, and $\sigma$.

We also remind ourselves that bond prices determine interest rates $R(t,r)$ by the formula $P = e^{-\int_T^t R(s) \, ds}$, i.e., $R(t,r) = -\frac{1}{\kappa} \ln P(t,r)$.

One of the first models of the class (1.1) has been proposed by Oldrich Vasicek in [14]. In this model, the short rate process is assumed to follow a stochastic differential equation: $d\sigma (t,r) = \lambda \gamma \sigma (t,r) \, dt + \sigma (t,r) \, dw$.

In one-factor models, term structure of interest rates is a function of the short rate and model parameters. However, it means that the parameters of the model are not uniquely determined. This is a simplification from the reality, as it can be seen in Fig. 1.2.

In one-factor models, term structure of interest rates is a function of the short rate and model parameters. However, it means that as soon as the parameters of the model are chosen, the term structure corresponding to a given short rate is uniquely determined. This is a simplification from the reality, as it can be seen in Fig. 1.2.

On NON-EXISTENCE OF A ONE FACTOR INTEREST RATE MODEL FOR VOLATILITY AVERAGED GENERALIZED FONG-VASICEK TERM STRUCTURES

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1. Overview of term structure modeling. The purpose of this paper is to study the generalized Fong-Vasicek two-factor interest rate model with stochastic volatility.

In this model the dispersion of the stochastic short rate (square of volatility) is assumed to be stochastic as well and it follows a non-negative process with volatility proportional to the square root of dispersion. The drift of the stochastic process for the dispersion is assumed to be in a rather general form including, in particular, linear function having one root (yielding the original Fong-Vasicek model, cf. [9]) or a cubic function having three roots (yielding a generalized Fong-Vasicek model for description of the volatility clustering, see e.g. [12]). We consider averaged bond prices with respect to the limiting distribution of stochastic dispersion. The averaged bond prices depend on time and current level of the short rate like it is the case in many popular one-factor interest rate model including, in particular, the Vasicek and Cox-Ingersoll-Ross model. However, as a main result of this paper we show that there is no such one-factor model yielding the same bond prices as the averaged values described above.

Key words. Stochastic term structure models, generalized Fong-Vasicek interest rate model, stochastic volatility, stochastic differential equation, averaging, limiting density.

AMS subject classifications. 60H05 65C30 91B28 91B70

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In this model, for a given market price of risk, the bond price is determined by the parameters $\kappa$, $\theta$, and $\sigma$.
models are introduced. In the two-factor models there are two sources of uncertainty yielding different term structure for the same short rate. They may depend on the value of the other factor. Moreover, two-factor models have more variety of possible shapes of term structures.

A general two-factor model with the factors $x$, $y$ is given by the system of SDEs:

\[ \begin{align*}
    dx &= \mu_x dt + \sigma_x dw_x, \\
    dy &= \mu_y dt + \sigma_y dw_y,
\end{align*} \]

where correlation between $dw_x$ and $dw_y$ is assumed to be constant $\rho \in [-1, 1]$, i.e. $E(dw_x dw_y) = \rho dt$.

The short rate is a function of these two factors, i.e. $r = r(x, y)$.

Let us denote by $P(t, x, y)$ the price of a zero coupon bond with maturity $T$, at the time $t$ when the values of the factors are $x$ and $y$. The FPE satisfied by the bond price, which reads (cf. [9], Chapter 7):

\[
\frac{\partial P}{\partial t} + \left(\mu_r - \kappa_r \sigma_r^2 - \frac{\theta_r}{2}\right) \frac{\partial P}{\partial r} + \kappa_r \sigma_r \rho \sigma_y \frac{\partial P}{\partial y} + \kappa_r \sigma_r \frac{\partial^2 P}{\partial y^2} = 0.
\]

The parameters $\lambda_r, \lambda_y$ stand for market prices of risk corresponding to factors $x$, $y$. The equation for the bond price is equipped by the terminal condition at $t = T, P(T, x, y) = 1$ for any $x, y$.

There are several ways of incorporating the second stochastic factor. Based on experimental data from real financial market it is reasonable to make an assumption that the market changes the volatility of the underlying process for the short rate. More precisely, the volatility of the stochastic process for the short rate is stochastic as well.

An empirical confirmation of such an assumption can be found e.g. in the recent paper by the authors [13]. In the so-called two-factor model with a stochastic volatility we allow the volatility to have a stochastic behavior driven by another stochastic differential equation. As an example of such a two-factor model one can consider the Fong-Vasicek model (cf. [8]) where the volatility stochastic follows a mean reverting Bessel-square root process. Another possibility is to consider the generalized Fong-Vasicek model in which the drift function is no longer linear but it may have cubic-like behavior having three distinct roots representing possible steady states for dispersion. By this way we can model so-called volatility clustering phenomena observed in real markets (see [13] for details). Now, as a consequence of the multi-dimensional Itô’s lemma the corresponding equation for the bond price is a linear parabolic equation in two space dimensions. These spatial dimensions correspond to the short rate and volatility.

Let us consider the process a Bessel square root process with a general drift function $\alpha$:

\[
\frac{dr}{r} = \alpha(r)dt + \sqrt{r}dw.
\]

It is well known that the density distribution of a stochastic process is a solution to the Fokker-Planck partial differential equation (see [4]). Recall that the cumulative distribution function $F = F(x, t) = P(X(t) \leq x)$ of the process $x(t)$ satisfying (1.8) and starting almost surely from the initial density $\nu_0$ can be obtained from a solution $F = \Phi(x)$ to the so-called Fokker-Planck equation for the density function:

\[
\frac{\partial F}{\partial t} = \frac{\partial}{\partial x} \left( -\theta(x) F \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \sigma^2(x) F \right).
\]

Here $\nu(\cdot; \cdot)$ denotes the Dirac delta function located at $\nu$. The limiting density $\nu(x)$ is obtained by setting $t \to \infty$ in the Fokker-Planck equation (1.9). A stationary solution satisfying $\nu(x) = 0$ for $x \leq 0$ is therefore a solution to the differential equation (see [15]).

