Equivalence between the
eigenvalue problem of
non-commutative harmonic
oscillators and existence of
holomorphic solutions of Heun
differential equations, eigenstates
degeneration and Rabi model

Masato Wakayama

MI 2013-10

( Received March 26, 2014 )
Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun differential equations, eigenstates degeneration and the Rabi model

Masato Wakayama
March 26, 2014

Abstract

The initial aim of the present paper is to provide a complete description of the eigenvalue problem for the non-commutative harmonic oscillator (NeHO), which is defined by a (two-by-two) matrix-valued self-adjoint parity-preserving ordinary differential operator [21], in terms of Heun’s ordinary differential equations, the second order Fuchsian differential equations with four regular singularities in a complex domain. This description has been achieved for odd eigenfunctions in Ochiai [18] nicely but missing up to now for the even parity, which is more important from the viewpoint of determination of the ground state. As a by-product of this study, using the monodromy representation of Heun’s equation, we prove that the multiplicity of the eigenvalue of the NeHO is at most two. Moreover, we give a condition for the existence of a finite-type eigenfunction (given by essentially a finite sum of Hermite functions) for the eigenvalue problem and an explicit example of such eigenvalues, from which one finds that doubly degenerate eigenstates of the NeHO actually exist even in the same parity. In the final section, as the second purpose of this paper, we discuss a connection between the quantum Rabi model [2, 15, 32] and the element of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ naturally arising from the NeHO through the oscillator representation. Precisely, an equivalent picture of the Rabi model drawn by a confluent Heun equation is obtained from the Heun operator defined by that element in $U(\mathfrak{sl}_2)$ under a (flat picture of non-unitary) principal series representation of $\mathfrak{sl}_2$.

2010 Mathematics Subject Classification: Primary 34L40, Secondary 81Q10, 34M05, 81S05.

Keywords and phrases: non-commutative harmonic oscillators, universal enveloping algebra, Heun differential equation, monodromy representation, quantum Rabi model, confluence process.

1 Introduction

In recent years, special attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, in other words, interacting quantum systems, like the quantum Rabi model, the Jaynes-Cumming (JC) model, etc., not only in mathematics [6, 7] but also in theoretical physics [15, 4, 2, 16, 29] and experimental physics (see e.g. [5, 33]). For instance, the quantum Rabi model [25] is known to be the simplest model used in quantum optics to describe interaction of light and matter and the JC model is the widely studied rotating-wave approximation of the Rabi model [15, 7]. The non-commutative harmonic oscillator (NeHO) $Q$ defined below has been expected to share/provide one of these Hamiltonians describing such quantum interacting systems.

The purposes of this paper are, in short, providing explicit descriptions of i) the eigenvalue problem of NeHO in terms of Heun’s operators, ii) the degeneration of eigenvalues and explicit examples, and iii) a connection between NeHO and the quantum Rabi model through the confluence process of Heun’s ODE using representation theory of Lie algebra $\mathfrak{sl}_2$. 

1
The normal form $Q$ of NeHOO ([23, 24, 21]) is given by
\[
Q = Q_{(\alpha, \beta)} = A \left( -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + J \left( x \frac{d}{dx} + \frac{1}{2} \right),
\]
where the mutually non-commuting (in general) coefficients $A$ and $J$ are given by
\[
A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

From the definition, $Q$ is obviously a parity-preserving differential operator. We assume that $\alpha, \beta > 0$ and $\alpha \beta > 1$ throughout the paper. The former requirement comes from the formal self-adjointness of the operator $Q$ relative to the natural inner product on $L^2(\mathbb{R}, \mathbb{C}^2)(= \mathbb{C}^2 \otimes L^2(\mathbb{R}))$. The latter guarantees that the eigenvalues of the eigenvalue problem $Q\varphi = \lambda \varphi$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) are all positive and form a discrete set with finite multiplicity. It should be first noted that, $Q$ is unitarily equivalent to a couple of quantum harmonic oscillators when $[A, J] = 0$, i.e. $\alpha = \beta$ holds, whence the eigenvalues are explicitly calculated as $\left\{ \sqrt{n^2 + \frac{1}{2}} \right\} n \in \mathbb{Z}_{\geq 0}$ having multiplicity 2 ([24], 1). Actually, when $\alpha = \beta$, there exists a structure behind $Q$ corresponding to the tensor product of the two dimensional trivial representation and the oscillator representation [10] of the Lie algebra $\mathfrak{sl}_2$ ([24]). However, when $\alpha \neq \beta$, representation theoretically, the apparent lack of an operator which commute with $Q$ (second conserved quantity) besides the Casimir operator, the image of generator of the center $Z\mathcal{U}(\mathfrak{sl}_2)$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. (Moreover, it has been shown that there is no annihilation/creation operators associated to NeHOO [22] when $\alpha \neq \beta$.) Therefore, the clarification of the spectrum in the general case where $\alpha \neq \beta$ is considered to be highly non-trivial (see [12, 20, 8], also references in [21]). It is, nevertheless, worth noticing that the spectral zeta function of $Q$ ([11]) which is essentially given by the Riemann zeta function if $\alpha = \beta$ yields a new number theoretic study including the subjects such as elliptic curves, modular forms, Eichler integrals and their natural generalization (see [13, 14] and references therein).

We have constructed the eigenfunctions and eigenvalues [24] in terms of continued fractions determined by a certain three terms recurrence relation, which can be derived from the expansion of eigenfunctions relative to a basis constructed by suitably twisting the classical Hermite functions. We say that the eigenfunction $\varphi(x)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ is of a finite-type if $\varphi(x)$ can be expanded by a finite number of this Hermite basis. The eigenvalue corresponding to the finite-type eigenfunction is said to be of finite-type. Otherwise, we say that the eigenvalues/eigenfunctions are of infinite-type. We denote $\Sigma_0$ (resp. $\Sigma_\infty$) the set of eigenvalues corresponding to eigenfunctions of finite (resp. infinite)-type. Since the operator $Q$ preserves the parity, we define $\Sigma_\pm$ to be the set of eigenvalues whose eigenfunctions are even/odd, that is, those satisfying $\varphi(-x) = \pm \varphi(x)$. Then there is a classification of eigenvalues: $\Sigma_0^\pm = \Sigma_0 \cap \Sigma_\pm$ (resp. $\Sigma_\infty^\pm = \Sigma_\infty \cap \Sigma_\pm$) corresponding to even/odd eigenfunctions of finite (resp. infinite)-type. In [24], it is shown that $\Sigma_0^\pm \subset \Sigma_\pm^\pm$ and the multiplicity of each $\lambda \in \Sigma_0^\pm$ (resp. $\Sigma_\infty^\pm$) is at most 2. This means that once the eigenvalue degenerates in the same parity, one of the eigenfunction is of the form $p(x) \times e^{-\alpha x^2}$, $p(x)$ being a polynomial and $\alpha$ a positive constant depending on the value $\sqrt{\alpha} \beta = 1$, and this resembles the situation of the (doubly) degenerating eigenvalues case for the Rabi model [15] (see also [2, 28, 32]). The spectral analysis of the Rabi model seems to be much simpler than the one of NeHOO, while the latter seems to share certain interesting properties the former has. Actually, one finds that the NeHOO gives essentially the Rabi model through the confinement limit procedure at the stage of Heun equations’ picture (see §5).

It is known [17] that the eigenvalues of NeHOO build a continuous curve with arguments $\alpha$ and $\beta$. It comes as an important problem to analyze the behavior of eigenvalue curves, in particular, a main issue of present day research, especially in mathematical physics, addresses the characterization of crossing/avoided crossing of eigenvalue curves (see e.g. [27, 6, 7]). From the observation in [17], since the eigenvalue curves are continuous, one can observe that $\Sigma_0^+ \cap \Sigma_\infty^- \neq \emptyset$ (see Figure 1 in [17], p.648; the graph of eigenvalue curves is drawn with respect to the variable $s = \beta/\alpha$ with a fixed $\alpha$; $\alpha = 3.0$). However, one does not know whether $\Sigma_0^+ \cap \Sigma_\infty$ (resp. $\Sigma_0^- \cap \Sigma_\infty^+$) is empty or not, while it is shown in [24] that $\Sigma_0^+ \cap \Sigma_\infty^- = \emptyset$. Therefore, the multiplicity of eigenvalue is at most 3 and may a priori reach 3. In this paper, we will solve one of the longstanding problems for the eigenstate degeneration of NeHOO, in particular, prove that $\Sigma_0^+ \cap \Sigma_\infty = \Sigma_0^- \cap \Sigma_\infty^+ = \emptyset$ (Theorem 1.2).
Generally, in harmonic analysis on the real line, even/odd eigenspaces have completely analogous structures. Also, in view of the description of the lowest eigenvalue, the study on even eigenstates is more important [8]. Moreover, we could not see any difference between the even/odd eigenspaces in the papers [23, 24]. However, in the complex domain picture drawn in [18], the odd part \( \Sigma^- \) corresponds to the second order equation given by Heun’s ordinary differential equation whereas the even part \( \Sigma^+ \) corresponds to the third-order equation (constructed by the same Heun operator). Therefore, working out a solution to this asymmetry has been desirable for a long time. In this paper, we prove that there exists a completely parallel structure of even eigenfunctions to that of the odd eigenfunctions. For readers’ convenience we state the results for the odd case obtained in [18] which provides an (exact) intertwiner for the oscillator representation corresponding to the even parity (see §2.2). The reason why one could not obtain in [18] the Heun operator (only the third order operator) in a parallel way. In fact, one of the main techniques to derive this correspondence is based on a brilliant idea developed in [18], but employing a modified Laplace transform different from that in [18] which provides an (exact) intertwiner for the oscillator representation corresponding to the even parity (see §2.2). The reason why one could not obtain in [18] the Heun operator (only the third order operator) in the even case is that the restriction of the modified Laplace transform in [18] to the even parity has only quasi-intertwing property.

