The ventilation of the toxic gaseous pollutant is the main interest of our study. Simultaneous measurement of passive tracer concentration and vertical wind component with high temporal resolution is very unique and allows us to directly derive the advective and turbulent vertical transport terms $\psi^\prime C^\prime$ and $\psi C^\prime w^\prime$ from the equation of advection (see below). The magnitude of the terms depends on the character of flow in the given plane and can vary significantly. There are not many papers concerning the concentration flux. Eckel and Robey [5] examined these factors in a laboratory atmospheric boundary layer flow and found that turbulent and advective transports are of the same magnitude. The magnitude of the turbulent transport of passive contaminant term was found significant due to the opposing of the mean flow within the plant canopy [Mayeux [5]].

2. Definitions. The advection equation for a scalar $u$, such as concentration of passive contaminant, is given by the vector of the scalar is expressed mathematically as

$$\nabla \cdot (\psi \mathbf{C}) + \frac{\partial}{\partial t} \int_{V} \psi \mathbf{C} \, dV = 0$$

where $\nabla$ is the divergence operator and $\psi$ is the velocity vector field. Any variable $\psi^\prime C^\prime$ within the urban canopy is of the form $\psi^\prime C^\prime$. The magnitude of the terms depends on the character of flow in the given plane and can vary significantly. The equation for advection of passive contaminant $\psi$ is given by

$$\nabla \cdot (\psi \mathbf{C}) + \frac{\partial}{\partial t} \int_{V} \psi \mathbf{C} \, dV = 0$$

where $\nabla^\prime$ is the mean and fluctuation part of vertical wind component, respectively, and $\psi^\prime C^\prime$ is the mean and fluctuation term of concentration of passive contaminant. The second term in the left-hand side of the equation 2.2 is the advective transport term. The second term is the mean value of the product of passive contaminant concentration and the mean vertical wind component. The third term is the mean value of the product of passive contaminant concentration and the mean vertical wind component. The third term is the mean value of the product of passive contaminant concentration and the mean vertical wind component.

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4. Experimental set-up. The experiment was carried out in the Boundary Layer Wind Tunnel at Wind Engineering Center of Tokyo Polytechnic University, Japan. The 34 m long facility provides test section equipped with translateable and the cross-section of the tunnel measuring 1.2 m in width and 1.5 m in height. Measuring instruments (see Fig. 3.1(b)) were placed on a controlled positioning system, which allowed to move the probes in all 3 directions and to rotate the 3-axis. The spires and roughness elements were used to model the urban atmospheric boundary layer in the scale 1:400.

The spires are two-dimensional structures placed at the very beginning of the development section of the wind-tunnel (just behind the stilling chamber). The roughness elements were used to model the urban canopy. The actual set-up consisted at least of 7 cubes. The actual number varied according to the set-up. The spires and roughness elements were placed in regular pattern. Spires and roughness elements together with one of the idealized urban canopies are shown in Fig. 3.1.

The research team of Wind Engineering Center of Tokyo Polytechnic University has developed new method for simultaneous measurement of velocity and concentration by the means of constant temperature thermistoranometer (CTA) and flame ionisation detection (FID) of the tracer gas (Yoshie et al., [7]). This set-up allows to measure the tracer concentration by the means of constant temperature thermo-anemometry (CTA) and flame ionisation detection (FID) of the tracer gas (Yoshie et al., [7]). The location was the same for all set-ups. Density of ethylene is 1.8 kg m⁻³ in standard atmospheric conditions and it can be considered as a passive tracer because its density is very close to standard air density (1.2 kg m⁻³). The molecular diffusivity is negligible compared to the turbulent diffusivity. The experimental conditions were carefully checked by series of measurements over the whole range of the wind speeds used in the experiments. Building Reynolds number \(Re_b\) were found for experiments with \(Q > 30 m^3/h\) and \(Re_b \approx 10^6\) (or 10² number, respectively). The dimensionless experimental conditions were: \(Q \approx 300 m^3/h\) (i.e. 18 l per hour) and \(Re_b \approx 10^3\) (or \(10^2\) number, respectively). For more detailed description see [11].

Tab. 3.1 Experimental conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q) (m³/h)</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>(Re_b)</td>
<td>10^6</td>
<td>10^6</td>
<td>10^6</td>
<td>10^6</td>
</tr>
<tr>
<td>(U_{ref})</td>
<td>5 m/s</td>
<td>5 m/s</td>
<td>5 m/s</td>
<td>5 m/s</td>
</tr>
<tr>
<td>(U_{max})</td>
<td>20 m/s</td>
<td>20 m/s</td>
<td>20 m/s</td>
<td>20 m/s</td>
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<tr>
<td>(H)</td>
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<td>0.29H</td>
<td>0.29H</td>
<td>0.29H</td>
</tr>
<tr>
<td>(H_0)</td>
<td>0.29H</td>
<td>0.29H</td>
<td>0.29H</td>
<td>0.29H</td>
</tr>
</tbody>
</table>

4. Modelled boundary layer. The logarithmic law describing mean vertical profile, \(c_\mu(z)\), in the lower part of the atmospheric boundary layer can be mathematically written as:

\[
c_\mu(z) = c_\mu(0) + \frac{1}{\kappa} \ln \left( \frac{z}{z_0} \right)
\]

where \(c_\mu(0)\) is friction velocity, \(\kappa\) is von Kármán constant, \(z_0\) is displacement height, and \(z\) is distance from ground. Another way how to describe the mean wind vertical profile in the power law:

\[
c_\mu(z) = \frac{1}{\kappa} \ln \left( \frac{z}{z_0} \right)
\]

5. Mean concentration field. Since the experiments were concluded in the scale, we need to define dimensionless or normalized variables to obtain values, which can be compared with the other experiments or numerical simulations. All the coordinates are given as a ratio of the building height, \(H\). The non-dimensional concentration is defined according to equation:

\[
c' = \frac{c_\mu}{c_\mu(0)}
\]

where \(c_\mu\) is measured concentration, \(H\) is the building height, \(U_{ref}\) is the average wind speed at 1.5 times building height (x=H/3m, or 42 m in full scale when the model scale is 1:400), and \(Q\) is the centre-stationary rate. The mean concentration distribution of the passive tracer gas at the height of 0.29H for the different urban set-ups is shown in Fig. 5.1. The figures have the same exponential concentration scale that ensures a clear insight to the plume structure.
at all positions as well as a direct comparison of the results for different set-ups. There were approximately 300 measurement points per set-up and they are depicted in Fig. 5.1 by small black diamonds. An interpolation was used to create the contour plots.

The difference between plane shapes are evident. The wind speed of the lowest point immediately after release, i.e. inside the street canyon where the point source was located, is distinctive for the aligned set-up. In this case the pollutant plume is in the range of $x/H = -0.64 - 0.64$ independently on the packing density. On the other hand, the pollutant concentration is much higher for the staggered set-up, with a peak at $x/H = -1.36$. The maximum concentration is much smaller within the urban canyon (i.e., the scale of vertical fluxes is different for advective and turbulent cases). The turbulent transport reaches the same magnitude as the advective flux at the roof top level and higher, where the mean value of vertical velocity and also the advective vertical flux becomes zero. The magnitude of both advective and turbulent transport is much higher in the case of staggered set-up with higher packing density (green and yellow in Fig. 5.1). However, the magnitude of the vertical turbulent transport is independent of the packing density and it is much smaller within the urban canopy (i.e., the scale of vertical fluxes is different for advective and turbulent cases). The turbulent transport reaches the same magnitude as the advective flux at the roof top level and higher, where the mean value of vertical velocity and also the advective vertical flux becomes zero. The magnitude of both advective and turbulent transport is much higher in the case of staggered set-up with higher packing density (green and yellow in Fig. 5.1). However, the magnitude of the vertical turbulent transport is independent of the packing density and it is much smaller within the urban canyon (i.e., the scale of vertical fluxes is different for advective and turbulent cases). 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numerical scheme can be used for such a detailed investigation of the saturation and capillary pressure behavior when dynamic capillary pressure is used instead of the static capillary pressure in the full two-phase flow system. Moreover, different models for the dynamic capillary pressure - saturation relationship are employed and their respective impacts are discussed.

In this section, the constitutive equations are introduced and the capillary pressure s is defined. Thorough definitions, descriptions, and examples can be found in [9], [20], [21], [22], or [23]. Then, the mathematical model and the numerical scheme are presented in Section 2. In the last section, the numerical scheme is validated using analytical and numerical solutions and the numerical experiments with dynamic capillary pressure models are discussed.