Concerning structural assumption made on the drift function $\alpha = R - B$ we shall henceforth assume the following hypothesis:

\[
(4) \quad \alpha = \alpha(y) > 0.
\]

Now it follows from the assumption $(4)$ made on the drift function $\alpha$, and [3], Lemma 2 that the stationary Fokker-Planck equation (1.10) has a solution $\nu$ that can be explicitly expressed as:

\[
\nu(y) = e^{-\gamma y} \exp \left( \int_{y_0}^{y} \frac{\theta(x)}{\sigma^2(x)} dx \right) C > 0.
\]

We show that the left hand side is a function of $y$. We denote it by $\nu(y)$, i.e. $\nu(y) = \nu_0(y)e^{-\gamma y}$. Then, the stationary bond price function $\nu(y)$ is a function $y$.

We know that $\frac{\partial P}{\partial t} + \kappa_r \sigma_r \rho \frac{\partial P}{\partial y} + \kappa_r \sigma_r \frac{\partial^2 P}{\partial y^2} = 0$ is the general form of the bond price function $P(t, r)$ for any $t \geq 0$ and $r \geq 0$.

A stationary solution satisfying $\nu(x) = 0$ for $x \leq 0$ is therefore a solution to the differential equation (see [15]).

\[
\frac{\partial^2 F}{\partial y^2} + \frac{\partial}{\partial y} \left( \frac{\theta(x)}{\sigma^2(x)} F \right) + \frac{\partial}{\partial y} \left( \frac{\theta(x)}{\sigma^2(x)} F \right) = 0.
\]

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A stationary solution satisfying $\nu(x) = 0$ for $x \leq 0$ is therefore a solution to the differential equation (see [15]).
On non-existence of a one factor interest rate model for volatility averaged term structures

Furthermore, by (1.10), we have
\[ \int_0^\infty \int_0^\infty \left[ \frac{\partial}{\partial y} \rho(y, p) \right] dy dp = \frac{1}{2} \int_0^\infty \rho(y) \left( 1 - e^\lambda y \right) dy \]

Therefore
\[ a'(T, y) = \left. \left( \frac{\partial}{\partial y} \rho(y, p) \right) \right|_{y=0} = \frac{1}{2} \lambda \int_0^\infty \rho(y) \left( 1 - e^\lambda y \right) dy \]

Now taking into account PDE (2.6) for the function \( A(T, y) \), we can recursively evaluate
\[ A(T, y) = \frac{1}{2} \lambda \int_0^T \rho(y) \left( 1 - e^\lambda y \right) dy \]

for any \( y > 0 \). Using the above expressions and the identities \( B(T) = \frac{\lambda}{1 - e^\lambda T} \) and \( B'(T) = \frac{\lambda^2}{1 - e^\lambda T} \), after straightforward computations we obtain
\[ \frac{\partial}{\partial y} A(T, y) = -a(T, y) \]

and by comparing (2.15) and (2.19) we obtain the expression for the constant \( K \) in terms of the model parameters as
\[ K = \frac{\lambda}{1 - e^\lambda T} \]

Computing the next derivative we end up with
\[ \frac{\partial}{\partial y} a''(T, y) = -a(T, y) + \lambda a'(y, T) \]

and by comparing (2.16) and (2.21) we can express the volatility \( \bar{\Omega} \) as
\[ \bar{\Omega} = \frac{\lambda}{1 - e^\lambda T} \]

Notice that the PDE for the averaged bond term now reads as follows:
\[ \frac{\partial}{\partial y} a''(T, y) = -a(T, y) + \lambda a'(y, T) \]

and by comparing (2.16) and (2.21) we can express the volatility \( \bar{\Omega} \) as
\[ \bar{\Omega} = \frac{\lambda}{1 - e^\lambda T} \]

In order to achieve contradiction we finally compute the fourth derivative as
\[ a''''(T, y) = 3\lambda^2 d - 4\lambda da + a^2 \lambda - 2 \int_0^y \left( \frac{\partial}{\partial y} \rho(y) \right) \left( 1 + \lambda y \right) dy \]

Comparing (2.17) and (2.22) we get the condition
\[ \sigma^2 = -\frac{1}{2} \lambda \int_0^y \rho(y) \left( 1 + \lambda y \right) dy \]

and concerning the initial data we assume
\[ \sigma_{\text{in}}, \alpha_0 \in C^2([0, T]) \]

and
\[ \sigma_{\text{in}}, \alpha_0 \in C^2([0, T]) \]

Moreover, we assume
\[ \sigma_{\text{in}}, \alpha_0 \in C^2([0, T]) \]

and we then have to consider the boundary condition for all \( T > 0 \).

More precisely, for \( y \to 0 \), we have
\[ \frac{\partial}{\partial y} A(T, y) = 0 \]

and for \( y \to \infty \), we have
\[ \frac{\partial}{\partial y} A(T, y) = \frac{\partial}{\partial y} \sigma_{\text{in}}(y) \]

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and for \( y \to \infty \), we have
\[ \frac{\partial}{\partial y} A(T, y) = \frac{\partial}{\partial y} \sigma_{\text{in}}(y) \]
Theorem 1.3. Assume that (1.9) is satisfied in addition to (1.5). Moreover, suppose that
\[ \sigma_0 \equiv \sigma_0(t) = 0 \quad \text{and} \quad \sigma_0 \equiv \sigma_0(t) = 0. \]
Then the initial-boundary-value problem (1.1)-(1.4) has a unique solution for all \( t > 0 \). Moreover, there are positive constants \( A \) and \( \mu \) which are dependent only on \( p, q, r, s \) and \( \eta_1 \), and a positive constant \( C \) depending on an initial data \((A_0(x), H_0(x))\) such that
\[ e^{-\min \{ A_0(x), H_0(x) \}} \leq e^{-\min \{ A_0(x), H_0(x) \}} \leq e^{-\min \{ A_0(x), H_0(x) \}} \leq C t^{\mu} \]
for all \( t > 0 \) and \( x \in \mathbb{R}^d \).