**Theorem 1.1.** There exist linear bijections:

For the even case:

\[
\text{Even} : \{ \varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q \varphi = \lambda \varphi, \varphi(-x) = \varphi(x) \} \sim \{ f \in \mathcal{O}(\Omega) \mid H^+_\lambda f = 0 \},
\]

For the odd case:

\[
\text{Odd} : \{ \varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q \varphi = \lambda \varphi, \varphi(-x) = -\varphi(x) \} \sim \{ f \in \mathcal{O}(\Omega) \mid H^-_\lambda f = 0 \},
\]

where \( \Omega \) is a simply-connected domain in \( \mathbb{C} \) (w-space) such that \( 0, 1 \in \Omega \) while \( \alpha \beta \not\in \Omega \), \( \mathcal{O}(\Omega) \) denotes the set of holomorphic functions on \( \Omega \), and \( H^\pm_\lambda = H^\pm_\lambda (w, \partial_w) \) are the Heun ordinary differential operators given respectively by

\[
H^+_\lambda (w, \partial_w) := \frac{d^2}{dw^2} + \left( \frac{1}{2} - p \right) \frac{d}{dw} + \left( \frac{1}{2} - q^+ \right) \frac{1}{w(w-1)(w-\alpha \beta)}\)
\]
and

\[
H^-_\lambda (w, \partial_w) := \frac{d^2}{dw^2} + \left( \frac{1}{2} - p \right) \frac{d}{dw} + \left( \frac{3}{2} - q^- \right) \frac{1}{w(w-1)(w-\alpha \beta)}.
\]

Here the numbers \( p = p(\lambda) \) and \( \nu = \nu(\lambda) \) are defined thorough the following relation:

\[
p = \frac{2\nu - 3}{4}, \quad \lambda = 2\nu \frac{\sqrt{\alpha \beta (\alpha \beta - 1)}}{\alpha + \beta}.
\]

The accessory parameters \( q^\pm = q^\pm(\lambda) \) in these Heun’s operators are expressed by the parameters \( \alpha, \beta \) and eigenvalue \( \lambda \) as

\[
q^+ = \left\{ \left( p + \frac{1}{2} \right)^2 - \left( p + \frac{3}{4} \right)^2 \left( \beta - \frac{\alpha}{\beta + \alpha} \right)^2 \right\}(\alpha \beta - 1) - \frac{1}{2} \left( p + \frac{1}{2} \right),
\]

\[
q^- = \left\{ p^2 - \left( p + \frac{3}{4} \right)^2 \left( \beta - \frac{\alpha}{\beta + \alpha} \right)^2 \right\}(\alpha \beta - 1) - \frac{3}{2} p^2. \quad \Box
\]

We note that these Heun operators \( H^\pm_\lambda (w, \partial_w) \) have four regular singular points, \( w = 0, 1, \alpha \beta \) and \( \infty \). The respective Riemann’s scheme of the operators \( H^\pm_\lambda (w, \partial_w) \) are expressed as

\[
H^+_\lambda : \begin{pmatrix}
0 & 1 & \alpha \beta & \infty \\
p + \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} - p & -p & -\left( p + \frac{1}{2} \right)
\end{pmatrix}
\]
and

\[
H^-_\lambda : \begin{pmatrix}
0 & 1 & \alpha \beta & \infty \\
p & p + 1 & -p - \frac{1}{2} & -p \\
0 & 0 & 0 & \frac{3}{2} \\
\frac{3}{2} - p & -p & -\left( p + \frac{1}{2} \right)
\end{pmatrix}.
\]
where each element of the first row indicates a regular singular point of $H^\pm_\lambda$ and those in the second and third rows are expressing the corresponding exponents.

As a corollary of this theorem we may actually provide examples of finite-type eigenvalues (and corresponding solutions). In other words, we have an even (resp. odd) polynomial solution when the parameter $p + \frac{1}{2} \in \mathbb{N}$ (resp. $p \in \mathbb{N}$) satisfies a certain algebraic equation obtained by the determinant of Jacobi’s (i.e. tri-diagonal) matrix (Proposition 4.1). It is also worth noticing that the ground state of the NcHO consists of only even functions [9], whence its simplicity follows from the result in [30]. The criterion for this simplicity (Theorem 1.1 in [30]) can be proved also by Theorem 1.1 above together with an upper bound estimate of the lowest eigenvalue given in [21] (Theorem 8.2.1) (see [30]). Furthermore, combining the results in Theorem 1.1 for even and odd eigenfunctions and using the monodromy representation of the corresponding Heun differential equations, we will show the following.

**Theorem 1.2.** Suppose $\alpha \beta > 1$. The multiplicity $m_\lambda$ of the eigenvalue $\lambda$ for the non-commutative harmonic oscillators $Q_{\alpha \beta}$ is at most 2. Moreover, when $\alpha \neq \beta$, $m_\lambda = 2$ holds if and only if either of the following two cases holds:

1. $\lambda \in \Sigma^+_0$ (resp. $\Sigma^-_0$); in this case one has a unique (up to scalar multiples) finite-type solution, i.e., a finite linear combination of even (resp. odd) twisted Hermite functions. Moreover, $\lambda$ is of the form $\lambda = 2\sqrt{\alpha \beta (\alpha \beta - 1)/(\alpha + \beta)} \left( 2L + \frac{1}{2} \right)$ for $L \in \mathbb{N}$.

2. $\lambda \in \Sigma^+_\infty \cap \Sigma^-_\infty$.

**Remark 1.1.** As for the precise definition of twisted Hermite functions in Theorem 1.2, see [24].

![Figure 1: Examples of configuration for doubly degenerations of the spectrum of NcHO](image)

In the final section, we will discuss a connection between the operator $\mathcal{R}$ (a degree 2 element of the universal enveloping algebra $U(sl_2)$ obtained naturally from NcHO through the oscillator representation of $sl_2$ (see Lemma 2.1) and the quantum Rabi model. Although the Rabi model has had an impressive impact on many fields of physics [5], only recently (in 2011) may this model be declared solved by D. Braak [2]. The Hamiltonian is given as

$$H_{\text{Rabi}}/\hbar = \omega a^\dagger a + \Delta \sigma_z + g(\sigma^+ + \sigma^-)(a^\dagger + a),$$

with $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ (see §5), by taking the confluence procedure of Heun’s equation (see [27, 26]). Actually, employing a (flat picture of) principal series representation $\varpi_\alpha$ of $sl_2$ (see Lemma 2.3), which is inequivalent to the oscillator representation when $\alpha \neq 1, 2$ gives the NcHO, one constructs the confluent Heun differential operator corresponding to the Rabi model from $\mathcal{R}$. The result in §5 amounts to saying that the Rabi model is considered to be a sort of confluent version of the NcHO.

### 2 Representation theoretic setting

#### 2.1 Oscillator representation of $sl_2$

Although there is no exact (continuous) symmetry on $Q$ ($\alpha \neq \beta$) described by the Lie algebra $sl_2$, as it is the case for the quantum harmonic oscillator, there seems to exist still a vague hidden (or
modified) $\mathfrak{sl}_2$-symmetry behind it, beside the parity $\mathbb{Z}_2$. Thus a formulation of the problem by the language of $\mathfrak{sl}_2$ is useful, as we have observed in [23, 24, 18]. Moreover, as we will see in §5, in order to observe the relation between the NcHO and the Rabi model, a viewpoint employing Lie algebra representation of $\mathfrak{sl}_2$ is important.

Let $H, E$ and $F$ be the standard generators of the Lie algebra $\mathfrak{sl}_2$ defined by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. $$

They satisfy the commutation relations


For the triplet $(\kappa, \epsilon, \nu) \in \mathbb{R}_+^3$, define a second order element $\mathcal{R}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ by

$$\mathcal{R} := \frac{2}{\sinh 2\kappa} \left\{ \left[ (\sinh 2\kappa)(E - F) - (\cosh 2\kappa)H + \nu \right](H - \nu) + (\epsilon \nu)^2 \right\} \in \mathcal{U}(\mathfrak{sl}_2).$$

Define also an element $\mathcal{R}'$ by

$$\mathcal{R}' = \frac{2}{\sinh 2\kappa} \left\{ (H - \nu) \left[ (\sinh 2\kappa)(E - F) - (\cosh 2\kappa)H + \nu \right] + (\epsilon \nu)^2 \right\} \in \mathcal{U}(\mathfrak{sl}_2).$$

We define the oscillator representation $\pi$ of $\mathfrak{sl}_2$ by

$$\pi(H) = x \partial_x + 1/2, \quad \pi(E) = x^2/2, \quad \pi(F) = -\partial_x^2/2,$

where $\partial_x = d/dx$. We will also denote the algebra homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ to the ring $\mathbb{C}[x, \partial_x]$ of differential operators by the same letter $\pi$. By this realization, the eigenvalue problem $Q \varphi(x) = \lambda \varphi(x)$ ($u \in L^2(\mathbb{R}, \mathbb{C}^2)$) turns to be solving the following equation.

$$[\pi(E + F) + J \pi(H) - \lambda I] \varphi(x) = 0,$$

As we have shown in [24] (see also [18]) this equation can be rewritten as

$$[\pi(E + F) + \frac{1}{\sqrt{\alpha \beta}} I \pi(H) - \lambda J A^{-1}] \tilde{\varphi}(x) = 0,$$

where $\tilde{\varphi}(x) = A^{1/2} \varphi(x)$.