1.1. Saturation. The fluid distribution in immiscible multiphase flow in porous media is described by the saturation $S_1 = \alpha_1$ which indicates the volumetric portion of the void space within the porous occupied by the fluid phase $1$. Hence, $\alpha_1$ is always between 0 and 1. The sum of saturations $\alpha_2 = 1 - \alpha_1$ is introduced. It represents the sum of phases that will be included in porous media only due to advection effects without the need for mixture models. Consequently, the effective saturation $S_e = S_1 - \lambda_1 \alpha_1$ that describes only volumetric portion of dispensible fluid phases is introduced as

$$S_e = \frac{S_1 - \lambda_1 \alpha_1}{1 - \lambda_1}$$

(1.1)

1.2. Capillary pressure. Following the standard definitions in literature, the capillary pressure $\tau_c(\alpha_2, \alpha_1, T)$ on the pore scale is defined as the difference between the non-wetting phase pressure $\tau_0(\alpha_1, T)$ and the wetting phase pressure $\tau_0(\alpha_2, T)$, i.e.,

$$\tau_c = \tau_0(\alpha_1, T) - \tau_0(\alpha_2, T)$$

(1.2)

On the macroscale, the capillary pressure has been commonly considered as a function of wetting phase saturation only and it has been widely used in model equations in literature [23], [24], [22], [12], [11], [10]. The following Brooks and Corey [4] capillary pressure - effective wetting phase saturation parameterization is used in the present two-phase flow model

$$\tau_c = \frac{\gamma \cos \theta}{\rho g} \left[ \frac{\alpha_1}{S_e} \right]^{n_c}$$

(1.3)

where $\gamma$ is the entry pressure and $\lambda_1$ - $\gamma$ describes the poor distribution of the grasic in porous media. The Brooks and Corey relationship (1.3) is suitable for modeling of flow in heterogeneous porous media because the difference in the entry pressure coefficients $\lambda_1$ in different porous materials captures the barrier effect that has been observed in experiments [20], [21], [17], [12]. At the main objectives of


NUMERICAL STUDY OF THE EFFECT OF DYNAMIC CAPILLARY PRESSURE IN POROUS MEDIUM

DRAZEN PUZII, ZHRISHNOK, TOSHIHIRO SAKAKI, TRISA H. ISLAMANGILO

Abstract. In order to investigate effects of the dynamic capillary pressure-saturation relationship on the multiphase flow through porous media, a new numerical model is proposed. This model is a extension of the work presented in [10]. The experiments with water and ethanol were performed for two different porous media (sand and glass spheres). Two different models are employed - the Brooks-Corey and van Genuchten models. The experimental data are used as boundary conditions for the numerical solution. The numerical results are compared with the experimental data. The numerical results show the importance of the dynamic capillary pressure in the modeling of multiphase flow in porous media.

1. Introduction. In the understanding and prediction of the flow of immiscible and incompatible fluids in porous media, a reliable model of capillary forces acting on the fluid is crucial. In part decades, various capillary pressure - saturation models were derived from laboratory measurements or capillary pressure-saturation relationships in porous media. However, there are reasons to believe that the classical approach cannot be used in the modeling of capillary pressure when the fluid content is in a near-wet model of the capillary pressure - saturation relationship is proposed, i.e., the dynamic capillary pressure [3], [14], [15].

This manuscript focuses on the implications of the dynamic capillary pressure - saturation relationship. The full implicit numerical scheme is proposed and validated using the porous analytical solutions for the static capillary pressure [10], [16], and [14]. By optimizing the experimental convergence of outcomes, it is shown that the numerical scheme is convergent and can be used for simulating flow in both homogeneous and heterogeneous porous media. Consequently, the inclusion of various models of dynamic capillary pressure are employed and compared to the static model of capillary pressure.

The two-phases system can be simplified to the Richards problem, where the pressure of the non-wetting phase ($\alpha_2 = 0$) is assumed to be constant. This is the case in [3], where the dynamic effects are modelled not to be relevant for the given structure of heterogeneous porous media. Other numerical approaches using the dynamic capillary pressure have been developed in [11], [20], or [21]. However, the relevance of using the dynamic capillary pressure in the full two-phase flow system of equations has not been verified yet. The presented fully implicit

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1. Mathematical model. The mathematical model describing the two-phase flow in a one-dimensional domain is presented in this section. The aim is to investigate how the inclusion of capillary pressure (1.3) affects the saturation (1.3) influences the numerical solution of the resulting system of equations.
2.1. Governing equations. The governing two-phase flow equations in one-dimensional domain \( [0, L] \) are given by the \( p \), \( S_t \), and \( S_w \) formulations:

\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial f}{\partial x} &= g, \\
\frac{\partial S_t}{\partial t} + \frac{\partial (\alpha \phi S_t)}{\partial x} &= g, \\
\frac{\partial S_w}{\partial t} + \frac{\partial (\alpha \phi S_w)}{\partial x} &= g, \\
\end{align*}
\]

where \( S_t, S_w, \) and \( \phi \) are the saturation and porosity, respectively. The initial and boundary conditions (2.1) are given for each experimental problem, separately.

2.2. Discrete problem. A standard finite volume discretization technique is used in order to determine approximate discrete solution \( \{S_t^j, S_w^j, \alpha, \phi\} \) of the problem (2.1), generally defined as \( \frac{x}{h} \Delta x \big|_{n} \), where \( x = 0, \ldots, L \) and \( T \) is the length of the domain and \( T_t \) is the final time of the simulation.

Since the nonlinear problem (2.1) involves the dynamic capillary pressure function defined in (1.4) that includes time derivative of \( S_t \), an implicit scheme is proposed in the following form:

\[
\begin{align*}
\frac{S_t^{j+1} - S_t^j}{\Delta t} + \frac{\alpha \phi (S_t^{j+1} - S_t^j)}{\Delta x} &= g, \\
\frac{S_t^{j+1} - S_t^j}{\Delta t} + \frac{\alpha \phi (S_t^{j+1} - S_t^j)}{\Delta x} &= g, \\
\frac{S_t^{j+1} - S_t^j}{\Delta t} + \frac{\alpha \phi (S_t^{j+1} - S_t^j)}{\Delta x} &= g, \\
\end{align*}
\]

where \( \alpha \) is a real number and the discrete Darcy velocities \( u \) are given as follows:

\[
\begin{align*}
u_{e,1}^{j+1} &= \frac{1}{\alpha \phi} \left[ S_t^{j+1} - S_t^j - \frac{\phi (S_t^{j+1} - S_t^j)}{k} \right], \\
\end{align*}
\]

\( \alpha \) is the saturation taken in the upstream direction with respect to the gradient of the phase potential \( \varphi \).

At the material interface, the continuity of capillary pressure (1.4) requires that there is a jump in saturation. Such discontinuous saturation \( S_t^I \) is represented by \( \xi^I \) and \( \xi^{I+1} \) in the Scheme 2.1, where \( t \) is the index of the scale located at the material interface. Therefore, the equation (1.6) is approximated by:

\[
\begin{align*}
\frac{S_t^{I+1} - S_t^I}{\Delta t} + \frac{\alpha \phi (S_t^{I+1} - S_t^I)}{\Delta x} &= g, \\
\frac{S_t^{I+1} - S_t^I}{\Delta t} + \frac{\alpha \phi (S_t^{I+1} - S_t^I)}{\Delta x} &= g, \\
\frac{S_t^{I+1} - S_t^I}{\Delta t} + \frac{\alpha \phi (S_t^{I+1} - S_t^I)}{\Delta x} &= g, \\
\end{align*}
\]

If \( \tau > 0 \), the interfacial saturations \( S_t^I \) and \( S_w^I \) can be computed analytically by inverting the static capillary pressure function \( p(\theta) \) defined in (1.3). Therefore, it is

Figure 2.2: Discretization for the saturation jump at material discontinuity.

The Ohj sand used was a porous medium in the following numerical simulations with water and air as a wetting and non-wetting fluid, respectively. Physical properties of the sand shown in Table 3.1 were measured during the laboratory experiments held in the Center for Experimental Study of Subsurface Environmental Processes, Colorado School of Mines. The details of the Ohj sand properties are shown in Table 3.3. Except for the pure advection problem, the numerical solution is computed also for the models of the dynamic capillary pressure (1.4). The value of the dynamic capillary coefficient \( \tau = \tau(x) \) was estimated as a result of the laboratory experiments, where capillary pressure and time evolution of the water saturation were measured. These functional models of the dynamic capillary coefficient \( \tau = \tau(x) \) were approximated, see Table 3.2. Additionally, the Stadler model (1.5) gives \( \tau = 3.3 \times 10^{-2} \) for the Ohj sand. In the following subsections, the numerical solutions using these dynamic capillary pressure models are compared to the referential numerical solutions computed with the static capillary pressure model \( \varphi = \varphi(\theta, \phi) \).

Unfortunately, no laboratory data is available for the case of a single heterogeneous porous medium described in Sections 3.4 and 3.5. In order to investigate solutions for the different models of the dynamic capillary coefficients \( \varphi = \varphi(\theta, \phi) \) in the heterogenous porous medium, a fictive, courser and Ohj’s introduced. No parameters are the same as for the Ohj sand except for the capillary pressure \( \varphi \), the intrinsic permeability \( K \), and \( \tau \), which are all multiplied by the factor 0.9 (see Table 3.1).

**Table 1.** Parameters of the porous medium used in the numerical simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>1.60 ( \times ) 10^11</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>( \alpha \phi )</td>
</tr>
<tr>
<td>Residual water saturation</td>
<td>( S_w^I )</td>
</tr>
<tr>
<td>Relative air entry pressure</td>
<td>( K_{r,\alpha} )</td>
</tr>
<tr>
<td>Relative entry pressure</td>
<td>( K_{r,\phi} )</td>
</tr>
<tr>
<td>Relative entry pressure of the most resistant</td>
<td>( K_{r,\phi} / K_{r,\alpha} )</td>
</tr>
</tbody>
</table>

Properties of the porous medium used in the numerical simulations.