As we mentioned above, it is assumed that a change in cells or tissue takes place in the region where the activator concentration is high. Therefore, nontrivial spatial patterns of the activator are expected to emerge. But in some numerical simulations, it is observed that a solution starting from an almost uniform initial value develops localization in the activator concentration for a while, but it oscillates and eventually converges uniformly to the trivial state at \( t \rightarrow \infty \). In fact, Wu and Li [12] proved that if \( \sigma_0(x) = 0 \) and \( \sigma_0(x) = 0 \) and \( \sigma \geq \sigma_0(x) > 0 \) uniformly on \( \mathbb{R}^d \) and \( t \rightarrow \infty \). We call such a phenomenon the collapse of patterns. Theorem 1.1 implies pattern never collapse as long as \( \sigma_0 \) is nontrivial, whereas for lack of uniform lower bound for the activator, Theorem 1.2 suggests that patterns in the activator can collapse even though \( \sigma_0 \) is nontrivial.

Hence, the activator concentration may fail to form spatial or spatial-temporal patterns if the system does not have a positive basic production term for the activator. In the following section, we will study this phenomenon to understand a role of the basic production terms in pattern formation.

Some remarks are in order. First, for the systematic study of global behavior of solutions of (1.1)-(1.4), it is important to know the behavior of solutions of the following kinetic system:

\[ \begin{align*}
\frac{dx}{dt} &= -u + x^p \sigma_0,
\frac{dv}{dt} &= -v + x^q \sigma_0.
\end{align*} \]

(1.10)

Here \( \sigma_0 \) and \( \sigma_0 \) are both nonnegative constants. When \( \sigma_0 = 0 \) and \( \sigma_0 = 0 \), we have obtained the complete understanding of all the behavior of solution orbits [9]. The case \( \sigma_0 > 0 \) is treated in an ongoing project.

Second, all the three theorems assume that \( r > p, q, s \), which is important to rule out the occurrence of finite time blow-up of solutions. Indeed, it has been shown in [5] and [8] that if \( p > r > q > s \), then there exist solutions of (1.1)-(1.4) with \( \sigma_0(x) = \sigma_0(x) = 0 \) which blow up in finite time.

Third, in [8] it is proved that if \( p > r > q > s \), then some solutions of (1.1)-(1.4) with \( \sigma_0(x) = \sigma_0(x) = 0 \) exist for all \( t > 0 \), but they are unbounded. By virtue of Theorem 5.1 all solutions are bounded if \( \sigma_0 \) is nontrivial.

2. Collapse of Patterns. The activator concentration may fail to form spatial or spatial-temporal patterns if the system (1.1)-(1.4) does not have a positive basic production term for the activator. In order to understand the mechanism of this phenomenon, we consider the behavior of a solution of the following system as \( t \rightarrow \infty \):

\[ \frac{dx}{dt} = A - A + x^p \sigma_0, \]

(2.1)

\[ \frac{dv}{dt} = D - H - H + v^q \sigma_0. \]

(2.2)

for \( x \in \mathbb{R}^d \) and \( t > 0 \), subject to the boundary condition and the initial condition

\[ A(0, \theta) = A_0(x), \quad H(0, \theta) = H_0(x), \quad v(x, 0) = v_0(x) \quad \text{for} \ x \in \mathbb{R}^d, \]

(2.3)

For the initial data, we assume (1.1) and (1.2). The exponents satisfy (1.5) and we assume

\[ \sigma_0(x) = 0 \quad \text{on} \quad \mathbb{R}^d. \]

There exists a solution of (2.1)-(2.4) for all \( t > 0 \) by Theorem 1.2 if \( p, q > r > s \). The main result of this paper is stated as follows:

Theorem 2.3. Let \( \sigma \geq \sigma_0(x) > 0 \) and assume that the initial data satisfies

\[ \min \{ H_0(x) \} \geq \frac{p - 1}{p - 1} \left( \min \{ A_0(x) \} \right)^{p - 1}. \]

Then the solution \((A, H, v)\) of (2.1)-(2.4) satisfies

\[ 0 < \min \{ A_0(x) \} \leq C t^{-\mu}, \quad \min \{ A_0(x) \} \leq \min \{ H_0(x) \} \leq C t^{-\mu}, \]

in which \( C \) is a positive constant depending on \((A_0(x), H_0(x))\), and \( z(x) \) is a solution of the problem

\[ D \cdot x + e - x \sigma_0(x) = 0 \quad \text{for} \ x \in \mathbb{R}^d, \]

(2.5)

\[ v(x, 0) = v_0(x) \quad \text{on} \ \mathbb{R}^d. \]

(2.6)

It is to be noted that in contrast to Theorem 1.2 and 1.3 we do not assume any further condition other than \( (1.5) \) in Theorem 2.1, yet we have a bounded solutions for all \( t > 0 \) by restricting initial data. To prove Theorem 2.3 we follow the approach due to [12].
Behavior of solutions to an activator-inhibitor system

where we put \( \sigma = \max_{x \in \Omega} \sigma(x) \). Because of (2.17), the right-hand side of (2.28) is estimated as follows:

\[
\frac{\partial v}{\partial t} \leq -\lambda v + C v^{-\gamma} \sigma.
\]

(2.28)

Multiplying both sides of (2.28) by \( v^\gamma \) and putting \( v^{\gamma+1} = W \), we have that

\[
\frac{\partial W}{\partial t} \leq -\lambda W + \frac{\sigma^2}{\gamma+1} + C \sigma^{\gamma+1} v^{-\gamma}.
\]

(2.21)

Using Lemma 2.5 below due to Monch and Takahashi [7], we obtain easily the following lemma:

Lemma 2.4. Suppose that \( \tau \geq q/(q-1) \) and (2.14) is satisfied. Then the initial value problem (2.18)(2.19) has a unique solution \( v(t) \) for all \( t > 0 \). Moreover, it satisfies that

\[
\max_{x \in \Omega} v(t) \leq \sigma.
\]

Now we are ready to obtain upper bounds on \( H(r, t) \). From the maximum principle, it is easy to see that

\[
H(r, t) \leq v(t)
\]

for \( x < r \), \( t > 0 \).

Moreover, it follows from Lemma 2.5 that

\[
\limsup_{t \to \infty} H(r, t) \leq \sigma.
\]

Putting the discussions above together, we obtain

Lemma 2.6. Let \( (A(r, t), H(r, t)) \) be a solution of (2.11). If \( r > q/(q-1) \) and

\[
\lim_{x \to \infty} h_0(x) \leq 1 - q/(q-1)
\]

then there exists \( C > 0 \) such that

\[
ev(h_0 - A(r, t)) \leq C v^{-\gamma} \text{ for all } x < r, t > 0.
\]

(2.27)

2.2. Proof of Theorem 2.1. It is easy to see from (2.25) that \( A(r, t) \to 0 \) uniformly on \( \Omega_{x_0} \), \( t \to +\infty \). We consider the behavior of \( H(r, t) \) as \( t \to +\infty \).