Now, as usual, let us realize the oscillator representation on the polynomial ring $\mathbb{C}[y]$ in place of $L^2(\mathbb{R})$ using the Cayley transform $C := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Define the annihilation operator $\psi = (x + \partial_x)/\sqrt{2}$ and creation operator $\psi^\dagger = (x - \partial_x)/\sqrt{2}$. Then one has $[\psi, \psi^\dagger] = 1$. Put $\varphi_0(x) := e^{-x^2/2} \in L^2(\mathbb{R})$. Then $\varphi_0$ is the ground state, that is, $\psi \varphi_0 = 0$. We define in general $\varphi_n := (\psi^\dagger)^n \varphi_0$, the Hermite functions. Then the set $\{ \varphi_n | n = 0, 1, 2, \ldots \}$ forms an orthogonal basis with $(\varphi_n, \varphi_m) = \sqrt{n!} \delta_{nm}$, (,) being the standard inner product of $L^2(\mathbb{R})$ (see e.g. [10]). We denote the set of all finite linear combinations of the Hermite functions $\varphi_n$ by $L^2(\mathbb{R})_{\text{fin}}$.

Let

$$T_C : L^2(\mathbb{R})_{\text{fin}} \to \mathbb{C}[y]$$

be the linear map defined by the property $T_C(\varphi_n) = y^n$. Then one immediately sees that $T_C(\psi^\dagger \varphi) = y T_C(\varphi)$ and $T_C(\psi \varphi) = \partial_y T_C(\varphi)$. Then, if we define the representation $(\pi', \mathbb{C}[y])$ of $\mathfrak{sl}_2$ by

$$\pi'(H) = y \partial_y + 1/2, \quad \pi'(E) = y^2/2, \quad \pi'(F) = -\partial_y^2/2,$$

one may easily show that $\pi'(CXC^{-1})T_C = T_C \pi(X)$ ($X \in \mathfrak{sl}_2$). Moreover, if we define a Fisher inner product on $\mathbb{C}[y]$ by $(f, g)_{\mathcal{F}} = \sqrt{n!} \delta_{mn} (f \psi^\dagger)(g \psi)$, then one finds that $(y^n, y^m)_{\mathcal{F}} = \delta_{m,n} \sqrt{n!}$, whence $T_C$ gives an isometry. If we denote the completion of $\mathbb{C}[y]$ with respect to this inner product by $\overline{\mathbb{C}[y]}$, then it follows that the map $T_C$ can be extended to an isometry between two Hilbert spaces $L^2(\mathbb{R})$ and $\overline{\mathbb{C}[y]}$.

The first claim of the following lemma follows immediately from [18] (Corollary 9 with Lemma 8), which translates the eigenvalue problem of $Q$ into a single differential equation. The second follows in a similar way.
Lemma 2.1. Assume \( \alpha \neq \beta \). Determine the triplet \((\kappa, \epsilon, \nu) \in \mathbb{R}^2_+ \) by the formulas
\[
\cosh \kappa = \sqrt{\frac{\alpha \beta}{\alpha - \beta}}, \quad \sinh \kappa = \frac{1}{\sqrt{\alpha \beta - 1}}, \quad \epsilon = \left| \frac{\alpha - \beta}{\alpha + \beta} \right|, \quad \nu = \frac{\alpha + \beta}{2\sqrt{\alpha \beta (\alpha - \beta - 1)}}.\]

Then the eigenvalue problem \( Q\varphi = \lambda \varphi \) (\( \varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \)) is equivalent to the equation \( \pi'(u) = 0 \) (\( u \in \mathbb{C}[y] \)). Let \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \text{Aut}(\mathbb{C}^2 \otimes L^2(\mathbb{R})) \). Then the eigenvalue problem \( KQK\varphi = \lambda \varphi \) (\( \varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \)) is equivalent to the equation \( \pi'(u) = 0 \) (\( u \in \mathbb{C}[y] \)).

Remark 2.1. We remark that the twist \( KQK \) and \( Q \) have the same spectrum.

Remark 2.2. Notice that \( \pi'(\mathbb{R}) \) is the third order differential operator and the recurrence equation (or its corresponding continued fraction) in [24] is equivalent to this third order differential equation. It is also worth noting that the construction of the transcendental function \( G^+(x) \) whose zeros give regular eigenvalues of the Rabi model in [2, 3] resembles to that of NeHo in [24].

Remark 2.3. The correspondence \( \varphi \leftrightarrow u \) in the lemma above can be given explicitly. For the readers’ convenience, we briefly summarize the correspondence given in [18].

Put
\[
S_{\pm} := E + F \pm \frac{i}{\sqrt{\alpha \beta}} H \in \mathfrak{sl}_2
\]
and
\[
\tilde{\varphi} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \left[ \begin{bmatrix} \sqrt{\alpha} \\ 0 \end{bmatrix} \right] \varphi.
\]
Define (invertible) transformations \( T \) and \( T' \) by
\[
(Tf)(x) = e^{(\sinh \kappa)x/2}((\cosh \kappa)\lambda f(\cosh \kappa x)) \quad \text{and} \quad (T'g)(y) = g\left(\frac{\cosh \kappa}{\cosh \kappa - \sinh \kappa} y\right).
\]
Set
\[
\left[ \begin{array}{c} u \\ \tilde{u} \end{array} \right] := T'TC T\tilde{\varphi}.
\]
Then one knows that whenever \( \alpha \neq \beta \) the eigenvalue problem \( Q\varphi = \lambda \varphi \) can be written as
\[
\begin{bmatrix} \frac{\sinh \kappa}{\cosh \kappa} \pi'(H) - \frac{\alpha + \beta}{2\alpha \beta} \lambda \\ -\frac{\alpha + \beta}{2\alpha \beta} \epsilon \lambda \end{bmatrix} \begin{bmatrix} u \\ \tilde{u} \end{bmatrix} = 0.
\]
Hence \( Q\varphi = \lambda \varphi \) is equivalent to the equation \( \pi'(u) = 0 \), and \( \tilde{u} = \frac{2\alpha^2}{(\alpha + \beta)^2} \sqrt{1 - \frac{1}{\alpha^2}} \pi'(H) - \frac{\alpha + \beta}{2\alpha \beta} \lambda \) (Lemma 8 and Corollary 9 in [18]).

2.2 Intertwiners arising from Laplace transforms

In order to obtain a complex analytic picture of the equation \( \pi'(\mathbb{R})u = 0 \) in Lemma 2.1 and observe a connection between NeHo and the Rabi model through Heun ODE, we introduce two representations of the \( \mathfrak{sl}_2 \).

Let \( a \in \mathbb{N} \). Define first the operator \( T_a \) acting on the space of Laurent polynomials \( \mathbb{C}[y, y^{-1}] \) (or \( y^2 \mathbb{C}[y] \)) by
\[
T_a := -\frac{1}{2} \partial_y^2 + \frac{(a - 1)(a - 2)}{2} \frac{1}{y^2}.
\]
Define a modified Laplace transform \( \mathcal{L}_a \) by
\[
(\mathcal{L}_a u)(z) = \int_0^\infty u(yz)e^{-\frac{z^2}{2}} y^{a-1} dy.
\]
Then, one finds that
\[
(\mathcal{L}_a T_a u)(z) = \left( -\frac{1}{2z} \partial_z + \frac{a - 1}{2z^2} \right) (\mathcal{L}_a u)(z) + \frac{1}{2z} u'(0) \delta_{a,1} - \frac{a - 1}{2z^2} u(0) \delta_{a,2},
\]
where \( \delta_{a,k} = 1 \) when \( k = a \) and 0 otherwise. This can be true whenever \( u(0), u'(0) \) and \( (L_a u)(z) \) exist.

We now define a representation \( \pi'_a \) of \( \mathfrak{sl}_2 \) on \( y^{a-1}\mathbb{C}[y] \) by

\[
\pi'_a(H) = \pi'(H), \quad \pi'_a(E) = \pi'(E), \quad \pi'_a(F) = T_a = \pi'(F) + \left( \frac{a-1}{2} \right) \frac{1}{y^2}.
\]

Moreover, introduce another representation of \( \mathfrak{sl}_2 \) on \( \mathbb{C}[z, z^{-1}] \) by

\[
\varpi_a(H) = z\partial_z + \frac{a}{2}, \quad \varpi_a(E) = \frac{1}{2}z^2(\partial_z + a), \quad \varpi_a(F) = -\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}.
\]

Then one easily verifies the following.

**Lemma 2.2.** Let \( a \neq 1, 2 \). Then one has

\[
L_a \pi'_a(X) = \varpi_a(X)L_a \quad (X \in \mathfrak{sl}_2).
\]

Furthermore, when \( a = 1 \) (resp. \( a = 2 \)) the restriction of \( L_1 \) (resp. \( L_2 \)) to the space of even (resp. odd) functions turns out to be an intertwiner between the two representations \( \pi'(= \pi'_1) \) (resp. \( \pi'_2 \)) and \( \varpi_1 \) (resp. \( \varpi_2 \)). Precisely, \( L_j \) \( (j = 1, 2) \) possesses the following quasi-intertwiner property:

\[
\begin{align*}
L_j \pi'(X) &= \varpi_j(X)L_j, \quad (for \ X = H, E), \\
(L_1 \pi'(F)u)(z) &= \varpi_1(F)(L_1 u)(z) + u'(0)/(2z), \\
(L_2 \pi'(F)u)(z) &= \varpi_2(F)(L_2 u)(z) - u(0)/(2z^2).
\end{align*}
\]

**Remark 2.4.** Remark that \( L_a \) defines an isometry. For instance, assume \( a = 1 \). If \( u(y) = \sum_{n=0}^{N} u_n y^n \in \mathbb{C}[y] \) then \( (L_1 u)(z) = \frac{1}{\sqrt{2}} \sum_{n=0}^{N} u_n \Gamma(\frac{a+1}{2})(\sqrt{2}z)^n \). Moreover, if one defines the inner product in \( z \)-space such that \( \{z^n | n \in \mathbb{N}\} \) forms an orthogonal basis and \( (z^n, z^n) = \frac{2\Gamma(\frac{a+1}{2})}{\Gamma(\frac{a-1}{2})} \), then \( L_1 \) is an isometry. The other cases are similar.