**Table 2.** Properties of the porous medium used in the numerical simulations.

<table>
<thead>
<tr>
<th>Model</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear model</td>
<td>( \varphi = 3.2 \times (1 - S_w) )</td>
</tr>
<tr>
<td>Linear model</td>
<td>( \varphi = 2.9 \times (1 - S_w) )</td>
</tr>
</tbody>
</table>

The representation of the dynamic capillary coefficient \( \tau(x) \) for the Ohj sand and factor index values \( \alpha \) for the model sand.

3.1. Pure advection. Assuming \( \varphi = 0 \), the system of equations (2.1) can be simplified into a single hyperbolic equation with \( \partial S_t^j / \partial x = 0 \), (see [17], [14]). In [5], Buckley and Leverett derived the analytical solution of such a problem using the modified method of characteristics. The problem description is shown in Table 3.1.

**Table 3.** Parameters of the porous medium used in the numerical simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>1.60 ( \times ) 10^11</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>( \alpha \phi )</td>
</tr>
<tr>
<td>Residual water saturation</td>
<td>( S_w^I )</td>
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<tr>
<td>Relative air entry pressure</td>
<td>( K_{r,\alpha} )</td>
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<tr>
<td>Relative entry pressure</td>
<td>( K_{r,\phi} )</td>
</tr>
<tr>
<td>Relative entry pressure of the most resistant</td>
<td>( K_{r,\phi} / K_{r,\alpha} )</td>
</tr>
</tbody>
</table>

The representation of the dynamic capillary pressure models are shown in Figure 3.1.

3.2. Pure advection in homogeneous medium. If no external forces act on the system, i.e., \( g = 0 \), the flow in the one-dimensional domain is governed only by capillarity and the system of equations (2.1) can be reformulated to study the Methods and Stand model problem formulation for the case of a bidirectional flow. The problem solution is shown in Table 3.5. Therefore, the Buckley and Leverett and Stedman semi-analytical solution for the pure capillary diffusion problem can be obtained as a benchmark solution for the numerical solution. The problem details are shown in Figure 3.5.

**Table 4.** Boundary conditions for the different governing equations: \( S_{t,\alpha} \) is the air saturation and \( S_{t,\phi} \) is the water saturation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source terms</td>
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</tr>
<tr>
<td>Source terms</td>
<td>( \Delta x )</td>
</tr>
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<td>Source terms</td>
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<tr>
<td>Source terms</td>
<td>( \Delta x )</td>
</tr>
<tr>
<td>Source terms</td>
<td>( \Delta x )</td>
</tr>
</tbody>
</table>

The representation of the dynamic capillary pressure models are shown in Figure 3.1.

3.3. Advection and capillary diffusion in heterogeneous medium. If a flux \( g = 0 \) is added at the downstream boundary of the domain, it is solved by both capillarity and advection. However, if the domain is placed horizontally, i.e., \( g = 0 \), the generalized McWhorter problem formulation can be used to
Boundary conditions: \( S_t(0,t) = 0.73 \) \( \forall x \in [0,4] \).

Problem setup: \( T = 1000, \quad L = 1, \quad \alpha = 0 \).

Porous medium: Homogeneous porous medium.

Sand: Oil-gray, Table 3.1.

3.4. Pure capillary diffusion in homogeneous medium. If the porous medium described in Section 3.2 has a single discontinuity in the material properties, respect to the static capillary pressure, which is in contradistinction to [38],

The numerical solutions of the pure capillary diffusion problem in homogeneous porous media, \( x = 1/16 \) cm and \( x = 1/4 \) cm, are measured in (\( \alpha \)), and the experimental order of convergence and eoc are measured in (\( \Delta \) and \( \alpha \)), respectively.

**Table 3.1**

<table>
<thead>
<tr>
<th>Condition</th>
<th>( m )</th>
<th>( \Delta t )</th>
<th>( m )</th>
<th>( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>0.77</td>
<td>0.72</td>
<td>0.78</td>
<td>0.74</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>0.77</td>
<td>0.70</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>( \frac{1}{8} )</td>
<td>0.80</td>
<td>0.72</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td>0.88</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
</tbody>
</table>

**Table 3.2**

<table>
<thead>
<tr>
<th>Condition</th>
<th>( m )</th>
<th>( \Delta t )</th>
<th>( m )</th>
<th>( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>0.77</td>
<td>0.72</td>
<td>0.78</td>
<td>0.74</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>0.77</td>
<td>0.70</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>( \frac{1}{8} )</td>
<td>0.80</td>
<td>0.72</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td>0.88</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
</tbody>
</table>

The numerical solutions of the pure capillary diffusion problem in homogeneous porous media, \( x = 1/16 \) cm and \( x = 1/4 \) cm, are shown in Figure 3.9.

3.5. Advection and capillary diffusion in heterogeneous medium. In 2008, Fleit et al. [38] generalized the van Duijn and de Neef problem formulation by the inclusion of the McWhorter and Sunada boundary fluxes in heterogeneous porous media.

The resulting problem formulation requires that the van Duijn and de Neef initial saturation distribution and the McWhorter and Sunada boundary fluxes are prescribed, see Table 3.3.

In Figure 3.9, the numerical solutions are compared to the semi-analytical solution and the experimental order of convergence is shown.

The numerical solutions of the problem for the dynamic capillary pressure model are shown in Figure 3.9.

3.6. Discussion of the results. The numerical scheme (2.1) was tested against five different analytical or semi-analytical solutions in the previous subsections. In all cases, the numerical solutions converge towards the analytical or semi-analytical solution and the order of convergence is shown in Figure 3.1, 3.2, 3.4, 3.5, and 3.6.

As expected, the numerical approximations of the discontinuities are not sharp even if the advection term dominates the flow (Figures 3.1, 3.3, and 3.5). This is caused by the numerical diffusion in the scheme (2.1) as it is already described in literature, see [38].

As shown in Figures 3.4 and 3.9, the jump in saturation across the interface in the
case of the heterogeneous porous medium is determined correctly and, moreover, the presence of a heterogeneity does not influence the experimental order of convergence. This will be an important referenced solution for further investigation of the experimental condition (1.6) in highly heterogeneous porous media.

The inclusion of the dynamic capillary pressure models (constant, linear, and logarithmic models of $\gamma$) is important in cases where there is a significant temporal change in the saturation $S$ since the temporal derivative of $S$ is multiplied by the dynamic effect coefficient $\gamma$, see (1.4). This occurs when the advection together with the capillary diffusion dominates the displacement as it is shown in Figures 3.3 and 3.5.

As shown in Figure 3.5, the use of the constant model for the dynamic effect coefficient $\gamma$ changes the monotonicity of the capillary pressure profile which may be physically meaningless. Therefore, the constant model requires further investigation of its validity. On the other hand, the use of the linear and the logarithmic models of $\gamma$ does not seem to be important in the heterogeneous porous medium since the air saturation and capillary pressure profiles are similar to the profile computed with the static capillary pressure (compare Figures 3.3 and 3.5).

In the case of a heterogeneous medium, the inclusion of the dynamic capillary pressure may substantially change the simulated evolution of the fluids since the entry pressure of the first porous media can be achieved sooner or later than in the static case as it is shown in Figures 3.7 and 3.9. This indicates that the conclusions published for the case of Richards equation in [1] may not hold for the full system of equation of the time-lag flow.

4. Conclusion. This manuscript presents a one-dimensional numerical scheme of two-phase immiscible and immovable flow that enables simulating non-static capillary pressure models in both homogeneous and heterogeneous porous media. The numerical scheme is validated and its order of convergence is estimated using the analytical and semi-analytical solutions for advection, advection and diffusion, and diffusion-dominated problems, respectively.

Laboratory measured parameters were used in the numerical simulation of the dynamic capillary pressure including three main models of the dynamic effect coefficient $\gamma = \gamma(S) = $ constant, linear, and logarithm. The numerical solutions for the dynamic capillary pressure show that the dynamic effect has a significant impact on the magnitude of the capillary pressure while the change in the saturation profiles may be considered negligible in some cases. The constant model of $\gamma$ showed rather unrealistic profile of the numerical approximation of the capillary pressure because the spatial monotonicity was different with respect to the results obtained with the static capillary pressure model.

Results of the simulation indicate that the dynamic effect may not be so important in drainage problems in a homogeneous porous medium, but, on the other hand, it is of a great importance in highly heterogeneous media where the capillary pressure flow through material interface.