Let \( \tau(z) \) be the unique solution of (2.25) and (2.26). Put \( W(x, t) = H(x, t) - \tau(z) \). Then it is a solution of the following problem:

\[
\frac{\partial W}{\partial t} = D W - W^2 + \frac{\sigma^2}{\gamma+1} + C \sigma^{\gamma+1} v^{-\gamma} \text{ for } x \in \Omega, t > 0.
\]

(2.26)

Let \( G(x, y; \rho, \gamma) \) be the Green function of

\[
\frac{\partial W}{\partial t} = D W - vW + \frac{\sigma^2}{\gamma+1} + C \sigma^{\gamma+1} v^{-\gamma} \text{ for } x \in \Omega, t > 0.
\]

(2.22)

where \( C \) is a positive constant depending on \( C_0, \gamma, U(0) \). Hence it follows that

\[
\max_{x \in \Omega} \tau(x) \leq C \sigma^{-\gamma} v^{-1/(\gamma-1)}
\]

(2.30)

which does not contain \( \tau \) unlike the case \( \sigma_n(x) = 0 \).

The following proposition explains why the collapse can occur for any \( \tau \) in this case.

Proposition 3.1. Let \( f(v, u, y), g(v, u, y) \) satisfy (3.5) and be differentiable with respect to \( u \) and \( y \) in \( u \leq c < \infty, 0 < v < \infty \), and Lipschitz continuous with respect to \( u \) and \( y \) (uniformly in \( x \)); moreover, they satisfy the inequalities\n
\[
\begin{align*}
0 \leq & f(x, u, y) \leq C_1 \sigma^m, \\
0 \leq & g(x, u, y) \leq C_2 \sigma^m, \\
0 \leq & \frac{\partial f}{\partial y}, \frac{\partial g}{\partial y} \leq C_3 v^{-\gamma}, \\
0 \leq & \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u} \leq C_4 v^{-\gamma}, \\
0 \leq & u(x, 0, y) = 1, v(x, 0, y) = 0.
\end{align*}
\]

(3.5)

where \( C_1, C_2, C_3, C_4 \) are positive constants independent of \( z \). As examples of \( g(v, u, y) \), we give (a) the Gray-Mei oxidant system with saturation:

\[
\begin{align*}
& f(x, u, y) = \sigma^m (u + \sigma^{-\gamma}), \\
& g(x, u, y) = v^{-\gamma},
\end{align*}
\]

where \( \sigma > 0 \), and \( \delta > 0 \) and \( \alpha > 0 \) are positive constants.

(a) the activator-inhibitor system proposed by MacWilliams [4]:

\[
\begin{align*}
& f(x, u, y) = \alpha u/\sqrt{1 + \alpha u^2}, \\
& g(x, u, y) = v^{-\gamma},
\end{align*}
\]

where \( \alpha > 0 \) and \( \beta > 0 \) are positive constants. MacWilliams used this model to simulate the lead-ion replacement experiment on lead.

Let \( \sigma_n(x) = 0 \) in (3.1). If we assume \( \max_{x \in \Omega} \sigma_n(x) > 0 \), we see that for any \( q > 0 \) satisfying \( c q < 1 \) there exists a positive number \( l \) such that

\[
\sigma_n(x) \geq \min_{x \in \Omega} \sigma_n(x) + l
\]

for \( x \in \Omega, t > 0 \). Let (U(t)) be a solution of the initial value problem

\[
\begin{align*}
& \frac{\partial u}{\partial t} = -\nu_1 \sigma^m + \nu_2 \sigma^m, \\
& u(0) = \max_{x \in \Omega} \sigma_n(x).
\end{align*}
\]

(3.27)

K. Sakukis and I. Toledo

where we put \( \gamma = [\min_{x \in \Omega} \sigma_n(x) + l]^{-1} \). We obtain that if \( \gamma \sigma_n(0)^{\gamma-1} < 1 \), then \( U(t) \) is monotone decreasing and satisfies

\[
\begin{align*}
& \lim_{t \to \infty} u(x, t) = v \leq C \sigma^{-\gamma} v^{-1/(\gamma-1)}, \\
& U(t) \leq C \sigma^{-\gamma} v^{-1/(\gamma-1)},
\end{align*}
\]

(3.30)

for all \( t > 0 \). Here, \( C \) is a positive constant depending on \( C_0, \gamma, U(0) \). Hence it follows that

\[
\begin{align*}
& \max_{x \in \Omega} \tau(x) \leq C \sigma^{-\gamma} v^{-1/(\gamma-1)}, \\
& \max_{x \in \Omega} v \leq C \sigma^{-\gamma} v^{-1/(\gamma-1)}.
\end{align*}
\]

(3.30)

The assertion is verified by showing that all the eigenvalues of \( L \) have negative real part.

4. Concluding Remarks. Let us consider the system (2.1) (2.4) with \( \sigma_n(x) \geq 0 \) on \( \Omega \). The deactivation of the collapse of patterns is that the activator concentration converges uniformly to the trivial state \( x \in \Omega \). Theorem 2.5 shows that the collapse of patterns occur if \( r > q/(q-1) \) and the initial data is restricted. Moreover, if we assume that \( \max_{x \in \Omega} \sigma_n(x) > 0 \), then collapse of patterns can occur for any \( \tau \), which has been mentioned in Section 3. Therefore, the result may be summarized as follows:

**Main production terms**

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Droplet motion on a plane

Based on a modification of Navier-Stokes equations, such as phase-field methods (see, e.g., [7] or [3]), but our model is substantially different. In [8], it was shown that if the motion is slow, then it can be modeled by a parabolic operator and the approximation of the flow by its steady-state solution. In the paper, we obtain a parabolic problem for the evolution of a scalar volume-preserving membrane with an obstacle and positive contact angle.