Since \( \varpi_a(E)z^{-a} = 0 \), one has the following second equivalence: the representation \( (\pi'_a, y^{2-a}\mathbb{C}[y^2]) \) can be considered as the Langlands quotient of the representations \( (\varpi_a, \mathbb{C}[z^2, z^{-2}]) \) or \( (\varpi_a, \mathbb{C}[z^2, z^{-2}]) \) depending on the parity of \( a \).

**Lemma 2.3.** The operator \( L_a \) gives the equivalence of irreducible modules of \( \mathfrak{sl}_2 \):

\[
(\pi'_a, y^{a-1}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a-1}\mathbb{C}[z^2]),
\]

\[
(\pi'_a, y^{2-a}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a}\mathbb{C}[z^2, z^{-2}]/z^{a-2}\mathbb{C}[z^{-2}]).
\]

Moreover, the Casimir operator \( Z_C := 4EF + H^2 - 2H \in Z(U(\mathfrak{sl}_2)) \) takes the value \( (a - 1)(a - 2) - \frac{3}{4} \) in both representations \( (\pi'_a, y^{a-1}\mathbb{C}[y^2]) \) and \( (\pi'_a, y^{2-a}\mathbb{C}[y^2]) \).

**Remark 2.5.** The lemma holds for \( a = 1, 2 \). Actually, for instance, by the quasi-intertwiner \( L_1 \), we obtain the equivalence between the odd part of the (oscillator) representation \( (\pi', y\mathbb{C}[y^2]) \) and the Langlands quotient of the representation \( (\varpi, z\mathbb{C}[z^2, z^{-2}]) \) of \( \mathfrak{sl}_2 \).

**Remark 2.6.** There is a symmetry \( a \leftrightarrow 3 - a \) for \( \pi'_a \). Actually, when \( a \notin \mathbb{Z} \), there is an equivalence between the two representations \( \pi'_a \) and \( \pi'_{3 - a} \) in a suitable setting.

### 2.3 Heun differential operators

Recalling the operator \( R \in U(\mathfrak{sl}_2) \), one observes (with \( \theta_z = z\partial_z \))

\[
\varpi_a(R) = \left\{ (z^2 + z^{-2} - 2 \coth 2\kappa)(\theta_z + \frac{1}{2}) + (a - \frac{1}{2})(z^2 - z^{-2}) + \frac{2\nu}{\sinh^2 2\kappa} \right\} (\theta_z + \frac{1}{2} - \nu) + \frac{2(\nu^2)}{\sinh 2\kappa}.
\]

Hence, conjugating by \( z^{a-1} \) one obtains the following lemma.
Lemma 2.4. For each integer $a$, one has
\[
z^{-a+1} \varpi_a(\mathcal{R}) z^{a-1} = \left\{ (z^2 + z^{-2} - 2 \coth 2\kappa)(\theta_z + a - \frac{1}{2}) + (a - \frac{1}{2})(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa} \right\}(\theta_z + a - \frac{1}{2} - \nu) + \frac{2(\nu)^2}{\sinh 2\kappa} \quad \square
\]

Furthermore, notice that the operators $\varpi_a(H)$, $\varpi_a(E)$ and $\varpi_a(F)$ are invariant under the symmetry $z \to -z$. This implies that the $\varpi_a(\mathcal{R})$ can be expressed in terms of the variable $z^2$. In fact, one has the following.

Proposition 2.5. Let $w := z^2 \coth \kappa$. Then the following relation holds.
\[
z^{-a+1} \varpi_a(\mathcal{R}) z^{a-1} = 4(\tanh \kappa) w(w-1)(w - \coth^2 \kappa) H^a(w, \partial_w),
\]
where $H^a(w, \partial_w)$ is the Heun differential operator given as
\[
H^a(w, \partial_w) = \frac{d^2}{dw^2} + \left( \frac{3 - 2\nu + 2a}{4w} + \frac{-1 - 2\nu + 2a}{4(w-1)} + \frac{-1 + 2\nu + 2a}{4(w - \coth^2 \kappa)} \right) \frac{d}{dw} + \frac{1}{2}(a - \frac{1}{2})(a - \frac{1}{2} - \nu) w - q_a \quad w(w-1)(w - \coth^2 \kappa).
\]

Here the appearing accessory parameter $q_a$ is given by
\[q_a = \left\{ -(a - \frac{1}{2} - \nu)^2 + (\nu)^2 \right\}(\coth^2 - 1) - 2(a - \frac{1}{2})(a - \frac{1}{2} - \nu).
\]

Similarly, for $\tilde{\mathcal{R}}$, one has $z^{-a+1} \varpi_a(\tilde{\mathcal{R}}) z^{a-1} = 4(\tanh \kappa) w(w-1)(w - \coth^2 \kappa) \tilde{H}^a(w, \partial_w)$ with
\[
\tilde{H}^a_{\lambda}(w, \partial_w) = \frac{d^2 f}{dw^2} + \left( \frac{-1 - 2\nu + 2a}{4w} + \frac{3 - 2\nu + 2a}{4(w-1)} + \frac{3 + 2\nu + 2a}{4(w - t)} \right) \frac{d}{dw} + \frac{1}{2}(a - \frac{1}{2})(a + \frac{1}{2} - \nu) w - q_a \quad w(w-1)(w - t).
\]

Proof. Since $w = z^2 \coth \kappa$, one notices that $z\partial_z = 2w\partial_w$. Put $t = \coth^2 \kappa$ for simplicity. Using the relations
\[
\begin{align*}
z^2 + z^{-2} - 2 \coth 2\kappa &= (\tanh \kappa) w^{-1}(w-1)(w - \coth^2 \kappa), \\
z^2 - z^{-2} &= (\tanh \kappa) w^{-1}(w^2 - \coth^2 \kappa), \\
2 / \sinh 2\kappa &= (\tanh \kappa)(\coth^2 - 1),
\end{align*}
\]

one obtains
\[
z^{-a+1} \varpi_a(\mathcal{R}) z^{a-1} = (\tanh \kappa) \left[ \left\{ w^{-1}(w-1)(w-t)(2w\partial_w + a - \frac{1}{2})^{w^{-1}(w-t)} + (a - \frac{1}{2}) w^{-1}(w-t) + (t-1)\nu \right\}^{w^{-1}(w-t)} + (t-1)(\nu)^2 \right].
\]

Taking into account the relation $[\partial_w, w] = 1$, one observes
\[
z^{-a+1} \varpi_a(\mathcal{R}) z^{a-1} = (\tanh \kappa) \left[ 4w(w-1)(w-t)\partial_w^2 + \left\{ (2a - 2\nu + 3)(w-1)(w-t) + (2a - 2\nu - 1)w(w-t) + (2a + 2\nu - 1)w(w-1) \right\} \partial_w + 2w(a - \frac{1}{2})(a - \frac{1}{2} - \nu) + \left\{ -(a - \frac{1}{2} - \nu)^2 + (\nu)^2 \right\} (t-1) - 2(a - \frac{1}{2})(a - \frac{1}{2} - \nu) \right].
\]
Eigenvalue problem of the NcHO and Heun’s ODE, and Rabi’s model

Factoring out the leading coefficient, one obtains the expression of $H^o(w, \partial_w)$. The expression of $H^o_{\lambda}(w, \partial_w)$ follows from the relation

$$z^{-a+1}\omega_a(\mathcal{R})z^{-a-1} - z^{-a+1}\omega_a(\mathcal{R})z^{-a-1}$$

$$= 4(\tanh \kappa) w(w-1)(w-\coth^2 \kappa) \left\{ -\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t} \right\} + \frac{a-\frac{1}{2}}{(w-1)(w-t)}.$$ 

This proves the proposition. \hfill \Box

3 Heun’s operators description for NcHO

The equivalence between the spectral problem of $Q$ and the existence/non-existence of holomorphic solutions of a Heun’s ODE in a certain complex domain is described in [18] for the odd parity. In the same way we have the equivalence for the even parity. Actually, the proof follows from the following quasi-intertwining property of the operator $\mathcal{L}_j$ resulting from Lemma 2.2 and the realization of the representation $\mathcal{R}_j$.

**Proposition 3.1.** The element $\mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2)$ satisfies the following equations:

$$(\mathcal{L}_1 \mathcal{R})(\mathcal{L}_1 u)(z) = \varpi_1(\mathcal{R})(\mathcal{L}_1 u)(z) + (\nu - \frac{3}{2})u'(0)z^{-1},$$

$$(\mathcal{L}_2 \mathcal{R})(\mathcal{L}_2 u)(z) = \varpi_2(\mathcal{R})(\mathcal{L}_2 u)(z) - (\nu - \frac{1}{2})u(0)z^{-2}.$$ 

In particular, the eigenvalue problem $Q \varphi = \lambda \varphi$ for the even and odd case is respectively equivalent to the equation

$$\varpi_1(\mathcal{R})(\mathcal{L}_1 u)(z) = 0 \ (\text{the even case}) \quad \text{and} \quad \varpi_2(\mathcal{R})(\mathcal{L}_2 u)(z) = 0 \ (\text{the odd case}).$$