5. List of symbols.

Acknowledgement. This work has been supported by:
In this case the cell-vertex Ron-Ho-Ni scheme described in [2], [5] was used. It is a linearized backward Euler method was used. Final system of equations was constructed using the least square method [6]. For the time discretization the linearized backward Euler method was used. This paper compares four different FVM schemes based on Lax-Wendroff approach. In this case the cell-vertex Ron-Ho-Ni scheme described in [2], [5] was used. It is a linearized backward Euler method was used. Final system of equations was constructed using the least square method [6]. For the time discretization the linearized backward Euler method was used. This paper compares four different FVM schemes based on Lax-Wendroff approach. In this case the cell-vertex Ron-Ho-Ni scheme described in [2], [5] was used. It is a linearized backward Euler method was used. Final system of equations was constructed using the least square method [6]. For the time discretization the linearized backward Euler method was used. This paper compares four different FVM schemes based on Lax-Wendroff approach. In this case the cell-vertex Ron-Ho-Ni scheme described in [2], [5] was used. It is a linearized backward Euler method was used. Final system of equations was constructed using the least square method [6]. For the time discretization the linearized backward Euler method was used.
Numerical Simulation of 3D Transonic Inviscid Flow over a Swept Wing

Fig. 4.1: Computational meshes.

a) \( \text{c}_p \) coefficient, C-mesh, top of the wing. b) Mach number, C-mesh, top of the wing.
c) Mach number, C-mesh, bottom of the wing.

Fig. 4.2: Method 2 - isolines of the \( \text{c}_p \) coefficient and Mach number.
a) \( \text{c}_p \) coefficient, Method 3, top of the wing. b) \( \text{c}_p \) coefficient, Method 4, top of the wing.
c) Mach number, Method 3, top of the wing. d) Mach number, Method 4, top of the wing.
e) Mach number, Method 3, bottom of the wing. f) Mach number, Method 4, bottom of the wing.

Fig. 4.3: Method 1. Isolines of \( \text{c}_p \) coefficient and Mach number.

Fig. 4.4: Methods 3 and 4, unstructured mesh. Isolines of \( \text{c}_p \) coefficient and Mach number alongside the wing.

Fig. 4.5: Methods 3 and 2, comparison with the experimental results. Behaviour of the \( \text{c}_p \) in the cut (x = spatial coordinate, y = chord): a) 20%, b) 44%, c) 65%, d) 80%, e) 90%, f) 95% of the wing span.
4. Numerical Results. As can be seen from the comparison between the num-
erical and experimental results in the case of 3D flow over the Delta HI wing (Figure 4.1 - 4.6), the schemes based on classical Lax-Wendroff approach still deliver satisfactory results, and can be used as a reliable tutorial simulation. Both the position and intensity of shockwaves are captured reasonably well, only the Method 2 pushed the shockwave to the trailing edge (Fig.4.5) - more than the other methods. The observed differences are highly probable consequence of un-
real nature of the chosen model, which is subject to severe turbulent behaviour at higher Mach numbers. However, the current model is able to model the shockwaves 
very accurately, which restrain their usability in three dimensional computation only to inviscid flows (both steady and unsteady). But considering the possibility of their communications as an implicit version, both can be successfully used for the turbulent simulations. The modern schemes deliver a good results too, particularly the Method 4 (WLSQR scheme) shows a very good correspondence with the experimental data. Method 3 manages the shockwaves more than the other methods, mostly because it is of sufficient density of the computational mesh.

5. Conclusion. Proposed FM schemes for numerical solution of stationary 3D transonic inviscid flow show reasonably good accuracy and efficiency. Although the intrinsical mathematical model have been chosen, the scheme turns able to capture important flow characteristics as the position and intensity of the shockwaves and proved themselves as a reliable potential simulation of inviscid cases. Some of the schemes however would need some further improvement (implicit form in the case of Modified Canon scheme, matrix-free GMRES in the case of WLSQR scheme).

Acknowledgement. This work was partially supported by the Research Plan VZ 1M06019/2010, VZ MSM 0021630513 and grant GACR 201/09/0812.

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[8] B. Hocking, An arbitrary Lagrangian-Eulerian scheme, method for transonic, laminar-

Appendix A: AMS subctegory classifications. 35C20 35B25 62P05 60H10 35K05

ON NON-EXISTENCE OF A ONE FACTOR INTEREST RATE MODEL FOR VOLATILITY AVERAGED GENERALIZED FONG–VASICEK TERM STRUCTURES

BE´ATIKA STEHL´IKOV´A AND DANIEL ˇSEVˇCOVIˇC

Abstract. We study the generalized Fong–Vasicek two-factor interest rate model with stochastic 
volatility. In this model discrete-time simulations are carried out as a numerical experiment with 
parameters in the range of realistic data and compare the averaged bond prices with the limiting 
structure of interest rates. Figure 1.1 shows the different shapes of term structures having two 
local maxima (UK). Interest rate models are often formulated in terms of stochastic differential 
equations (SDE) for the instantaneous interest rate or its present price. The yield of bonds, as a function of maturity, forms a term 
structure of interest rates. It describes a functional dependence between the time to maturity of a discount bond 
and its present price. The yield of bonds, as a function of maturity, forms a term 
structure of interest rates. Figure 1.1 shows the different shapes of term structures observed 
for description of the volatility clustering, see e.g. [12]). We consider averaged bond prices with respect to the limiting distribu-
tion of stochastic dispersion. The averaged bond prices depend on time and current level of the short rate like it is the case in many popular one-factor interest rate models. By simulations, we show that for the same model parameters 
yielding the same local maxima as the averaged values described above.

Key words. two-factor term structure models, generalized Fong–Vasicek interest rate model, stochastic volatility, stochastic differential equations, averaging, limiting density.

AMS subject classifications. 35C20 35B25 62P05 60H10 35K05

1. Overview of term structure modeling. The purpose of this paper is to study the 
generalized Fong–Vasicek two-factor interest rate model with stochastic volatility. In this model the shapes of the stochastic short rate (shape of volatility) is assumed to be stochastic as well as the volatility which is a non-negative process with volatility proportional to the square root of dispersion. The drift of the stochastic process for the dispersion is assumed to be in a rather general form including, in particular, linear function having one root (yielding the original Fong–Vasicek model, cf. [8]) or a cubic like function having three roots (yielding a generalized Fong–Vasicek model for description of the volatility clustering, see e.g. [12]). We consider averaged bond prices with respect to the limiting distribution of stochastic dispersion. The averaged bond prices depend on time and current level of the short rate like it is the case in many popular one-factor interest rate models. By simulations, we show that for the same model parameters yielding the same local maxima as the averaged values described above.

The term structure of bond prices (as yields) is a function of time to maturity, state variable like e.g. instantaneous interest rate or its local model parameters. It describes a functional dependence between the time to maturity of a discount bond and its present price. The yield of bonds, as a function of maturity, forms a term 
structure of interest rates. Figure 1.1 shows the different shapes of term structures observed 
for description of the volatility clustering, see e.g. [12]). We consider averaged bond prices with respect to the limiting distribution of stochastic dispersion. The averaged bond prices depend on time and current level of the short rate like it is the case in many popular one-factor interest rate models including in particular the Vasicek and Cox-Ingersoll-Ross model. Moreover, as a main result of this paper we show that there is no such one-factor model yielding the same bond prices as the averaged values described above.

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Graph of Appl. Math. and Statist., Comenius University Bratislava, Mlynska Dolina, Slovakia
The parabolic PDE is derived by constructing a riskless portfolio of bonds and using Itô's lemma. When the risky asset price \( S_t \) and the short rate \( r_t \) are modeled as stochastic processes, the price dynamics of a derivative asset can be described by a PDE. For example, for a short rate \( r_t \) driven by a Wiener process, the price of a bond \( P(t, r) \) can be modeled by the Black-Scholes PDE:

\[
\frac{1}{2} \sigma^2 r^2 \frac{\partial^2 P}{\partial r^2} + \left( \mu - \frac{1}{2} \sigma^2 \right) r \frac{\partial P}{\partial r} - \frac{\partial P}{\partial t} = 0
\]

where \( \sigma \) is the volatility of the short rate. The function \( \sigma \) can be a function of the short rate, leading to different term structures for the same short rate. They may depend on the volatility of the stochastic process for the short rate, which is modeled as a stochastic process, often following a normal distribution.

More precisely, the volatility of the stochastic process for the short rate is stochastic as well, being driven by another stochastic process, often following a normal distribution. We recall that a stochastic process \( X_t \) is a Wiener process, and its increments \( \Delta X_t = X_t - X_0 \) are independent and normally distributed for any \( t > 0 \) and any \( 0 < s < t \).

The term structure of interest rates can be modeled using the Fokker-Planck equation (1.9):

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 r^2 \frac{\partial^2 P}{\partial r^2} + \left( \mu - \frac{1}{2} \sigma^2 \right) r \frac{\partial P}{\partial r} = 0
\]

where \( P(t, r) \) is the price of a bond at time \( t \) with a short rate \( r \). The function \( \sigma(r, t) \) determines the trend in evolution of the short rate, \( \rho \) the value of stochastic fluctuations. The price of a discount bond \( P(t, r) \) at time \( t \) when the value of short rate is \( r \) is known to be a solution of the partial differential equation:

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 r^2 \frac{\partial^2 P}{\partial r^2} + \left( \mu - \frac{1}{2} \sigma^2 \right) r \frac{\partial P}{\partial r} = 0
\]

with the terminal condition \( P(T, r) = 1 \) for any \( r \geq 0 \) where \( T > 0 \) is a maturity of the bond. Here \( \mu \) stands for the overall market price of risk. The above linear parabolic PDE is derived by constructing a riskless portfolio of bonds and using Itô's lemma for evaluating differentials of a stochastic process that is unbiased by buying or selling bonds with different maturities. We refer the reader for a comprehensive overview of term-structure modeling to the book by Heath [8]. Chapter 7 for details.