We focus on the analysis of such problems. We propose construction of approximate solutions by a variational method and show their convergence to a unique weak solution. Approximation of variational principle to constrained problems of this type is effective. Another method that was successful in abstract analysis of constrained evolutionary problems relies on the technique of subdifferentials and Yosida type approximations. This method is effective. Another method that was successful in abstract analysis of constrained evolutionary problems is the technique of subdifferentials and Yosida type approximations. This method is effective. Another method that was successful in abstract analysis of constrained evolutionary problems is the technique of subdifferentials and Yosida type approximations. This method is effective. Another method that was successful in abstract analysis of constrained evolutionary problems is the technique of subdifferentials and Yosida type approximations. This method is effective.

1. Introduction. Many applications require proper understanding of motion of liquid droplets on a surface. These include micro- and nano-fluidics, development of surfaces with special wetting properties, effective heat transfer, spraying pesticides on plant leaves or paint to surfaces, printing, etc. The possibility of changing the properties of the surface using nanomachines has been reported recently. This method can be used to produce a gradient in the surface tension. The equilibrium contact angle $\theta$ of a drop depends on the properties of the liquid and of the material on which the drop is lying. It is described by Young’s equation

$$\gamma \cos \theta = \gamma_1 + \gamma_2 \cos\phi,$$

where $\gamma$ is the solid surface tension, $\gamma_1$ the liquid surface tension and $\gamma_2$ the solid/liquid interfacial surface tension. If we create a wettability gradient on the underlying surface, the drop stretches in the direction of greater wettability, which may result into translation of the drop.

Although many experiments and measurements of droplets have been done, there is no well-established analytic model to describe the dynamical aspects of drop motion. Many papers have been devoted to analyzing the shapes of steady drops on horizontal and inclined surfaces. Works dealing with the motion of droplets often make some kind of steady or quasi-steady assumptions. The authors of [9] take a similar approach as [3] or [7] and develop a model for a drop that does not change its shape and moves steadily overcoming shear exerted by the solid surface. They consider a small droplet and rely on the lubrication approximation of de Gennes ([6]).

Taking into account the principle of surface tension and the features of positive contact angle, another natural and reasonable design for the model of moving drop is to approximate the drop by a film, representing the surface of the drop. The film can be then filled with a fluid behaving in accordance with a model of fluid dynamics, and these two models can be converted into the other. This approach is similar to the group of methods.

2. Model. In this section, we derive a model for a droplet moving due to the difference in contact angles. We approximate the drop by its surface and assume that the area density of the surface is constant and that the surface tension is homogeneous. We also consider only the cases when the contact angle $\theta$ is smaller than $90^\circ$, as in Figure 2.1. Then surface tension, contact angle and volume preservation become the main aspects determining the shape of the moving drop.

\[ \theta > \theta^* \quad \text{and} \quad \theta < \theta^* \]

From $\theta < 90^\circ$, we can describe the surface as a scalar function $u: (0, T) \times \mathbb{R}^2 \rightarrow \mathbb{R}$, where $(0, T)$ is the time interval and $\mathbb{R}^2$ is the domain where the motion is considered. The plane, on which the drop rests, corresponds to the level set of the function $u$. The domain $D \subset \mathbb{R}^2$ is taken bounded but large enough so that the drop does not touch its boundary during the motion. The boundary of the set $\{ u = 0 \} = \{ (x, t) \in (0, T) : u(x, t) = 0 \}$ will be called free boundary.

Let us consider $\gamma_1$ and $\gamma_2$ for the characteristic function of the set $\{ u \geq 0 \}$ and simply denote the notation for surface tensions:

\[ \gamma_1 = \gamma_s, \quad \gamma_2 = \gamma_l - \gamma_s \]
Droplet motion on a plane

proportional to $u_\varepsilon^2$ and noting that the resistance force acting against the vertical motion of the film is proportional to the speed of the film, we get the following relation

$$\chi_{\varepsilon} \partial_t u_\varepsilon + \text{sm} - \eta \Delta u_\varepsilon = \eta \Delta \gamma - \gamma u_\varepsilon \gamma - \gamma_{\varepsilon} u_\varepsilon.$$

Here, $\delta$ is proportional to zero density of the region constituting the membrane, $\alpha$ is a drag coefficient and $\lambda$ is a Lagrange multiplier originating in the volume constraint. We have replaced the characteristic function in the second term of $(2.5)$ by a function $\chi_{\varepsilon} \in C(\mathbb{R})$ satisfying

$$\chi_{\varepsilon}(x) = \begin{cases} 0, & x \leq \varepsilon, \\ 1, & x \geq \varepsilon. \end{cases}$$

$(\gamma_{\varepsilon}(x)) \leq C/\varepsilon$ for $x \in (0, \infty)$ (see Figure 2.2). The purpose of the smoothing is to avert the appearance of delta function in the equation. This is a parabolic problem with free-boundary condition. The solution of this equation, as it is, seems very sensitive to the nonlocal multiplier $\lambda$, the smoothing parameter $\varepsilon$ and the volume constraint, which grows proportional to the membrane area density.

FIG. 2.2. Smoothing of characteristic function.

If we consider a long time-scale motion $(\partial_{\varepsilon}\varepsilon) \ll |\mu(\varepsilon)|$, it can be sufficiently approximated by the following parabolic equation:

$$u_\varepsilon = \Delta u_\varepsilon - \gamma (x) \gamma u_\varepsilon - \gamma_{\varepsilon} u_\varepsilon,$$

where we have put $\gamma_{\varepsilon} \equiv 1$ in order to simplify the subsequent development. The specific form of the time-dependent function $\lambda$ can be derived by volume-preserving Lagrangian:

$$\lambda = \int_{\Omega_<} \left( \gamma(x) u_\varepsilon + \gamma_{\varepsilon} u_\varepsilon \right) \, dx.$$  

(2.7)

This is a parabolic problem with free-boundary condition $x \partial_{\varepsilon} u_\varepsilon > 0$ and a complicated term $\lambda$ having the form of the integral of the unknown function. Multiplying equation (2.6) by $\varepsilon$ and integrating over $\Omega$, we see that any solution of (2.6) preserves volume. Because of the nonlocal multiplier, the solution of this equation, as it is, seems very sensitive to the nonlocal multiplier $\lambda$, the smoothing parameter $\varepsilon$ and the volume constraint, which grows proportional to the membrane area density.