Here

$$\varpi_1(\mathcal{R}) = \left\{ (z^2 + z^{-2} - 2 \coth 2\kappa)(\theta_z + \frac{1}{2}) + \frac{1}{2}(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa} \right\}(\theta_z + \frac{1}{2} - \nu) + \frac{2(\nu^2)(\nu)}{\sinh 2\kappa},$$

$$\varpi_2(\mathcal{R}) = \left\{ (z^2 + z^{-2} - 2 \coth 2\kappa)(\theta_z + \frac{1}{2}) + \frac{1}{2}(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa} \right\}(\theta_z + \frac{1}{2} - \nu) + \frac{2(\nu^2)(\nu)}{\sinh 2\kappa},$$

where we have put $\theta_z = \frac{\alpha}{\beta \kappa}$. \hfill \Box

We now prove Theorem 1.1. For the even parity one has

$$\varpi_1(\mathcal{R}) = 4(\tanh \kappa) w(w-1)\left((w-\alpha\beta)H^o_{\lambda}(w, \partial_w),$$

where $H^o_{\lambda}(w, \partial_w)$ is the Heun differential operator given by (1.1) in the Introduction, that is

$$H^o_{\lambda}(w, \partial_w) = \frac{d^2}{dw^2} + \left( \frac{1}{2} - p - \frac{1}{2} - p + \frac{1}{w-1} + \frac{1}{w-\alpha\beta} \right) \frac{d}{dw} + \frac{1}{w(w-1)}(w-\alpha\beta)^{\frac{1}{2}}(p+\frac{1}{2})$$

where $p = \frac{2\nu^2}{\sinh 2\kappa}(\nu = \frac{\alpha + \beta}{2\sqrt{\alpha\beta}})$. The accessory parameter $q^+$ is given by

$$q^+ = \left\{ \left( p + \frac{1}{2} \right)^2 - \left( p + \frac{3}{4} \right)^2 \left( \frac{\beta - \alpha}{\beta + \alpha} \right)^2 \right\}(\alpha\beta - 1) - \frac{1}{2}(p + \frac{1}{2}).$$

By its expression, $H^o_{\lambda}(w, \partial_w)$ is a second-order linear differential operator with four regular singular points $0, 1, \alpha\beta$ and $\infty$ on $\mathbb{P}^1(\mathbb{C})$. Notice that the parameter $\nu$ designates the exponents. From these observations, we may summarize the properties of the operator $\varpi_1(\mathcal{R})$ as follows.

**Proposition 3.2.** The second-order linear differential operator $\varpi_1(\mathcal{R})$ with rational coefficients in $z$ has six singular points $z = 0, \pm 1/\sqrt{\alpha\beta}, \pm \sqrt{\alpha\beta}, \infty$. Here, all these six points are of regular singular type. The exponents of those singularities can be read from the following Riemann scheme:

$$\varpi_1(\mathcal{R}) : \left( \begin{array}{cccccccc}
0 & 1/\sqrt{\alpha\beta} & -1/\sqrt{\alpha\beta} & \sqrt{\alpha\beta} & -\sqrt{\alpha\beta} & \infty & \infty & z \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
2p+1 & p + \frac{3}{2} & p + \frac{3}{2} & -p & -p & -2p - 1 & & \end{array} \right).$$ 

\hfill \Box
Notice that since the operator $\varpi_1(\mathcal{R})$ is regular singular at the origin, any formal series solution of $f \in \mathbb{C}[z]$ of the equation $\varpi_1(\mathcal{R})f = 0$ converges to a holomorphic function near the origin. Therefore, using a discussion similar to that in [18] about $L^2$-conditions on $\mathbb{R}$ (or convergence conditions in $\mathbb{C}[y]$) and analytic continuation to a simply-connected region $\Omega'$ containing $0, \pm 1/\sqrt{\alpha \beta}$ in the $z$-space, one can conclude that the spectral problem for the non-commutative harmonic oscillator $Q$ is equivalent to that of finding all the holomorphic solutions $f(z) \in \mathcal{O}(\Omega')$ of the differential equation $\varpi_1(\mathcal{R})f = 0$. This obviously gives the assertion in Theorem 1.1.

Remark 3.1. For the readers’ convenience, we recall the Riemann scheme of the operator $\varpi_2(\mathcal{R})$ from [18]:

$$\varpi_2(\mathcal{R}) : \begin{pmatrix} 0 & 1/\sqrt{\alpha \beta} & -1/\sqrt{\alpha \beta} & \sqrt{\alpha \beta} & -\sqrt{\alpha \beta} & \infty & 2 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 2p+1 & p+1 & p+1 & -(p+\frac{1}{2}) & -(p+\frac{1}{2}) & -2p-1 \end{pmatrix} ; z.$$ (3.2)

4 Degeneration of eigenstates

In this section, we discuss degeneration of eigenstates, that is, focus on eigenvalues of finite-type and their multiplicities. We give an example of finite-type eigenvalues and the proof of Theorem 1.2, which claims that the multiplicity of any eigenvalue of $Q$ is at most 2 and actually reaches 2 in the same parity.

4.1 Polynomial solutions of $\varpi_1(\mathcal{R})f = 0$

Recall first Theorem 1.1 in [24] that the finite-type eigenvalues are of the form

$$\lambda = 2\frac{\sqrt{\alpha \beta (\alpha \beta - 1)}}{\alpha + \beta} (N + \frac{1}{2}) \quad (N \in \mathbb{Z}_{\geq 0}).$$

This implies that $\nu = \lambda \delta \cosh \kappa = N + \frac{1}{2}$ if we have a polynomial solution of $\varpi_1(\mathcal{R})f = 0$. Suppose that $p(z) = \sum_{n=0}^{L} a_n z^{2n}$ ($a_L \neq 0$) is a polynomial solution of the equation $\varpi_1(\mathcal{R})f = 0$ with $\nu = N + \frac{1}{2}$. Since

$$\varpi_1(\mathcal{R})z^{2n} = \left\{ (z^2 + z^{-2} - 2 \coth 2\kappa)(2n + \frac{1}{2}) + \frac{2\nu}{\sinh 2\kappa} \right\} (2n + \frac{1}{2} - \nu) z^{2n} + \frac{2(\nu)^2}{\sinh 2\kappa} z^{2n}$$

$$= (2n + 1)(2n - N)z^{2n+2} + \left\{ -2 \coth 2\kappa (2n + \frac{1}{2}) + \frac{2N + 1}{\sinh 2\kappa} \right\} (2n - N) + \frac{\epsilon^2(2N+1)^2}{2\sinh 2\kappa} z^{2n} + 2n(2n - N)z^{2n-2},$$

one observes

$$\varpi_1(\mathcal{R})p(z) = \sum_{n=1}^{L+1} a_{n-1} (2n - 1)(2n - 2 - N)z^{2n}$$

$$+ \sum_{n=0}^{L} a_n \left\{ -2(2n + \frac{1}{2}) \coth 2\kappa + \frac{2N + 1}{\sinh 2\kappa} \right\} (2n - N) + \frac{\epsilon^2(2N+1)^2}{2\sinh 2\kappa} z^{2n}$$

$$+ \sum_{n=0}^{L-1} a_{n+1} 2(n + 1)(2n + 2 - N)z^{2n} = 0.$$ 

If we look at the coefficient of $z^{2L+2}$ then $a_L(2L + 1)(2L - N) = 0$, whence necessarily $N = 2L$ if $p \neq 0$. Therefore the condition $N$ to be even is necessary for having an even polynomial solution of $\varpi_1(\mathcal{R})f = 0$, i.e., a finite-type eigenfunction of $Q$ by Corollary 3.1.
Eigenvalue problem of the NcHO and Heun’s ODE, and Rabi’s model

Now we assume that $N = 2L$. Then we have

$$\begin{align*}
-2a_{L-1}(2L-1) + a_L \frac{\varepsilon^2(4L^2+1)}{2 \sinh 2\kappa} &= 0, \\
-4a_{L-2}(2L-3) + a_{L-1} \left[ -2 \left\{ -2(2L-\frac{3}{2}) \coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa} \right] &= 0, \\
A_{n}(n-1)(2n-2 - 2L) + a_n \left( 2 \left\{ -2(2n + \frac{3}{2}) \coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} (n-L) + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa} \right) + 4a_{n+1}(n+1 - L) &= 0 \quad (1 \leq n \leq L-2), \\
a_0 \left( -2L \left\{ -\coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa} \right) + 4a_1(1-L) &= 0.
\end{align*}$$

Hence, if

$$\begin{align*}
-4(-\cosh 2\kappa + 5) + 25\varepsilon^2 &= 0 \quad \text{(4.1)}
\end{align*}$$

holds, then the polynomial $p(z) = a_0 + a_1 z^2 = a_0 \left( 1 + \frac{\sinh 2\kappa}{2 \sinh 2\kappa} \right) z^2$ is a non-trivial solution of $\varpi_1(\mathcal{R})p = 0$. We now observe the existence of solutions for (4.1). For simplicity we put $\alpha = 1$ and $\beta > 1$. Since $\varepsilon^2 = \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2$ and $\cosh 2\kappa = \frac{\alpha^2 + 1}{\alpha - 1}$, the equation (4.1) turns to be

$$\begin{align*}
25 \left( \frac{\beta - 1}{\beta + 1} \right)^2 + 4 \frac{\beta + 1}{\beta - 1} - 20 &= 0.
\end{align*}$$

Define next the cubic polynomial

$$f(\beta) = 25(\beta - 1)^3 + 4(\beta + 1)^3 - 20(\beta + 1)^2(\beta - 1).$$

Then, since $f(1) > 0$, $f(2) < 0$, $f(8) < 0$, $f(9) > 0$, it follows immediately that we have 2 solutions of (4.1), one in the interval $(1, 2)$ and another one in $(8, 9)$. This shows that there exists a pair $(\alpha, \beta)$ such that $Q\varphi = 5 \frac{\sqrt{\alpha \beta} \alpha (\alpha - \beta)}{\alpha + \beta} \varphi$ and $\varphi(-x) = \varphi(x)$.

The general theorem in [24] implies that the multiplicity of the eigenvalue $5 \frac{\sqrt{\alpha \beta} \alpha (\alpha - \beta)}{\alpha + \beta}$ is 2 for this $Q = Q(\alpha, \beta)$. Hence, the eigenvalue curves can be actually crossing as the numerical graph in [17] (see Figure 1 on p.648) has indicated.