![Image 1](https://example.com/image1.png)

**Figure 1.1.** Examples of yield curves of governmental bonds: Australia, Brazil, Japan, United Kingdom (30th May 2008). Source: http://www.bloomberg.com

![Image 2](https://example.com/image2.png)

**Figure 1.2.** Examples of real EURIBOR term structures. Source: http://www.euribor.org

We also consider conditional that bond prices determine interest rates \( r(t) \) by the formula \( P = e^{-r(t)T} \), i.e.

\[
P(t, r) = \frac{1}{T} \log P(t, r).
\]

One of the first models of the class (1.1) has been proposed by Oldrich Vasicek in [13]. In this model, the short rate process is assumed to follow a stochastic differential equation:

\[
\frac{dr}{r} = \left( \theta - \frac{r}{\kappa} \right) \, dt + \sigma \, dw
\]

where \( \theta, \kappa, \sigma > 0 \) are positive constants. Here \( r \) stands for volatility of random fluctuations of the short rate process. The term structure of the process \( P(t, r) \) represents a mean reverting process with a limit \( \theta \) referred to as long term interest rate. The speed of reversion is given by the parameter \( \kappa > 0 \). In this model, for a constant market price of risk \( \lambda \), the corresponding PDE for bond prices is:

\[
\frac{1}{2} \sigma^2 r^2 \frac{\partial^2 P}{\partial r^2} + \left( \mu - \frac{1}{2} \sigma^2 \right) r \frac{\partial P}{\partial r} - \frac{\partial P}{\partial t} = 0
\]

which has an explicit solution \( P(T, r) \) satisfying the terminal condition \( P(T, r) = 1 \) for any \( r \geq 0 \). It has the form:

\[
P(t, r) = e^{-(r-\theta)t + \frac{1}{2} \sigma^2 T^2 (r-\theta)^2}.
\]

In a rich variety of several other models in which the SDE for the short rate may be determined by other factors than one factor interest rate model for volatility averaged term structures.
2. Approximating with respect to stochastic volatility and relation to one-factor models. Knowing the density distribution \( f \) of the stochastic volatility, we can safely perform averaging of the bond price and the term structure with respect to volatility. Unlike the short rate whole is known from the market data on daily basis, the volatility of the short rate process is stochastic. The exact value of the stochastic volatility is not observable on the market, we can just observe its statistical properties. Therefore such a volatility averaging is of special importance for practitioners.

We shall consider the following model with stochastic volatility:

\[
\begin{align*}
\frac{dy}{dt} &= a(y(t)) - b_1 \sigma(t) y(t) + \sigma(t) \epsilon(t), \\
\text{and by comparing (2.15) and (2.19) we obtain the expression for the constant terms of the model parameters as}
\end{align*}
\]

\[
\begin{align*}
\text{with uncorrelated increments } \epsilon(t) \text{ and } \sigma(t) \text{ of a Wiener process. The market prices of risk are assumed to have a form } \phi(t, \sigma(t)) = \phi_1(t) + \phi_2(t) \sigma(t), \\
\text{the bond price } P(t, \sigma(t)) \text{ satisfies the following PDE:}
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial \sigma^2} - r \frac{\partial P}{\partial y} + \rho \sigma \frac{\partial^2 P}{\partial y \partial \sigma} - \frac{1}{2} \rho^2 \sigma^2 \frac{\partial^2 P}{\partial \sigma^2 \partial y} = 0,
\end{align*}
\]

with the terminal condition \( P(T, \sigma) = 1 \) for any \( y \). The explicit solution can be written as:

\[
\begin{align*}
&\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial \sigma^2} - r \frac{\partial P}{\partial y} + \rho \sigma \frac{\partial^2 P}{\partial y \partial \sigma} - \frac{1}{2} \rho^2 \sigma^2 \frac{\partial^2 P}{\partial \sigma^2 \partial y} = 0,
\end{align*}
\]

\[
\begin{align*}
\text{with the terminal condition } A(T, \sigma) = 1 \text{ for any } y \end{align*}
\]

\[
\text{The function } f_B = \Phi(t, y) = \Phi_2(t) + \Phi_3(t) y \\
\text{satisfies the stationary Fokker–Planck equation (1.10). The averaged bond prices in respect to the limiting distribution volatility is given by:
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \pi = \frac{\partial P}{\partial y} \left. \frac{\partial P}{\partial \sigma} \right|_{\sigma = 0}
\end{align*}
\]

In what follows, we shall denote by \( \bar{\pi} \) the averaged value of the function \( \pi : (t, \sigma) \rightarrow \bar{\pi} \) with respect to the limiting density function \( f_B(t, y) \). Notice that it is the same functional dependence as for the bond prices in one-factor models, including, in particular, solutions to (1.5) given by (1.15). However, the way there is no

end one-factor model (yielding the same bond prices as those of averaged bond prices \( P(t, \sigma(t)) \) given by the averaging of the one-factor model.

Now we are in a position to state our main result of the paper. We are going to prove that there is no one-factor interest rate model for the corresponding bond prices that is identical with the volatility averaged bond prices \( P(t, \sigma(t)) \) for any \( t \in [0,T] \) and \( \sigma(t) \). Suppose to the contrary that \( P = P(t, \sigma(t)) \) is a bond price from one-factor model in which the short rate is assumed to follow a general SDE

\[
\begin{align*}
\text{Now we consider the constant one-factor interest rate model for volatility averaged term structures}
\end{align*}
\]

However, the latter equality can not be satisfied for a general class of model parameters. Indeed, setting \( \lambda = 0 \), we obtain \( \sigma^2 = \sigma^2(\mu(f)) \), which is not possible as \( f(\sigma^2) = 0 \) for \( \mu = 0 \).

Summarizing, we have the following theorem:

**Theorem 2.1.** Consider the generalized Fong–Huang–Lau two-factor model with stochastic volatility (2.20) and the averaged bond price \( P(t) \) with respect to the limiting distribution of the stochastic volatility. Then there is no one-factor interest rate model (2.8) with corresponding PDE for the bond price (1.9) satisfying the same bond prices as the averaged ones \( P(t, \sigma(t)) \) from the two-factor model.

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## REFERENCES


Behavior of solutions to an activator-inhibitor system

Abstract. We consider a reaction-diffusion system which is subject to a changing boundary. The equation is modified such that a change in cells or tissue takes place in the region. The activator concentration is high. The activator concentration on the boundary at time $t=0$ is in $\Omega$ and, in addition to $t=0$, $\sigma_{0}(x)$ is a positive constant. The constants $\alpha_{0}(x)$ and $\sigma_{0}(x)$ are chosen.

Key words. activator-inhibitor system, pattern formation.

AMS subject classifications. 35K57, 35K40.

1. Introduction and Statement of Results. In the celebrated paper [5, 6], A. M. Turing found that the reaction-diffusion model for pattern formation suggests the existence of an activator-inhibitor system that is subject to a changing boundary. To the best of our knowledge, this is the first time such a system has been considered. Let $\Omega_{0}$ denote the unit outer normal to $\Omega$ and $\Omega = \Omega_{0} + \partial \Omega_{0}$ be the physical domain. We denote the unit outer normal vector to $\partial \Omega_{0}$ by $\nu_{0}$. The Laplace operator is defined in $\Omega_{0}$.

We consider the following activator-inhibitor system proposed by Gierer and Meinhardt:

$$\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - u^{3} + v^{2} + \sigma_{0}(x), \\
\frac{\partial v}{\partial t} &= \Delta v - \alpha v + \frac{\alpha v^{3}}{1 + v^{2}},
\end{align*}$$

(1.1)

for $x \in \Omega$ and $t > 0$, subject to the boundary condition and the initial condition:

$$\begin{align*}
\frac{\partial u}{\partial \nu_{0}}(x, t) &= f_{1}(u, v), \\
\frac{\partial v}{\partial \nu_{0}}(x, t) &= g_{1}(u, v),
\end{align*}$$

(1.2)

for $x \in \partial \Omega$ and $t > 0$, and $u(x, 0) = u_{0}(x)$, $v(x, 0) = v_{0}(x)$ for $x \in \Omega_{0}$.

Here, $f_{1}$ and $g_{1}$ are positive constants. The exponents $\alpha_{1}, \alpha_{2}, \beta_{1}, \gamma_{1}$, and $\gamma_{2}$ are positive integers.

Theorem 1.1. Assume that (1.1) is subject to (1.2). Moreover, suppose that

$$\begin{align*}
\sigma_{0}(x) &< 0, \\
\alpha_{1}(x) &> 0, \quad \text{and} \quad \gamma_{1}(x) > 0.
\end{align*}$$

Then the initial-boundary value problem (1.1)-(1.4) has a unique solution for all $t > 0$.