4. Model equation and its properties. In the previous section, we have obtained a model equation for droplet motion. Impposing appropriate initial and boundary conditions, we have the following problem:

$$\text{Find } u: (0, T) \times \Omega \to \mathbb{R} \text{ satisfying }$$

$$u_t(0, \varepsilon) = \Delta u(x) \gamma(x) - \gamma u_\varepsilon(x) + \gamma_{\varepsilon}(x) u_\varepsilon(x) \quad \text{in } Q_T, \quad t \in (0, T),$$

$$\frac{\partial u}{\partial \nu}(t, \varepsilon) = 0 \quad \text{on } (0, T) \times \partial \Omega,$$

$$u(\varepsilon, 0) = u_0(\varepsilon) \quad \text{on } \mathbb{R}^N.$$  

(3.1)

where $Q_T = (0, T) \times \Omega \subset \mathbb{R}^{N+1}$.

Remark. It is possible to add also a general "outer force" term $f(x, u, u_\varepsilon, u_{\varepsilon})$ as a right hand side of model equation (1.1). Moreover, we could regularize the formula, so that we can make use of the "cut-off at zero" argument.

First, we shall formulate the main principle for our equation. Let us consider the set $Q_\varepsilon(t) = \{ x \in \mathbb{R}^N : u(x, \varepsilon, t) > 0 \}$. If it is smooth, then it is an open set. Moreover, from the definition of $u_\varepsilon$, we see that $u(0, \varepsilon, t) \subset Q_\varepsilon(t)$ for $0 < \varepsilon < \infty$. This function satisfies assumptions necessary to carry out the proof in the subsequent section. In a coupled model considering also the motions of the fluid film, the fluid in the present section, we mention some features of the model equation (3.1), especially the relation that holds on the free boundary when the smoothing parameter $\varepsilon$ is taken to zero.

4.1. Uniform Energy Estimate. Let us consider the equation

$$u_t = \Delta u - \gamma u_\varepsilon - \gamma_{\varepsilon} u_\varepsilon,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } (0, T) \times \partial \Omega,$$

$$u(0, \varepsilon) = u_0(\varepsilon) \quad \text{on } \mathbb{R}^N.$$  

(4.1)

where $\gamma \in C^0([0, \infty))$ and $u_0 \in L^2(\Omega) \cap H^1_0(\Omega)$.

Remark. It is possible to add also a general "outer force" term $f(x, u, u_\varepsilon, u_{\varepsilon})$ as a right hand side of model equation (1.1).

4.2. Uniform Convergence of Approximation. In this section, we shall formulate the main principle for our equation. Let us consider the set $Q_\varepsilon(t) = \{ x \in \mathbb{R}^N : u(x, \varepsilon, t) > 0 \}$. If it is smooth, then it is an open set. Moreover, from the definition of $u_\varepsilon$, we see that $u(0, \varepsilon, t) \subset Q_\varepsilon(t)$ for $0 < \varepsilon < \infty$. This function satisfies assumptions necessary to carry out the proof in the subsequent section. In a coupled model considering also the motions of the fluid film, the fluid in the present section, we mention some features of the model equation (3.1), especially the relation that holds on the free boundary when the smoothing parameter $\varepsilon$ is taken to zero.

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In the present section, we mention some features of the model equation (3.1), especially the relation that holds on the free boundary when the smoothing parameter $\varepsilon$ is taken to zero.
where $V^3$ is given by (4.2). We note that $V^3 \in L^2(0,T)$ and all the integrals in the above equation have sense for $u^N$ with the stated regularity.

To solve this problem, we make use of the mentioned variational method. The results are summarized in the following theorem.

**Theorem 4.2.** There exists a weak solution of the above approximate problem satisfying

$$u^N \geq 0,$$  

the perturbed volume constraint

$$\int_0^T \lambda_1(u^N)^q \, dt = V$$  

and the following estimate

$$\int_0^T \lambda_1(u^N)^q + \frac{1}{q} |\nabla u^N|^p \, dt + \frac{1}{q} \int_0^T \lambda_1(u^N)^q \, dt \leq C(u_*^{\lambda})$$

for all $t \in (0,T)$, where $C(u_*)$ does not depend on $\delta$.

Moreover, the solution is uniformly bounded in $L^2(0,T)$ and uniformly Hölder continuous on $Q_T$ with respect to the parameter $\delta$.

The rough structure of the proof, based on the result [10], is to use a minimizing argument. First of all, put

$$\{ u \in H^1(Q_T) : \text{with the stated regularity} \}.$$

and the following estimates:

$$\int_\Omega (u_0 - u)^2 \, dx \leq C \int_\Omega |\nabla u_0|^2 \, dx$$

for a.e. $t \in (0,T)$ and

$$\int_\Omega |\nabla u|^2 \, dx \leq C \int_\Omega (u_0 - u)^2 \, dx$$

where $C$ is a constant independent of $\delta$.

We prove Theorem 4.2 by sending $\delta$ to zero. Above all, we have to show that there exists a minimizer of $J_\delta$, that the functions $u^N$ are bounded in a certain norm and that they converge to a weak solution of the smooth problem (4.1). Here we omit the technical proof since it can be recovered from [10] by careful modification of the constant.

Now we prove the main result - existence of weak solutions to (3.1)-(3.3), the meaning of which is explained in the following definition.

**Definition 4.3.** A function $u$ belonging to the space $W^{1,2}(Q_T) \cap C^2(\bar{Q}_T \setminus \{0,1\})$ is called a weak solution to (3.1) if, provided it satisfies the initial condition (3.2) and the identities

$$\int_0^T \int_\Omega (u_{\delta,t} + \nabla u_{\delta} \cdot \nabla \varphi) \, dx \, dt = \int_0^T \int_\Omega (u_{\delta} - u_{\delta,0}) \, dx \, dt$$

for all $\varphi \in C^2(\Omega \setminus \{0,1\})$.

**Theorem 4.4.** There exists a unique weak solution to the problem (3.1)-(3.3) that is Hölder continuous on $\Omega \times [0,T]$.

Proof. Recalling (4.16) we pass to a subsequence $u^N$ converging weakly in $L^2(Q_T)$.