In general, we define the tri-diagonal $(L + 1) \times (L + 1)$-matrix $B_{2L}(\alpha, \beta) = (B_{ij})_{0 \leq i, j \leq L}$ by

$$\begin{align*}
B_{0,0} &= \left[ -2L \left\{ -\coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa} \right], \\
B_{n-1,n} &= \left( 2n-1 \right)(2n-2 - 2L), \\
B_{n,n} &= \left[ 2 \left\{ -2(2n + \frac{3}{2}) \coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} (n-L) + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa} \right], \\
B_{n+1,n} &= 4(n+1)(n+1 - L) = 0 \quad (n = 1, 2, \ldots, L-2), \\
B_{L-2,L-1} &= -4(2L-3), \\
B_{L-1,L-1} &= -2 \left\{ -2(2L-\frac{3}{2}) \coth 2\kappa + 4 \frac{L+1}{\sinh 2\kappa} \right\} + \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa}, \\
B_{L-1,L} &= -2(2L-1), \\
B_{L,L} &= \frac{\varepsilon^2(4L+1)^2}{2 \sinh 2\kappa}.
\end{align*}$$

Note that $B_{ij} = 0$ if $|i - j| > 1$.

Since there can not be two independent polynomial solutions of $\varpi_1(\mathcal{R})f = 0$, we notice that the rank of the matrix satisfies $L \leq \text{rank}(B_{2L}(\alpha, \beta)) \leq L + 1$. Clearly one has the following.

**Proposition 4.1.** Let $L \in \mathbb{N}$. If $\alpha, \beta (\alpha \neq \beta)$ satisfy the algebraic equation $\det(B_{2L}(\alpha, \beta)) = 0$, then $\lambda = 2 \frac{\sqrt{\alpha \beta} \alpha (\alpha - \beta)}{\alpha + \beta} \left( 2L + \frac{1}{2} \right) \in \Sigma^+_\alpha$.

**Remark 4.1.** Since $\alpha \neq \beta$, one has that the coefficient $B_{L,L} \neq 0$. Thus, if we set $B_{2L}(\alpha, \beta) = (B_{ij})_{0 \leq i, j \leq L-1}$ we may consider the equation $\det(B_{2L}(\alpha, \beta)) = 0$ in place of $\det(B_{2L}(\alpha, \beta)) = 0$.

**Remark 4.2.** The odd cases corresponding to Proposition 4.1 can be established in the same way.
4.2 Proof of Theorem 1.2

We now prove that the multiplicity \( m_\lambda \) of the eigenvalue \( \lambda \) of \( Q \) is at most 2. Since \( Q \) is unitarily equivalent to a couple of quantum harmonic oscillators when \( \alpha = \beta \) ([23, 24]), one may assume that \( \alpha \neq \beta \). Since one knows from [24] the multiplicity of each eigenvalue is at most 3, it is enough to show that \( m_\lambda \neq 3 \) for every \( \lambda \).

Suppose \( m_\lambda = 3 \). Then we have either \( \lambda \in \Sigma_0^+ \cap \Sigma_\infty^+ \) or \( \lambda \in \Sigma_0^- \cap \Sigma_\infty^- \). We now assume \( \lambda \in \Sigma_0^+ \cap \Sigma_\infty^- \). This implies that we have \( \dim_{\mathbb{C}} \{ f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0 \} = 2 \) and \( \lambda \) is of the form \( \lambda = 2 \Sigma^{(2)}(\alpha \beta \bar{\alpha} \bar{\beta})^{-1} (2L + \frac{1}{2}) \) for some \( L \in \mathbb{N} \). Recall the relation \( p = L - \frac{1}{2} \). Then, it follows that the parameter \( p \) in the Riemann’s scheme of the Heun operator \( H_\lambda^+ \) satisfies \( p + \frac{1}{2} \in \mathbb{N} \).

Let \( f_1(w) \) and \( f_2(w) \) respectively be a polynomial and a holomorphic solution of \( H_\lambda^+(w, \partial_w) f = 0 \), respectively. Let \( \tilde{f}_j(z) (j = 1, 2) \) be the respective solutions of \( w_1(\mathcal{R}) \tilde{f}(z) = 0 \), that is, \( \tilde{f}_j(w) = \tilde{f}_j(z) (w = z^2 \coth \kappa) \). Put \( u_j = \mathcal{L}_1^{-1} \tilde{f}_j \). Note that \( u_1 \) is an even polynomial in \( \mathbb{C}[y] \). We may construct a constant term free even solution \( u^+ = \mathbb{C}[y] \) of \( \pi^+(\mathcal{R}) u^+ = 0 \) by a suitably chosen linear combination of \( u_1 \) and \( u_2 \). Then, by Corollary 3.1, one verifies that \( w_2(\mathcal{R}) (\mathcal{L}_2 u^+)(z) = 0 \). If we put \( \tilde{g}(z) = (\mathcal{L}_2 u^+)(z) \) and define \( g^+(w) \) by the equation \( g^+(w) = \tilde{g}(z) \), then \( g^+(w) \in \mathcal{O}(\Omega) \) is a solution of \( H_\lambda^+ g^+ = 0 \). Therefore, by assumption we conclude that \( \dim_{\mathbb{C}} \{ f \in \mathcal{O}(\Omega) \mid H_\lambda^+ g = 0 \} = 2 \).

Now we recall the Riemann scheme of \( H_\lambda^- \)

\[
\begin{pmatrix}
0 & 1 & \alpha \beta & \infty & 0 \\
0 & 0 & 0 & \frac{3}{2} & -p \\
p & p + 1 & -p + \frac{1}{2} & -p
\end{pmatrix}
\]

and the monodromy representation of the differential equation \( H_\lambda^- g = 0 \). We consider a base point near the origin. Take a basis of local solutions at this point and denote the monodromy matrix around the singular points \( 0, 1, \alpha \beta, \infty \) by \( A_0, A_1, A_2, A_3 \), respectively. Note that \( A_0 A_1 A_2 A_3 = I \). The existence of two dimensional holomorphic solutions on \( \Omega \) implies that \( A_0 = A_1 = I \), that is, both 0 and 1 are apparent singular points. The monodromy matrices \( A_2 \) and \( A_3 \) have two distinct eigenvalues, 1 and \( -1 \), and thus are semisimple. Then, since \( A_3 = A_2^{-1} \), the monodromy representation factors through the cyclic group of order two. We then choose a basis such that \( A_2 = A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Then, the solution corresponding to the vector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is invariant under the monodromy representation. It follows that this solution is meromorphic on \( \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{ \infty \} \), whence it actually is a rational function of \( w \). However, since \( p \in \mathbb{N} - \frac{1}{2} \), from the Riemann scheme of \( H_\lambda^- \), one finds that no such rational solution can exist. This contradicts the claim \( \dim_{\mathbb{C}} \{ f \in \mathcal{O}(\Omega) \mid H_\lambda^- g = 0 \} = 2 \), hence the assumption. Therefore we have \( \Sigma_0^+ \cap \Sigma_\infty^- = \emptyset \). Similarly one can show \( \Sigma_0^- \cap \Sigma_\infty^+ = \emptyset \). This completes the proof of the theorem.

4.3 Heun polynomials for \( H_\lambda^+ f = 0 \)

In this subsection assume again that \( \alpha \neq \beta \). As in the odd case studied in [19], from \( H_\lambda^+ f = 0 \), one can determine a shape of the solution corresponding to the eigenvalue \( \Sigma_0^+ \). In the terminology in [26] (see, p. 41) these solutions are given by Heun polynomials. We first recall the Riemann scheme of the Heun equation \( H_\lambda^+ f = 0 \).

\[
\begin{pmatrix}
0 & 1 & \alpha \beta & \infty & 0 \\
0 & 0 & 0 & \frac{3}{2} & -p \\
p + \frac{1}{2} & p + \frac{3}{2} & -p & -(p + \frac{1}{2})
\end{pmatrix}
\]

The Heun polynomial, which we denote by \( H_p \) (see [26]) is, by definition, a solution of the Heun equation given by the form

\[
H_p(w) = w^{\sigma_1} (w - 1)^{\sigma_2} (w - \alpha \beta)^{\sigma_3} p(w),
\]

where \( p(w) \) is a polynomial in \( w \), and \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are, each of them, one of the two possible exponents at the corresponding singularity. Notice that \( p + \frac{1}{2} \in \mathbb{N} \) if the corresponding \( \lambda \) is in \( \Sigma_0^+ \).
Eigenvalue problem of the NcHO and Heun’s ODE, and Rabi’s model

Theorem 4.2. Suppose that \( \dim \{ f \in \mathcal{O}(\Omega) \mid H_p^+ f = 0 \} = 2 \). Then, there exist Heun polynomials \( H_{p_1}(w) \) and \( H_{p_2}(w) \) such that \( \{ f \in \mathcal{O}(\Omega) \mid H_p^+ f = 0 \} = \mathbb{C} H_{p_1} \oplus \mathbb{C} H_{p_2} \). More precisely, \( H_{p_1}(w) \) is equal to a polynomial \( p_1(w) \) of degree at most \( p + \frac{1}{2} \) and \( H_{p_2}(w) = (w - \alpha \beta)^{-\nu} p_2(w) \), \( p_2(w) \) being a polynomial of degree at most \( p - \frac{1}{2} \), and these polynomials \( p_j(w) (j = 1, 2) \) are unique up to scalar multiples.

The proof of this theorem can be done in a way similar to that in [19]. We thus omit the proof. Furthermore, as in [19], we have two converse statements of Theorem 4.2. The first one is the following.

Theorem 4.3. Suppose that the Heun equation \( H_p^+ f = 0 \) has a solution of the form \( q(w)(w - \alpha \beta)^{\frac{1}{2}} \) at the origin, where \( q(w) \) is a non-zero rational function. Then, one has \( \dim \{ f \in \mathcal{O}(\Omega) \mid H_p^+ f = 0 \} = 2 \).

The second is the case where one has a rational solution of the equation \( H_p^+ f = 0 \).