Moreover, it is proved that if $\beta_{1} > \alpha_{2}$ then $\beta_{1}/\alpha_{2}$ and $\gamma_{2} - \beta_{1}$ are positive integers.

Theorem 1.2. Assume that (1.1) is subject to (1.2). Moreover, suppose that

$$\begin{align*}
\alpha_{1}(x) &> 0, \\
\gamma_{1}(x) &> 0, \quad \text{and} \quad \gamma_{2}(x) > 0.
\end{align*}$$

Then the initial-boundary value problem (1.1)-(1.4) has a unique solution for all $t > 0$.

2. Collapse of Patterns. The activator concentration may fail to form spatial or spatial-temporal patterns of the system (1.1)-(1.4) does not have a positive basic production term for the activator. In order to understand the mechanism of this phenomenon, we consider the behavior of a solution of the following system at $t \to +\infty$:

$$\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - A^{2}u + \sigma_{0}(x), \\
\frac{\partial v}{\partial t} &= \Delta v - \alpha v + \frac{\alpha v^{3}}{1 + v^{2}},
\end{align*}$$

(2.1)

for $x \in \Omega$ and $t > 0$, subject to the boundary condition and the initial condition:

$$\begin{align*}
\frac{\partial u}{\partial \nu_{0}}(x, t) &= f_{1}(u, v), \\
\frac{\partial v}{\partial \nu_{0}}(x, t) &= g_{1}(u, v),
\end{align*}$$

(2.2)

for $x \in \partial \Omega$ and $t > 0$, and $u(x, 0) = u_{0}(x)$, $v(x, 0) = v_{0}(x)$ for $x \in \Omega_{0}$.

The initial condition, we assume (1.7) and (1.8). The exponents satisfy (1.5) and we assume

$$\begin{align*}
\sigma_{0}(x) &< 0, \\
\alpha_{1}(x) &> 0, \quad \text{and} \quad \gamma_{1}(x) > 0.
\end{align*}$$

There exists a solution of (2.1)-(2.4) for all $t > 0$ by Theorem 1.2 if $\beta_{1} > \alpha_{2}$. The main result of the paper is stated as follows:

Theorem 2.1. Let $e > 0$ satisfy $\gamma_{1}(x) > e > 0$ and assume that the initial data satisfies

$$\begin{align*}
\max_{x \in \Omega_{0}} u_{0}(x) &> 0, \\
\max_{x \in \Omega_{0}} v_{0}(x) &> 0.
\end{align*}$$

Then the solution $(u(x, t), v(x, t))$ of (2.1)-(2.4) satisfies

$$\begin{align*}
\max_{x \in \Omega_{0}} u(x, t) &< 0, \\
\max_{x \in \Omega_{0}} v(x, t) &> 0.
\end{align*}$$

Moreover, we assume

$$\begin{align*}
\sigma_{0}(x) &< 0, \\
\alpha_{1}(x) &> 0, \quad \text{and} \quad \gamma_{1}(x) > 0.
\end{align*}$$

and concerning the initial data we assume

$$\begin{align*}
\max_{x \in \Omega_{0}} u_{0}(x) &> 0, \\
\max_{x \in \Omega_{0}} v_{0}(x) &> 0.
\end{align*}$$

and

$$\begin{align*}
\max_{x \in \Omega_{0}} u(x, t) &< 0, \\
\max_{x \in \Omega_{0}} v(x, t) &> 0.
\end{align*}$$

It is to be noted that in contrast to Theorems 1.1 and 2.1 we do not assume any further condition other than (1.7) in Theorem 1.1, yet we have a bounded solution for all $t > 0$ for satisfying initial data.

K. Suzuki and I. Takagi

[5] and [6] if $\beta_{1} > \alpha_{2}$, then there exist solutions of (1.1)-(1.4) with $\sigma_{0}(x) = 0$ which blow up in finite time.

Theorem 2.2. Let $e > 0$ satisfy $\gamma_{1}(x) > e > 0$ and assume that the initial data satisfies

$$\begin{align*}
\max_{x \in \Omega_{0}} u_{0}(x) &> 0, \\
\max_{x \in \Omega_{0}} v_{0}(x) &> 0.
\end{align*}$$

Then the solution $(u(x, t), v(x, t))$ of (2.1)-(2.4) satisfies

$$\begin{align*}
\max_{x \in \Omega_{0}} u(x, t) &< 0, \\
\max_{x \in \Omega_{0}} v(x, t) &> 0.
\end{align*}$$

It is to be noted that in contrast to Theorems 1.1 and 2.1 we do not assume any further condition other than (1.7).
2.1. Boundedness of solutions.

2.1.1. Lower bounds. First, we estimate $A_{i}(t)$ and $H(x,t)$ from below. Let $g\left(\epsilon_{i}\right)$ be a solution of the initial-boundary value problem:

$$
\begin{align*}
\frac{\partial g}{\partial t} & = \epsilon_{i}^{2} \Delta g + \frac{\partial g}{\partial x} \\
\frac{\partial g}{\partial x} & = g - A_{i}(t) \\
\frac{\partial g}{\partial x} & = 0
\end{align*}
$$

and $g(\epsilon_{i})$ be a solution of

$$
\begin{align*}
\epsilon_{i}\frac{\partial g}{\partial t} & = \epsilon_{i}^{2} \Delta g + \frac{\partial g}{\partial x} \\
\frac{\partial g}{\partial x} & = g - H(x,t) \\
\frac{\partial g}{\partial x} & = 0
\end{align*}
$$

and $g(\epsilon_{i})$ is the solution for all $t > 0, x \in \Omega$.

The following lemma is obtained:

Lemma 2.2. The solution $A_{i}(t), H(x,t)$ of (2.1)–(2.4) satisfies $A_{i}(t) \geq g(\epsilon_{i}), H(x,t) \geq g(\epsilon_{i})$ for all $t > 0, x \in \Omega$.

This lemma can be proved easily by using the maximum principle. Moreover, we see that the solution (2.7)–(2.9) satisfies $\epsilon_{i}^{-1} A_{i}(x,t) \leq \epsilon_{i}^{-1} \|A_{i}(x,t)\|_{\infty}$ for all $x \in \Omega, t > 0$ and the solution (2.10)–(2.11) is given by

$$
\begin{align*}
\epsilon_{i}^{-1} \|A_{i}(x,t)\|_{\infty} \leq \|\epsilon_{i}^{-1} A_{i}(x,t)\|_{\infty} \leq \epsilon_{i}^{-1} \|A_{i}(x,t)\|_{\infty}
\end{align*}
$$

2.1.2. Upper bounds. In order to derive an estimate of $A_{i}(t)$ from above, we need the following lemma. For simplicity, we put $\psi = \sup_{x \in \Omega} H(x,t)$.

Lemma 2.3. Let $\tau > \theta(p+1)$ and $\theta_{m}$ be a positive number satisfying $\frac{\epsilon_{i}^{-1}\theta_{m}}{\epsilon_{i}^{-1}\psi} \leq \frac{1}{1 - \epsilon_{i}^{-1}\psi} \epsilon_{i}^{-1}\psi$.

Then the problem

$$
\begin{align*}
\frac{\partial \psi}{\partial t} & = \epsilon_{i}^{2} \Delta \psi - \psi + \theta_{m} \psi \quad \text{in } (0,\infty) \times \Omega \\
\frac{\partial \psi}{\partial t} & = 0 \quad \text{on } \partial \Omega
\end{align*}
$$

has a unique solution for all $t > 0$ and there exists a positive constant $C$ such that $\psi(t) \leq C \epsilon_{i}^{-1}$.
Behavior of solutions to an activator-inhibitor system

Here we have used the estimates \( R_s(f) \geq e^{-\alpha} \min\{R_s(x)\} \) and (2.27). It follows from the assumption (1.5) and \( \tau > \gamma|p - l| \) that

\[
0 < \frac{\alpha}{\gamma} \left( -\frac{1}{\tau} \right) \Delta \left( -\frac{1}{\tau} \right) > 0.
\]

Therefore,

\[
\text{max} \{ R_s(x) \} < \text{const}< \frac{\alpha}{\gamma} \left( -\frac{1}{\tau} \right) \Delta \left( -\frac{1}{\tau} \right)
\]

where \( C \) depends on \( f_s \) and \( R_s \).

3. Collapse of patterns in a more general case. The assertion in Theorem 2.1 holds in a more general case: that is, we consider the following system which generalizes (1.1)–(1.4) slightly:

\[
\begin{align*}
\frac{\partial u}{\partial t} & = \Delta u + \gamma f(u, w, v), \\
\frac{\partial w}{\partial t} & = \Delta w + \gamma f(w, u, v), \\
\frac{\partial v}{\partial t} & = \Delta v + \gamma f(v, u, w),
\end{align*}
\]

where \( \gamma \) is a positive constant. The following proposition explains why the collapse can occur for any \( \tau > 0 \) in this case:

**Proposition 3.1.** Let \( f(u, w, v) \) satisfy (1.5) and be differentiable with respect to \( (u, w, v) \), \( 0 < u, w, v < \infty \), and \( f(u, w, v) \) be continuous in \( (u, w, v) \). (Here, the case \( u, w, v = 0 \) is excluded for the Gierer–Meinhardt system 2.) Assume that \( |a| \), \( |b| < \infty \), and \( |c| < \infty \). Then the stationary solution \( (u_0, w_0, v_0) = (0, 0, 0) \) is not asymptotically stable. Here the initial values \( u(x, y, t = 0) \) and \( v(x, y, t = 0) \) are assumed to be positive.