Thus, we cannot use approximations by functions from $C^0(\Omega \times [0,T])$. Moreover, we cannot apply any approximation technique. Indeed, $u^N$ is bounded in $L^2(0,T; \mathcal{L}(\Omega))$, strongly in $L^2(0,T; \mathcal{L}(\Omega))$, uniformly strongly in $L^2(0,T; \mathcal{L}(\Omega))$.

Now, the form of $\bar{u}_{\delta}$ is admissible function in (4.3). Then we get

$$\int_\Omega (u_0 - u)^2 \, dx \leq C \int_\Omega |\nabla u_0|^2 \, dx$$

for all $u \in H^1(Q_T)$.

Hence, taking $\delta$ to zero in (4.15) yields

$$\int_\Omega |\nabla u|^2 \, dx \leq C \int_\Omega (u_0 - u)^2 \, dx$$

which immediately implies $\bar{u}_{\delta} \equiv (\bar{u}_{\delta})$ almost everywhere in $(0,T)$.

The uniqueness follows from the uniqueness of the solution obtained by the method of variational inequalities (see Remark below).

**Remark.** We show that the weak solution constructed above is the same as the solution obtained by the method of variational inequalities. More precisely, we prove that our solution satisfies the relation

$$\int_\Omega \frac{|u(x,t) - u_{\delta,x}(x,t)|^2}{|u(x,t)|^2} \, dx \leq C \int_\Omega (u_0 - u)^2 \, dx$$

for all $u \in H^1(Q_T)$.

Since solution of (4.16) is unique (this can be seen taking two solutions $u, v \in H^1$ setting $z = \phi(x,t)$ and $z = \psi(x,t)$ in the corresponding relations for $\phi, \psi$, adding the resulting inequalities and using Gronwall's lemma), we conclude that there is a unique weak solution in the sense of Definition 3.3, which is identified in the sense of Temam's approximation.

To start with, take any $u \in C^0(\bar{Q}; \mathcal{L}(\Omega))$ and define $\phi_0$ as

$$\phi_0(x,t) = \frac{u(x,t)}{||u(x,t)||},$$

Then for $\epsilon \in (0,1)$ the function $u^{\epsilon}(x,t) = u(x,t) / \epsilon$ is nonnegative and has volume $\epsilon V$, thus it is admissible variant for the functional (4.2), yielding

$$\int_\Omega (|\nabla u^{\epsilon}|^2 + |u^{\epsilon} - u_{\delta,x}|^2) \, dx = 0.$$ 

Letting $\epsilon \to 0$ gives

$$\int_\Omega (|\nabla u|^2 + |u - u_{\delta,x}|^2) \, dx = 0.$$

Using an analog for (4.15) in the limit as $\epsilon \to 0$, we find

$$\int_\Omega (|\nabla u|^2 + |u - u_{\delta,x}|^2) \, dx = 0.$$

Hence, by the lower semicontinuity of the Dirichlet integral, we obtain

$$\int_\Omega (|\nabla u|^2 + |u - u_{\delta,x}|^2) \, dx = 0.$$ 

Results (4.11) and the same reasoning as above finally give (4.16).
Program Part I
Takachiho, Miyazaki, Japan (September 1 - 4, 2008)

August 31 (Sun.)
Welcome to Takachiho and Registration

September 1 (Mon.)
September 2 (Tue.)

Chair: Michal Beneš
9:30 — 10:30 Ryo Kobayashi
10:40 — 11:30 Tatsuki Kawakami
11:30 — 12:00 Tatsunari Sakurai
lunch

Chair: Seiro Omata
14:00 — 14:30 Kila Ra Bopucapció
14:40 — 15:10 Hélder Munakata
15:20 — 15:50 Ivo Opalí
free discussion
coffee break
Chair: Teppei Tsuchiya
16:20 — 16:30 Hironori Seru
17:00 — 17:30 Jun Mach
17:40 — 18:10 Kanako Suzuki
Welcome dinner & free discussion
dinner & free discussion

September 3 (Wed.)
September 4 (Thu.)

Chair: Tatsuyuki Nakaki
10:00 — 10:30 Naoyuki Ishimura
10:40 — 11:10 Boitse Shikhlov
11:20 — 11:50 Daniel Sevcovic
more to Miyazaki city by conference bus
lunch

Chair: Jiri Mikyucky
14:00 — 14:30 Atsushi Suzuki
14:40 — 15:10 Tatjewski Nákosl
destination of the bus
15:20 — 15:50 Yohki Kusak
Gepa
coffee break
Chair: Petr Krbomal
16:20 — 16:50 Takahiti Tanaka
central Miyazaki located
17:00 — 17:30 Karel Svouluk
close to some hotels
17:40 — 18:10 Hiroshi Kori
of participants
Welcome dinner & free discussion
dinner & free discussion

Program Part II
University of Miyazaki, Japan (September 4 - 7, 2008)

September 4 (Tue.)
Welcome to Miyazaki city and Registration

September 5 (Wed.)
September 6 (Thu.)

Chair: Masato Kimura
9:30 — 10:30 Peter Frindel
10:40 — 11:10 Katsuhiko Hiramatsu
lunch
Chair: Tatsuki Kawakami
11:20 — 12:00 Tatsuki Kawakami
a visit to a multi-fan wind tunnel
break conference lunch
Chair: Daniel Sevcovic
14:00 — 14:30 Seiwa Onuma
14:40 — 15:10 Hidetsuki Nakamura
15:20 — 15:50 Masaki Kusum
A posteriori estimate
Chair: Eduardo Lagani
free discussion
16:10 — 16:40 Petr Fremeisíček
16:50 — 17:20 Jiri Mikyucky
17:30 — 18:00 Raděk Furlík
welcome dinner
Chair: Eduardo Lagani
free discussion
16:10 — 16:40 Petr Fremeisíček
16:50 — 17:20 Jiri Mikyucky
17:30 — 18:00 Raděk Furlík
welcome dinner
Chair: Michal Beneš
October 10 — 11:00 Olga Dvihlevi
11:20 — 12:50 Marinka Rovenski
lunch
Chair: Peter Fremeisíček
14:00 — 15:30 pasteur session
Raděk Furlík, Peter Golub
Masaki Kusum, Jan Mach, Petr Fremeisíček
Pavel Strachota, Atsushi Suzuki
Tatsuki Tsuchiya, Shigetoshi Yázaki
coffee break
Chair: Michal Beneš
15:50 — 16:20 Petr Krbomal
16:30 — 17:00 Michal Beneš
Announcements