Theorem 4.4. Assume \( p + \frac{1}{2} \in \mathbb{N} \). Suppose that the Heun equation \( H_p^+ f = 0 \) has a non-zero rational solution of \( w \) at the origin. Then, one has \( \dim \{ f \in \mathcal{O}(\Omega) \mid H_p^+ f = 0 \} = 2 \).

The proofs of these theorems are similar to those in [19] and left to the readers. Note that once the condition of Theorem 4.3/4.4 holds, all the assertions in Theorem 4.2 follow.

5 Connection with the Rabi model via confluence process

In this section we will observe the relation between the NcHO and the quantum Rabi model. Precisely, we find that the Rabi model (see [15, 2, 7, 32]) can be obtained from the Heun differential equation defined by \( \mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2) \) by a suitable choice of a triple \((\kappa, \varepsilon, \nu) \in \mathbb{R}^3 \).

The quantum Rabi model is define by the Hamiltonian

\[
H_{\text{Rabi}}/\hbar = \omega \psi^\dagger \psi + \Delta \sigma_z + g \sigma_x (\psi^\dagger + \psi).
\]

Here \( \psi = (x + \partial_x)/\sqrt{2} \) (resp. \( \psi^\dagger = (x - \partial_x)/\sqrt{2} \)) is the annihilation (resp. creation) operator for a bosonic mode of frequency \( \omega \), \( \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \), \( \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) are the Pauli matrices for the two-level system, \( 2\Delta \) is the energy difference between the two levels, and \( g \) denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set \( \hbar = 1 \) and \( \omega = 1 \).

In order to observe the relation between the NcHO and the Rabi model, we will consider the confluent Heun differential equation which is derived by the standard confluence procedure from the Heun differential equation defined by \( \mathcal{R} \) in Lemma 2.1 via the representation \( \pi'_u (\cong \pi_u) \) of \( \mathfrak{sl}_2 \). Roughly speaking, our observation shows that the quantum Rabi model can be obtained by a confluence process by \( \mathcal{R} \) through their respective Heun’s pictures:

\[
\begin{array}{c}
\text{Heun ODE} \\
\mathcal{R} \quad \overset{\text{confluence process}}{\longrightarrow} \\
\text{Confluent Heun ODE} \sim \text{Rabi model}
\end{array}
\]

In this picture, under the action defined by the representation (a flat picture of principal series) \( \pi'_u \) on \( \mathbb{C}[y, y^{-1}] \) and \( \pi_u \) on \( \mathfrak{sl}_2 \) (see §5.1 below), which is not equivalent in general to the oscillator representation \( \pi' \), \( \mathcal{R} \) provides a target Heun operator for obtaining the confluent Heun operator corresponding the Rabi model through the Laplace transform \( \mathcal{L}_u \).

5.1 Confluent Heun equations derived from the Rabi model

From now we assume \( a \in \mathbb{R} \), not necessarily an integer. The analysis of the quantum Rabi model has extensively used the Bargmann representation of bosonic operators which is realized by the
following Bargmann transform $B$ (from real coordinate $x$ to complex variable $z$) \[1, 28\].

\[
(Bf)(x) = \sqrt{2} \int_{-\infty}^{\infty} f(x)e^{2\pi xz-\pi^2 x^2-\frac{z^2}{2}} dx.
\]

Here the Bargmann space is by definition a Hilbert space of entire functions equipped with the inner product

\[
(f|g) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)}g(z)e^{-|z|^2}d(\text{Re}(z))d(\text{Im}(z)).
\]

The main advantage is simply due to the fact that

\[
\psi^\dagger = (x-\partial_x)/\sqrt{2} \to z \quad \text{and} \quad \psi = (x+\partial_x)/\sqrt{2} \to \partial_z.
\]

**Remark 5.1.** This makes the Rabi model to be a first order differential operator. The same situation, however, does not appear for NeHO. This explains one of the reasons why the analysis for NeHO is rather difficult.

Then the Schrödinger equation \[H_{\text{Rabi}}\phi = E\phi\] of the quantum Rabi model is reduced to the following 2nd order differential equation:

\[
\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0,
\]

where

\[
p(z) = \frac{(1-2E-2g^2)x - g}{z^2 - g^2}, \quad q(z) = \frac{-g^2 z^2 + gz + E^2 - g^2 - \Delta^2}{z^2 - g^2}.
\]

Write \[f(w) = e^{-g^2}\phi(x)\], where \[x = (g + z)/2g\]. Substituting $f$ into the equation above, one finds the function $\phi$ satisfies the following confluent Heun equation (by a calculation similar to that in [32]). Then one has $H_{\text{Rabi}}^{\text{Rabi}}\phi = 0$, where

\[
H_1^{\text{Rabi}} := \frac{d^2}{dx^2} + \left(-4g^2 + \frac{1-(E+g^2)}{x} + \frac{1-(E+g^2+1)}{x-1}\right)\frac{d}{dx} + \frac{4g^2(E+g^2)x + \mu}{x(x-1)},
\]

with the accessory parameter $\mu = (E+g^2)^2 - 4g^2(E+g^2) - \Delta^2$.

Setting $f(z) = e^{g^2}\phi(x)$, where \[x = (g - z)/2g\], one obtains another equation as

\[
H_2^{\text{Rabi}} := \frac{d^2}{dx^2} + \left(-4g^2 + \frac{1-(E+g^2+1)}{x} + \frac{1-(E+g^2)}{x-1}\right)\frac{d}{dx} + \frac{4g^2(E+g^2+1)x + \mu}{x(x-1)}.
\]

**Remark 5.2.** Each equation $H_j^{\text{Rabi}}\phi = 0 \ (j = 1, 2)$ has a one dimensional family of analytic solutions. The suitable linear combination of these analytic solutions gives symmetric (resp. anti-symmetric) solutions (cf. [32]).

### 5.2 Confluence process of the Heun equation

Put $t = \coth^2 \kappa (>1)$. The Heun operator $H^\alpha(w, \partial_w)$ derived from $\varpi_\alpha(R)$ is give by

\[
H^\alpha(w, \partial_w) = \frac{d^2}{dw^2} + \left(\frac{3-2\nu+2\alpha}{4w} + \frac{-1-2\nu+2\alpha}{4(w-1)} + \frac{-1+2\nu+2\alpha}{4(w-t)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w - qa}{w(w-1)(w-t)}.
\]

The corresponding generalized Riemann scheme ([27]) is expressed as

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & t & \infty \\
\frac{1+2\nu-2a}{4} & \frac{5+2\nu-2a}{4} & \frac{5-2\nu-2a}{4} & \frac{-1-2\nu+2a}{4} \\
\end{pmatrix}^T ; \quad w \quad qa.
\]
Here the first line indicates the $s$-rank of each singularity (see [27]). Replace $a$ (resp. $\nu$) by $a + p$ (resp. $\nu + p$) at the expression of $H^u(w, \partial_w)$ above. It then follows that

$$A := \frac{1}{4}(-1 - 2\nu + 2a), \quad B := a + p + \frac{1}{2}, \quad C := \frac{1}{4}(3 - 2\nu + 2a) = 1 + A, \quad D := A.$$  

Then

$$w(w - 1)(w - t)H^u(w, \partial_w) = w(w - 1)(w - t)\partial_w^2 + \left[C(w - 1)(w - t) + Dw(w - t) + (A + B + 1 - C - D)w(w - 1)\right]\partial_w + ABw - qa.$$  

Let us consider a confluence process of the singular points at $w = t$ and $w = \infty$ ([27] p.100, Table 3.1.2). The corresponding process is given by $t := \rho^{-1}, B := rp^{-1}$ and $\rho \to 0$ (equivalently $p \to \infty$):

$$- \lim_{\rho \to 0} w(w - 1)(w - t)\rho H^u(w, \partial_w) = w(w - 1)\partial_w^2 + \left[C(w - 1) + Dw - rw(w - 1)\right] - rAw + \lim_{\rho \to 0} \rho qa.$$  

Now we take $\varepsilon = k\rho$ for some constraint $k$. Then

$$\lim_{\rho \to 0} \rho qa = \lim_{\rho \to 0} \left\{ - (a - \frac{1}{2} - \nu)^2 + (\varepsilon(\nu + p))^2 \right\}(1 - \rho) - \left\{ 2\rho(a + p - \frac{1}{2}) \right\} \cdot (a - \frac{1}{2} - \nu) = -(2A)^2 - 4A + k^2.$$  

Hence one obtains the following confluent Heun equation.

$$\frac{d^2 \phi}{dw^2} + \left[ - r + \frac{1 + A}{w} + \frac{A}{w - 1} \right] \frac{d\phi}{dw} + \frac{-rAw - (2A)^2 - 4A + k^2}{w(w - 1)} \phi = 0,$$

whose generalized Riemann’s scheme is given as

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \infty \\ 0 & 0 & A \\ -A & 1 - A & 1 + A \\ 0 & 1 & t \end{pmatrix} \quad \text{with} \quad A = \frac{1}{4}(-1 - 2\nu + 2a).$$

Notice that $w = \infty$ is an irregular singularity with $s$-rank 2 (see e.g. [27], p.33).

Let us compare this equation with the confluent Heun operator $H^u_{\text{Rabi}}$ for the Rabi model above. Then, if we take $r = 4q^2, A = - (E + g^2)$ with a suitable choice of $k$ (i.e. $k^2 = 5A^2 + 4(1 - g^2)A - \Delta^2$) in this equation gives the latter.