**Proof.** Since \( \frac{\partial f}{\partial u}(0, 0, 0) = 0 \), \( \frac{\partial f}{\partial w}(0, 0, 0) = 0 \), and \( \frac{\partial f}{\partial v}(0, 0, 0) = 0 \), the Invariance principle around the stationary solution \( (u_0, w_0, v_0) = (0, 0, 0) \) becomes

\[
\begin{align*}
\left| \frac{\partial f}{\partial u} \right| u = 0, w = 0, v = 0, & \quad \text{for } x \in \Omega, \\
\left| \frac{\partial f}{\partial w} \right| u = 0, w = 0, v = 0, & \quad \text{for } x \in \Omega, \\
\left| \frac{\partial f}{\partial v} \right| u = 0, w = 0, v = 0, & \quad \text{for } x \in \Omega.
\end{align*}
\]

The assertion is verified by showing that all the eigenvalues of \( E \) have negative real parts.

4. Concluding Remarks. We consider the system (2.1)–(2.4) with \( p(x, y) \geq 0 \). The definition of the collapse of patterns is that the activator concentration converges uniformly to the trivial state \( 0 \) in Theorem 2.1 shows that the collapse of patterns occur if \( g(x, y) = p(x, y) \) and the initial data is restricted. Moreover, we assume that \( \min\{u_0, w_0, v_0\} \geq 0 \), then collapse of patterns can occur for any \( \tau \), which has been mentioned in Section 2. Therefore, the results may be summarized as follows:

**Modeling and Analysis of Droplet Motion on a Plane**

**References**

based on a modification of Navier-Stokes equations, such as phase-field methods (see, e.g. [7] or [10]) but our model is substantially different. In [6] it was shown that if the motion is slow, then it can be modeled by a parabolic operator and the approximation of the drop by the surface is sufficient. In this way, we obtain a parabolic problem for the evolution of a scalar volume-preserving membrane with an obstacle and positive contact angle.

Here we focus on the analysis of each problem. We propose construction of approximate solutions by a numerical method and show their convergence to a unique weak solution. Application of variational principles to constrained problems of this type is effective. Another method that was successful in abstract analysis of constrained evolution problems relies on the technique of elliptic and Young’s approximation. Although this framework is able to solve a large class of problems, it suffers some disadvantages which we try to overcome by introducing a different approach. The most substantial contribution of the new proof is that, contrary to elliptic/differential, it is applicable to numerical computations (for an example of numerical results, see [3]). Another feature of our method is the absence of assumptions on convexity, which is in essence indispensable for the definition of subdifferentials. This fact is significant in regard of future consideration of sharp contact assumptions on convexity, which are in essence indispensable for the definition of subdifferentials. This linearization of the minimal surface operator is relevant only from the mathematical point of view but not necessary as far as numerical computation is concerned.

First, we shall formally discuss the maximum principle for our equation. Let us consider the function $u>0$, and integrate over $\Omega$, we see that any solution of (3.6) preserves volume.

$\int_{\Omega} \frac{\partial u}{\partial t} + \Delta u - \sqrt{\gamma}\chi_\ast u = 0$.

Here $\gamma$ is a Lagrange multiplier originating in the volume constraint. This linearization of the minimal surface operator is relevant only from the mathematical point of view but not necessary as far as numerical computation is concerned.

Proposition 3.1. Let us suppose there exists a classical solution $u>0$ to (3.1)–(3.5). We also assume that $\gamma\geq 1$, where $\gamma$ is the minimum of $\gamma$ and $\gamma_0$. Then $\gamma\geq \frac{\sqrt{\gamma}}{\gamma_0}$, which we also denote as $\gamma>0$, so that the drop is small so that it is not influenced by gravitational forces, it has the shape of a spherical cap (see [3]). Mathematically, this can be shown using Sobolev’s partition function and supersmooth inequality in the framework of BV functions. In this case, we use the well-known Young’s equation for the contact angle $\theta > 0$.

$$u = u_0 + \frac{\sqrt{\gamma}}{\gamma_0} v$$

by explicitly minimizing functional (2.3) under condition (2.2). If we assume that the minimum of $u_0$ is smooth, we can also consider (2.2) for nonuniform distribution of $\gamma$ (i.e., nonsmooth drops).

3. Model. In this section, we derive a model for a droplet moving due to the difference in contact angles. We approximate the drop by its surface and assume that the area density of the surface is constant and that the surface tension is homogeneous. We also consider only the case when the contact angle $\theta$ is smaller than $90^\circ$, as in Figure 2.1. The motion of the system at zero volume preservation becomes the main aspect of the description of the moving drop.

Droplet motion on a plane

The essence of a drop can be written in the following way:

$$E = \frac{1}{2} \int_{\Omega} \beta |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla u|^2 (1 - \nabla u) \cdot (1 - \nabla u) d\Omega$$

where $E$ is the volume of the drop. If $\beta = 0$, then the drop is small so that it is not influenced by gravitational forces, it then has the shape of a spherical cap (see [3]). Mathematically, this can be shown using Sobolev’s partition function and supersmooth inequality in the framework of BV functions. In this case, we use the well-known Young’s equation for the contact angle $\theta > 0$.

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3.1. Droplet motion on a plane

Droplet motion on a plane

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by explicitly minimizing functional (2.3) under condition (2.2). If we assume that the minimum of $u_0$ is smooth, we can also consider (2.2) for nonuniform distribution of $\gamma$ (i.e., nonsmooth drops).

Because assuming $\theta < \theta_0, \theta < \theta_0, \gamma < \gamma_0$, we consider hydrophilic surfaces, where the value of $\gamma_0$ is close to $\gamma_0$, which gives small gradients in the shape of the drop and makes it possible to approximate.

This linearization of the minimal surface operator is relevant only from the mathematical point of view but not necessarily as far as numerical computation is concerned. Note that, in contrast to our previous results, see [6] (and integrating over $\Omega$, we see that any solution of (3.6) preserves volume.

$$\int_{\Omega} \frac{\partial u}{\partial t} + \Delta u - \sqrt{\gamma}\chi_\ast u = 0$$

Here $\gamma$ is a Lagrange multiplier originating in the volume constraint. This linearization of the minimal surface operator is relevant only from the mathematical point of view but not necessarily as far as numerical computation is concerned. Note that, in contrast to our previous results, see [6] (and integrating over $\Omega$, we see that any solution of (3.6) preserves volume.

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$\int_{\Omega} \frac{\partial u}{\partial t} + \Delta u - \sqrt{\gamma}\chi_\ast u = 0$.
The simplifying notation $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is used here. Applying Green's formula, we derive an equation with only first order derivatives of $u'$ and then take $t$ to zero. Noting that $\mathcal{A}_u (x, t) = \mathcal{A}_u (x, t)$ and assuming that $\mathcal{A}_u (x, t) = \mathcal{A}_u (x, t)$, we have for the right-hand side of (3.5),
\[
\int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) \delta \nu \, dx - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx = 0.
\]
(3.6)

Noting that $\mathcal{A}_u (x, t) \equiv \mathcal{A}_u (x, t)$, the error term normal to the set $\{ t > 0 \} \cap \{ x \in \Omega \}$ is being the time-dependent component.

As for the left-hand side of (3.5), we can proceed in the following way:
\[
\int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx = \int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx
\]

Under the notation $Dv \equiv \{ u_1, \ldots, u_n, \}$, the error term normal can be expressed as $x = Dv (x, t)$. Hence, on $\{ t \geq 0 \}$ we get $x = Dv (x, t)$ and
\[
\int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx = \int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx.
\]
By (3.6) and (3.7), we conclude that
\[
\int_{\partial \Omega} \left[ \mathcal{A}_u (x, t) \delta \nu \right] \cdot \nabla u' (x, t) \, dS - \int_{\Omega} \mathcal{A}_u (x, t) \frac{\partial u}{\partial t} \delta \nu \, dx = 0.
\]

So almost all points $(x, t) \in (0, 1) \times \Theta$.

Let us study the solution between the free boundary condition (3.8), which we have just formally derived, and Young's equation (2.3). Using (2.3), we find
\[
\exists \lambda > 2 \quad \left( \int_{-1}^1 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \, dx \right) = \frac{\int_{\Omega} \left( u'(x, t) \right)^2 \, dx}{\int_{\Omega} \mathcal{A}_u (x, t) \, dx} \geq 0,
\]
where $\mathcal{A}_u (x, t) \equiv \mathcal{A}_u (x, t)$ and that by the approximation (2.4) we have introduced an error of order $O(h^2)$ in the contact angle.