List of talks
Patterns and Computations in Biological Systems – True Slime Mold –
Ryo Kobayashi, Hiroshima University
Application of Anisotropic Diffusion in MR Tactography
Pavel Strachota, Czech Technical University in Prague
Onset and stability of stationary pattern in a spatially discrete excitable system
Tatsunari Sakurai, Chiba University
Dispersion of a Passive Pollutant within and above Various Urban Canopies
Kila Ra Bopucapício, Academy of Sciences of the Czech Republic / Tokyo Polytechnic University
Solutions to nonlinear diffusion problems by semiloin reaction-diffusion systems
Hidetsuki Munakata, University of Toyama
Hybrid wave injection method in 3D seismic wave propagation as one of boundary conditions, application in finite differences
Ivo Opalí, Disaster Prevention Research Institute DPRI, Kyoto University / Charles University
A Paradoxical Outbreak Caused by Harvesting/Thinning in Discrete Population Dynamics
Hironori Seno, Hiroshima University
Qualitative study of the Gray-Scott model
Jan Mach, Czech Technical University in Prague
Behavior of solutions to an activator-inhibitor system with basic production terms
Kanako Suzuki, Tohoku University
How to unify the total/local length-constraints of the gradient flow for the bending energy of plane curves
Tatsuki Tsuchiya, Saitama University
The gradient flow for bending energy of plane curves under the local-length constraint: A formulation applying the Fredholm alternative
Yuki Miyamoto, Saitama University
On the global behavior of the gradient flow for the bending energy under the local-length constraint for the initial curve whose rotation number is not necessarily one
Fumito Sato, Saitama University

Singualr nonlinear partial differential equation for the risk preference
Naoyuki Ishimura, Hitotsubashi University

Averaging in two-factor interest rate models
Beáta Stehlíková, Comenius University

Higher order estimates for the curvature and nonlinear stability of stationary solutions for curvature flow with triple junction
Daníel Ševčovič, Comenius University

An iterative substructuring algorithm with congruent subdomains
Atsushi Suzuki, Kyushu University

A singular limit method to free boundaries in oil reservoir
Tatsuyuki Nakaki, Kyushu University

A rigorous treatment of the perturbation theory for many electron system
Yuhei Kashima, Hokkaido University

Computer simulation on a phase field model for crack evolution
Takeshi Takahashi, Hiroshima Kokusai Gakuin University

Globally constrained evolutionary problems
Karel Švadlenka, Kanazawa University

Synchronization engineering
Hiroshi Kuri, Osaka University

Finite volume method for some advection dominated problems
Peter Frolík, Slovak University of Technology in Bratislava

Finite volume methods for some advection dominated problems
Hiroshi Kori, Ochanomizu University

Comparison of Methods for Curve Evolution [poster]
Masato Kimura, Kyushu University

Finite volume methods for some advection dominated problems
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Finite volume method for modelling flow in porous media with dynamic capillary pressure
Radek Fučík, Czech Technical University in Prague

Error estimates of some boundary flux quantities
Daixue Tagami, Kyushu University

Observing snow crystals growth by using Hiramatsu Apparatus
Kazuhiko Hiramatsu, Hokkaido Asahikawa Higashi Senior High School

Turbulence generation by a multi-fan type wind tunnel
Shigehira Ozono, University of Miyazaki

Computational reconstruction of vertebrate early embryogenesis by nonlinear PDE methods
Karol Mikula, Slovak University of Technology in Bratislava

Nonlinear tensor anisotropic diffusion in coherence enhancing image filtering
Olga Dhlbliková, Slovak University of Technology in Bratislava

A finite volume method for 4D image segmentation
Marina Remelkivá, Slovak University of Technology in Bratislava

Fully Implicit 1D Numerical Scheme for Modelling Two-Phase Flow with Dynamic Capillary Pressure [poster]
Radek Fučík, Czech Technical University in Prague

Integral Solutions for Two-Phase Unsteady Flow in Homogeneous Porous Medium [poster]
Radek Fučík, Czech Technical University in Prague

Interactions between steady and oscillatory convection in mushy layers [poster]
Peter Guba, Comenius University

Finite difference methods for Allen-Cahn / Cahn-Hilliard equations [poster]
Masato Kurna, Kyushu University

Numerical methods for solving the Gray-Scott model [poster]
Jan Mach, Czech Technical University in Prague

Comparison of Methods for Curve Evolution [poster]
Petr Paul, Czech Technical University in Prague

Finite volume method for the Allen-Cahn equation with anisotropy [poster]
Pavel Strachota, Czech Technical University in Prague

Finite element computation of subducting plate near trench [poster]
Atsushi Suzuki, Kyushu University

Global structure of stationary solutions of a shadow system for adsorbateinduced phase transition model [poster]
Tobis Tsujikawa, University of Miyazaki

Relationship of local projection methods for convection-diffusion equations to residual-based stabilizations
Petr Knobloch, Charles University

Numerical solution of non-local anisotropic Allen-Cahn equation
Michal Beneš, Czech Technical University in Prague

Entropy dissipations methods for the sign-changing solutions of the heat equations with the nonlinear boundary condition
Tatsuki Kawakami, Tohoku University

Mathematical theory and numerical calculations of collision, peeling and attached droplets
Seiro Omata, Kanazawa University

Numerical computation for bubble motion by the Bence-Merriman-Osher algorithm
Hideaki Nakagawa, Kanazawa University

Numerical calculations and their application to fluid / membrane interactions
Masaki Kazuma, Kanazawa University

Comparison of Modified TVD MacCormack and WLSQR scheme in the Case of 2D and 3D Inviscid Transonic Flow in External Aerodynamics
Petr Farmánek, Czech Technical University in Prague

Efficient Implementation of Discontinuous Galerkin Method on Rectangular Grids
Jiří Mikyška, Czech Technical University in Prague

Fully Implicit Numerical Scheme for Modelling Flow in Porous Media with Dynamic Capillary Pressure
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Nonlinear Tensor Anisotropic Diffusion in Coherence Enhancing Image Filtering
Olga Dhlbliková, Slovak University of Technology in Bratislava

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