**Remark 5.3.** Recall the operator $\tilde{R} \in \mathcal{U}(\mathfrak{sl}_2)$. Then, one has the confluent Heun operator from the Heun operator $H^u_{\tilde{R}}(w, \partial_w)$ corresponding to $\omega_{\tilde{R}}(\tilde{R})$ as

$$\tilde{H}^u_{\tilde{R}}(w, \partial_w) \rightarrow \frac{d^2 \phi}{dw^2} + \left[ - r + \frac{1 + A}{w} + \frac{A}{w - 1} \right] \frac{d\phi}{dw} + \frac{-r(1 + A)w - (2A)^2 - 4A + k^2}{w(w - 1)}.$$

A confluence procedure for $\omega_{\tilde{R}}(\tilde{R})$, similar to the one we have taken in the case of $\omega_{\tilde{R}}(\tilde{R})$, yields $H^u_{\text{Rabi}}$ in the preceding subsection.

**Remark 5.4.** One can find an element $K$ (resp. $\tilde{K}$) $\in \mathcal{U}(\mathfrak{sl}_2)$ of order two such that $\omega_K(K)$ (resp. $\omega_{\tilde{R}}(\tilde{K})$) essentially (i.e. up to the accessory parameter) provides the confluent Heun operator $H^u_{\text{Rabi}}$ (resp. $H^u_{\text{Rabi}}$) [31].

**Acknowledgement:** This work is partially supported by Grand-in-Aid for Scientific Research (B) No. 21340011 of JSPS. The author wishes to thank Fumio Hiroshima and Takashi Ichinose for stimulating discussion, providing also useful information on the Rabi model and related works, which has led him a special attention to the Rabi model. He also thanks Alberto Parmeggiani for many valuable comments. Further, he would like to acknowledge the Department of Mathematics, Indiana University, especially Mihai Ciucu and Richard Bradley, for giving him an opportunity at the Distinguished Lecture Series to deliver the recent research work on the NeHO including materials of this paper a part and their warm hospitality in Spring 2013.
References


Masato Wakayama
Institute of Mathematics for Industry,
Kyushu University
744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
wakayama@imi.kyushu-u.ac.jp
List of MI Preprint Series, Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1  Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2  Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3  Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4  Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5  Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p-adic field

MI2008-6  Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7  Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8  Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9  Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2008-10  Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11  Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12  Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the $L^2$ a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13  Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
MI2008-14  Takashi NAKAMURA
   Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15  Takashi NAKAMURA
   Some topics related to Hurwitz-Lerch zeta functions

MI2009-1  Yasuhide FUKUMOTO
   Global time evolution of viscous vortex rings

MI2009-2  Hidetoshi MATSUI & Sadanori KONISHI
   Regularized functional regression modeling for functional response and predictors

MI2009-3  Hidetoshi MATSUI & Sadanori KONISHI
   Variable selection for functional regression model via the $L_1$ regularization

MI2009-4  Shuichi KAWANO & Sadanori KONISHI
   Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5  Toshiro HIRANOUCHI & Yuichiro TAGUCHII
   Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6  Kenji KAJIWARA & Yasuhiro OHTA
   Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7  Yoshiyuki KAGEI
   Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8  Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
   Nonlinear regression modeling via the lasso-type regularization

MI2009-9  Takeshi TAKAISHI & Masato KIMURA
   Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10  Shingo SAITO
   Generalisation of Mack’s formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11  Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
   Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12  Tetsu MASUDA
   Hypergeometric $\pi$-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13  Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
   A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14  Yasunori MAEKAWA
   On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
Large time behavior of the semigroup on $L^p$ spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

Spectrum in multi-species asymmetric simple exclusion process on a ring

Non-linear algebraic differential equations satisfied by certain family of elliptic functions

Local Instability of an Elliptical Flow Subjected to a Coriolis Force

Sparse functional principal component analysis via regularized basis expansions and its application

Semi-supervised logistic discrimination via regularized Gaussian basis expansions

Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

Recent progress in value distribution of the hyperbolic Gauss map

On very accurate enclosure of the optimal constant in the a priori error estimates for $H^2_0$-projection

Ramification of local fields and Fontaine’s property (Pm)

Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance
MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA  
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI  
Hecke’s zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA  
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA  
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA  
Projective reduction of the discrete Painlevé system of type($A_2 + A_1$)$^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI  
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA  
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA  
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA  
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA  
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI  
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI  
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI  
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI  
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA  
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO  
Abstract collision systems on groups
MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres

MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space

MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field

MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling

MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

MI2010-21 Hiroki MASUDA
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test

MI2010-22 Toshimitsu TAKAESU
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs

MI2010-23 Takahiro ITO, Mitsuhiko FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Composition, union and division of cellular automata on groups

MI2010-24 Toshimitsu TAKAESU
A Hardy’s Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra
Toshimitsu TAKAESU
On the Essential Self-Adjointness of Anti-Commutative Operators

Reiichiro KAWAI & Hiroki MASUDA
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling

Chikashi ARITA & Daichi YANAGISAWA
Exclusive Queueing Process with Discrete Time

Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Motion and Bäcklund transformations of discrete plane curves

Takanori YASUDA, Masaya YASUDA, Takeshi SHIMOYAMA & Jun KOGURE
On the Number of the Pairing-friendly Curves

Chikashi ARITA & Kohei MOTEGI
Spin-spin correlation functions of the $q$-VBS state of an integer spin model

Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and spike detection via Gaussian basis expansions

Nobutaka NAKAZONO
Hypergeometric $\tau$ functions of the $q$-Painlevé systems of type $(A_2 + A_1)^{(1)}$

Yoshiyuki KAGEI
Global existence of solutions to the compressible Navier-Stokes equation around parallel flows

Nobushige KUROKAWA, Masato WAKAYAMA & Yoshinori YAMASAKI
Milnor-Selberg zeta functions and zeta regularizations

Kissani PERERA & Yoshihiro MIZOGUCHI
Laplacian energy of directed graphs and minimizing maximum outdegree algorithms

Takanori YASUDA
CAP representations of inner forms of $Sp(4)$ with respect to Klingen parabolic subgroup

Chikashi ARITA & Andreas SCHADSCHNEIDER
Dynamical analysis of the exclusive queueing process

Yasuhide FUKUMOTO& Alexander B. SAMOKHIN
Singular electromagnetic modes in an anisotropic medium

Hiroki KONDO, Shingo SAIITO & Setsuo TANIGUCHI
Asymptotic tail dependence of the normal copula

Takehiro HIROTSU, Hiroki KONDO, Shingo SAIITO, Takuya SATO, Tatsushi TANAKA & Setsuo TANIGUCHI
Anderson-Darling test and the Malliavin calculus

Hiroshi INOUE, Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling via Compressed Sensing
MI2011-5 Hiroshi INOUE
Implications in Compressed Sensing and the Restricted Isometry Property

MI2011-6 Daeju KIM & Sadanori KONISHI
Predictive information criterion for nonlinear regression model based on basis expansion methods

MI2011-7 Shohei TATEISHI, Chiaki KINJYO & Sadanori KONISHI
Group variable selection via relevance vector machine

MI2011-8 Jan BREZINA & Yoshiyuki KAGEI
Decay properties of solutions to the linearized compressible Navier-Stokes equation around time-periodic parallel flow
Group variable selection via relevance vector machine

MI2011-9 Chikashi ARITA, Arvind AYYER, Kirone MALLICK & Sylvain PROLHAC
Recursive structures in the multispecies TASEP

MI2011-10 Kazunori YASUTAKE
On projective space bundle with nef normalized tautological line bundle

MI2011-11 Hisashi ANDO, Mike HAY, Kenji KAJIWARA & Tetsu MASUDA
An explicit formula for the discrete power function associated with circle patterns of Schramm type

MI2011-12 Yoshiyuki KAGEI
Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow

MI2011-13 Vladimír CHALUPECKÝ & Adrian MUNTEAN
Semi-discrete finite difference multiscale scheme for a concrete corrosion model: approximation estimates and convergence

MI2011-14 Jun-ichi INOGUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves

MI2011-15 Hiroshi INOUE
A generalization of restricted isometry property and applications to compressed sensing

MI2011-16 Yu KAWAKAMI
A ramification theorem for the ratio of canonical forms of flat surfaces in hyperbolic three-space

MI2011-17 Naoyuki KAMIMURA
Matroid intersection with priority constraints

MI2012-1 Kazufumi KIMOTO & Masato WAKAYAMA
Spectrum of non-commutative harmonic oscillators and residual modular forms

MI2012-2 Hiroki MASUDA
Mighty convergence of the Gaussian quasi-likelihood random fields for ergodic Levy driven SDE observed at high frequency
A Weak RIP of theory of compressed sensing and LASSO

Hamiltonian bifurcation theory for a rotating flow subject to elliptic straining field

On the maximal number of exceptional values of Gauss maps for various classes of surfaces

Hamiltonian bifurcation theory for a rotating flow subject to elliptic straining field

Topological Measurement of Protein Compressibility via Persistence Diagrams

Solutions to a $q$-analog of Painlevé III equation of type $D_7^{(1)}$

A new approach to the Pareto stable matching problem

Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow

Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

Adaptive basis expansion via the extended fused lasso

On simplicity of the lowest eigenvalue of non-commutative harmonic oscillators

On the convergence rates for the compressible Navier-Stokes equations with potential force

A Counter-example to Thomson-Tait-Chetayev’s Theorem

A unified view of topological invariants of barotropic and baroclinic fluids and their application to formal stability analysis of three-dimensional ideal gas flows

Asymptotics for functionals of self-normalized residuals of discretely observed stochastic processes

On Counting Output Patterns of Logic Circuits

RIPless Theory for Compressed Sensing
MI2013-6 Hiroshi INOUE
   Improved bounds on Restricted isometry for compressed sensing

MI2013-7 Hidetoshi MATSUI
   Variable and boundary selection for functional data via multiclass logistic regression modeling

MI2013-8 Hidetoshi MATSUI
   Variable selection for varying coefficient models with the sparse regularization

MI2013-9 Naoyuki KAMIYAMA
   Packing Arborescences in Acyclic Temporal Networks

MI2013-10 Masato WAKAYAMA
   Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun’s differential equations, eigenstates degeneration, and Rabi’s model