**Theorem 4.2.** There exists a weak solution of the above approximate problem satisfying
\[
\mathcal{A} \geq \mathcal{A}_0, \quad (4.4)
\]
and the following estimate:
\[
\left\{ \begin{array}{l}
\left[ e^{\gamma} u \right]_{x=0} = 0, \\
\int_{\Omega} \mathcal{A}_u (x, t) \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) \, dx \leq C (\Omega) \left( \int_{\Omega} \mathcal{A}_u (x, t) \, dx \right) \left( \int_{\Omega} \left( u'(x, t) \right)^2 \, dx \right)
\end{array} \right.
\]
for some $t \in (0, T)$, where $C (\Omega)$ does not depend on $t$.

Moreover, the solutions are uniformly bounded in $[0, T]$ and uniformly Hölder continuous on $\Omega$ with respect to the parameter $t$.

The rough structure of the proof, based on the result (10), is to use a min-max method for a time-discretized functional in order to construct approximate solutions [see (5)], and to show that these approximations converge to a weak solution. We divide the time interval $(0, T)$ continuously into $N$ subdivisions of length $\theta = T/N, N \in N$. For each $0 \leq \theta \leq N$ we construct an approximate solution $u^u (x, t)$ in the following manner.

First of all, put $u^u (x, t) = u_0$ and for $x = 1, 2, \ldots, N$, find a minimum $\mathcal{A}_u (x, t) \equiv \mathcal{A}_u (x, t)$ of the functional
\[
\mathcal{A}_u (x, t) = \left( \int_{-1}^1 \left( \frac{\partial u^u (x, t)}{\partial x} \right)^2 \, dx \right) + \left( \int_{\Omega} \mathcal{A}_u (x, t) \, dx \right)
\]
(4.7)
in the admissible function set
\[
\mathcal{A}_u (x, t) = \left\{ u \in \mathcal{A}_u (x, t) \cap \mathcal{A}_u (x, t) \mid \mathcal{A}_u (x, t) \right\}
\]
(4.8)

This functional is called the discrete Morse flow corresponding to (4.11). We remark that both in (4.4) are (4.7), the role of a penalty is played by the smoothed characteristic function of the uniform constraint on the set $\mathcal{A}_u (x, t)$. The minimizing principle then yields the estimate (4.4).

As the next step, we approximate the minimum $u^u (x, t)$, $x = 0, 1, 2, \ldots, N$ in time, i.e., we introduce the following functions (see Figure 4.2):
\[
u^u (x, t), \quad u^u (x, t) \leq 0, \quad (x, t) \in \Omega, x = 1, 2, \ldots, N.
\]
(4.9)

\[
u^u (x, t), \quad u^u (x, t) = 0, \quad (x, t) \in \Omega, x = 1, 2, \ldots, N, \quad t \in \Omega.
\]
(4.10)

We prove Theorem 4.2 by sending $\lambda$ to zero. Above all, we have to show that there exists a minimum $u^u (x, t)$ of the functional $\mathcal{A}_u (x, t) \equiv \mathcal{A}_u (x, t)$ that converges to a weak solution of the problem (3.1). Here we omit the technical proof since it can be recovered from [10] by careful modification of the discontinuous Morse flow variational technique. Since we also obtain the uniform convergence of approximate solutions with respect to the smoothing parameter $\lambda, \theta$ shall finally be able to construct a weak solution to the original problem.

To begin with, we introduce the approximate problem parameterized by $\lambda > 0$:
\[
u^u (x, t) = 0, \quad \mathcal{A}_u (x, t) \equiv \mathcal{A}_u (x, t)
\]
(4.11)

where
\[
u^u (x, t) = \lambda \left[ e^{\gamma} u \right]_{x=0} + \left( \int_{\Omega} \mathcal{A}_u (x, t) \, dx \right)
\]
and $\mathcal{A}_u (x, t) = \lambda \left[ e^{\gamma} u \right]_{x=0}$ is a smoothing of the characteristic function $\mathcal{A}_u (x, t)$ (see Figure 4.1).

Note that the denominator in (4.2) is positive.
Also, by the uniform convergence (4.12), we see that $s^\alpha > 0 = x_0$ for $s$ small enough. Consequently, we have for small $s$ the key identity

$$\int_0^1 (\phi(\varepsilon u^\delta) + \phi(\varepsilon u^\delta_v)) \, dx = \int_0^1 (\phi(u^\delta) + \phi(u^\delta_v)) \, dx.$$

The definition of support of test function $\eta$ becomes relevant here, leading now back to the characteristic function in (3.1). Finally, due to an estimate on an approximate Lagrange multiplier, we have the uniform boundedness of $\lambda^\alpha$ in $L^2(\Omega)$. Thus, restricting a subsequence, there exists a function $\lambda^\alpha \in L^2(\Omega)$ such that $\lambda^\alpha \rightharpoonup \lambda$ weakly in $L^2(\Omega)$.

We have arrived at the following identity:

$$\int_0^1 (\phi(u^\delta_a) - \phi(u^\delta_b)) \, dx = \int_0^1 \lambda^\alpha_\delta \, dx.$$

Now, the form of $\lambda$ would be caused if we could put

$$\lambda^\alpha(x) = \lambda^\alpha_\delta(x) = 0 \quad \text{for} \quad |x| = \varepsilon_0.$$

in (4.13). We cannot do directly because this function does not have compact support (and $|x| = \varepsilon_0$). We also cannot apply any approximation technique. Indeed, we only know that function $\lambda$ is a Bocher continuous $H^2$-function, which is not good enough information to get necessary regularity (local pathologic continuity) of the boundary of $|x| = \varepsilon_0$. Thus, we cannot use approximations by function $\lambda^\alpha_\delta(x)$ because of (4.13).

We notice that (4.14) is an admissible function in (4.3). Then we get

$$\int_0^1 \lambda^\alpha_\delta \, dx = 0.$$

Since solution of (4.13) is unique (this can be seen using fixed point theorems $\lambda \in C^0$, setting $z = x$ in (4.16) and $z = u$ as the corresponding role for $x$, adding the resulting inequalities and using Gronwall’s lemma), we conclude that there is a unique weak solution in the sense of Definition 2, which is identical to the unique solution in the sense of Tent’s approximation.

To start with, take any $u \in L^2(\Omega; T; R^N(0))$ and define function $\delta_t^\alpha$, then

$$\int_0^1 \delta_t^\alpha \, dx = 0.$$

for $\delta_t^\alpha \in [0, 1]$. Then $\delta_t^\alpha \to 0$, the function $\delta_t^\alpha \rightarrow \delta_t$ is nonnegative and has volume $V$, then as an admissible solution for the functional (4.3), adding

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<table>
<thead>
<tr>
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<th>September 5 (Fri.)</th>
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<td>9:15 – 9:30</td>
<td>Opening</td>
<td>Lunch</td>
</tr>
<tr>
<td>Chair: Shigetoshi Yazaki</td>
<td>10:00 – 10:30 Daisuke Tagami</td>
<td>Lunch conference lunch</td>
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<tr>
<td>9:30 – 10:30</td>
<td>Peter Frøkiař</td>
<td>10:40 – 11:10 Kazuhiko Hiramatsu</td>
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<tr>
<td>10:30 – 11:10</td>
<td>Petr Pauš</td>
<td>11:20 – 12:00 Shigetoshi Yazaki</td>
</tr>
<tr>
<td>11:30 – 12:00</td>
<td>Tatsuki Kawakami</td>
<td>a visit to a multi-fan wind tunnel</td>
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<tr>
<td>Chair: Masato Kimura</td>
<td>10:00 – 10:30 Daisuke Tagami</td>
<td>Lunch conference lunch</td>
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<tr>
<td>9:15 – 9:20</td>
<td>Opening</td>
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<tr>
<td>9:30 – 10:30</td>
<td>Peter Frøkiař</td>
<td>10:40 – 11:10 Kazuhiko Hiramatsu</td>
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<td>10:30 – 11:10</td>
<td>Petr Pauš</td>
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<tr>
<td>Chair: Daniel Sevcik</td>
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<tr>
<td>14:00 – 14:30</td>
<td>Seiro Omata</td>
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<tr>
<td>14:40 – 15:10</td>
<td>Hideaki Nakagawa</td>
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<tr>
<td>15:20 – 15:50</td>
<td>Masaki Kosuma</td>
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<tr>
<td>16:10 – 16:40</td>
<td>Petr Furmanek</td>
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<tr>
<td>16:50 – 17:20</td>
<td>Jiří Miška</td>
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<tr>
<td>17:30 – 18:00</td>
<td>Radek Fučík</td>
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<tr>
<td>September 7 (Sun.)</td>
<td>September 8 (Mon.)</td>
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<td>10:00 – 10:30</td>
<td>Karel Míška</td>
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<td>10:40 – 11:10</td>
<td>Olga Drblíková</td>
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<td>14:00 – 15:30</td>
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<td>Pavel Strachota, Atsushi Suzuki</td>
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<td>Tohru Tojikawa, Shigetoshi Yazaki</td>
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<td>September 9 (Tue.)</td>
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<td>Petr Knobloch</td>
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<tr>
<td>16:30 – 17:00</td>
<td>Michal Beneš</td>
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<tr>
<td>17:30 – 18:00</td>
<td>Announcements</td>
<td>Lunch</td>
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